

## Fourier Transform

### Reference

- Chapter 2.2 – 2.5, Carlson, Communication Systems

Using the Fourier series, a signal over a finite interval can be represented in terms of a complex exponential series. If the function is periodic, this representation can be extended over the entire interval  $(-\infty, \infty)$ .

$$f(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t} \qquad F_n = \frac{1}{T} \int_0^T f(t) e^{-jn\omega_0 t} dt$$

Fourier transforms.1

## Fourier Transform

On the other hand, Fourier transform provides the link between the time-domain and frequency domain descriptions of a signal. Fourier transform can be used for both periodic and non-periodic signals.

$$f(t) = \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega \qquad F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$$

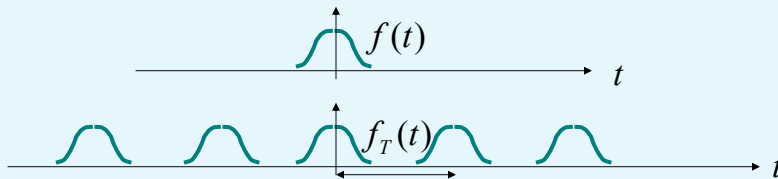
Fourier transforms.2

## Example

The the last example of the previous lecture shows that if the period ( $T$ ) of a periodic signal increases, the fundamental frequency ( $\omega_0=2\pi/T$ ) becomes smaller and the frequency spectrum becomes more dense while the amplitude of each frequency component decreases. The shape of the spectrum, however, remains unchanged with varying  $T$ . Now, we will consider a signal with period approaching infinity.

Suppose we are given a non-periodic signal  $f(t)$ . In order to applying Fourier series to the signal  $f(t)$ , we construct a new periodic signal  $f_T(t)$  with period  $T$ .

Fourier transforms.3



The original signal can be obtained back again by letting the period  $T \rightarrow \infty$ , that is,

$$f(t) = \lim_{T \rightarrow \infty} f_T(t)$$

Fourier transforms.4

The periodic function  $f_T(t)$  can be represented by an exponential Fourier series.

$$f_T(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t} \quad \text{where}$$

$$F_n = \frac{1}{T} \int_{-T/2}^{T/2} f_T(t) e^{-jn\omega_0 t} dt \quad \text{and} \quad \omega_0 = 2\pi / T$$

As the magnitude of the Fourier coefficients go to zero when the period is increased, we define

$$\omega_n \equiv n\omega_0 \quad \text{and} \quad F(\omega_n) \equiv TF_n$$

The Fourier series pair become

$$f_T(t) = \sum_{n=-\infty}^{\infty} \frac{1}{T} F(\omega_n) e^{j\omega_n t} \quad \text{where} \quad F(\omega_n) = \int_{-T/2}^{T/2} f_T(t) e^{-j\omega_n t} dt$$

Fourier transforms.5

The spacing between adjacent lines ( $\Delta\omega$ ) in the line spectrum of  $f_T(t)$  is

$$\Delta\omega = 2\pi / T$$

Therefore, we have

$$f_T(t) = \sum_{n=-\infty}^{\infty} F(\omega_n) e^{j\omega_n t} \frac{\Delta\omega}{2\pi} \quad (1)$$

Now as  $T$  becomes very large,  $\Delta\omega$  becomes smaller and the spectrum becomes denser. In the limit, the discrete lines in the spectrum of  $f_T(t)$  merge and the frequency spectrum becomes continuous.

Fourier transforms.6

Mathematically, the infinite sum (1) becomes an integral

$$\lim_{T \rightarrow \infty} f_T(t) = \lim_{T \rightarrow \infty} \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} F(\omega_n) e^{j\omega_n t} \Delta\omega$$

$$\Rightarrow f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega \quad \leftarrow \text{Inverse Fourier transform of } F(\omega)$$

Similarly,

$$F_n = \frac{1}{T} \int_{-T/2}^{T/2} f_T(t) e^{-jn\omega_o t} dt$$

$$\Rightarrow F(\omega_n) = \int_{-T/2}^{T/2} f_T(t) e^{-j\omega_n t} dt \quad \because \omega_n \equiv n\omega_o \text{ and } F(\omega_n) \equiv TF_n$$

$$\Rightarrow \lim_{T \rightarrow \infty} F(\omega_n) = \lim_{T \rightarrow \infty} \int_{-T/2}^{T/2} f_T(t) e^{-jn\omega_o t} dt$$

$$\Rightarrow F(\omega) = \int_{-T/2}^{T/2} f_T(t) e^{-jn\omega_o t} dt \quad \leftarrow \text{Fourier transform of } f(t)$$

Fourier transforms.7

Operators are often used to denote the transform pair.

$$\mathfrak{F}\{f(t)\} \quad \text{Fourier transform of } f(t)$$

$$\mathfrak{F}^{-1}\{F(\omega)\} \quad \text{Inverse Fourier transform of } F(\omega)$$

$$f(t) = \mathfrak{F}^{-1}[\mathfrak{F}\{f(t)\}]$$

Fourier transforms.8

## Singularity functions

- This is a particular class of functions which are useful in signal analysis.
- They are mathematical idealization and, strictly speaking, do not occur in physical systems.
- Good approximation to certain limiting condition in physical systems. For example, a very narrow pulse.

Fourier transforms.9

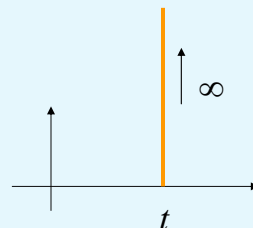
## Singularity functions

### Impulse function

- This function has the property exhibited by the following integral:

$$\int_a^b f(t)\delta(t-t_0)dt = \begin{cases} f(t_0) & a < t_0 < b \\ 0 & \text{elsewhere} \end{cases} \quad (2)$$

for any  $f(t)$  continuous at  $t = t_0$ ,  $t_0$  is finite. All the properties can be derived from this definition.



Fourier transforms.10

## Singularity functions

Properties of the impulse function

### Amplitude

All values of  $\delta(t)$  for  $t \neq t_0$  are zero.

The amplitude at the point  $t = t_0$  is undefined.

### Area (strength)

If  $f(t) = 1$ , (2) becomes

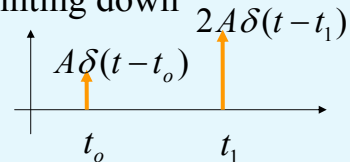
$$\int_a^b \delta(t - t_0) dt = 1 \quad a < t_0 < b$$

Therefore  $\delta(t)$  has unit area. Similarly,  $A\delta(t)$  has an area of A units.

Fourier transforms.11

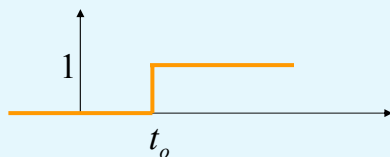
### Graphic representation

To display the impulse function at  $t = t_0$ , an arrow is used to avoid to display the amplitude. The area of the impulse is designated by a quantity in parentheses beside the arrow or by the height of the arrow. An arrow pointing down indicates negative area.



### Relation to the unit step function

The unit step function is defined by



$$u(t - t_0) = \begin{cases} 1 & t > t_0 \\ 0 & t < t_0 \end{cases}$$

Fourier transforms.12

Using (2) and letting  $a = -\infty, b = t$ , and  $f(t) = 1$

$$\int_a^b f(t)\delta(t-t_o)dt$$

$$\Rightarrow \int_{-\infty}^t \delta(\tau-t_o)d\tau = \begin{cases} 1 & t > t_o \\ 0 & t < t_o \end{cases} \Rightarrow \int_{-\infty}^t \delta(\tau-t_o)d\tau = u(t-t_o)$$

Therefore the integral of the unit impulse function is the unit step function. The converse can also be shown by differentiating both sides of the above equation.

$$\int_{-\infty}^t \delta(\tau-t_o)d\tau = u(t-t_o)$$

$$\Rightarrow \frac{d}{dt} \int_{-\infty}^t \delta(\tau-t_o)d\tau = \frac{d}{dt} u(t-t_o)$$

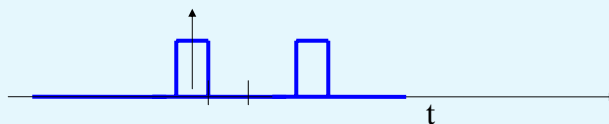
$$\Rightarrow \delta(t-t_o) = \frac{d}{dt} u(t-t_o)$$

Fourier transforms.13

### Spectral density function $F(\omega)$

$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)e^{j\omega t} d\omega$  represents  $f(t)$  as a continuous sum of exponential functions with frequencies lying in the interval  $(-\infty, \infty)$ . The relative amplitude of the components at any frequency  $\omega$  is proportional to  $F(\omega)$ .  $F(\omega)$  is called the spectral density function of  $f(t)$ .

- Each point on the  $F(\omega)$  curve contributes nothing to the representation of  $f(t)$ ; it is the area that contributes.

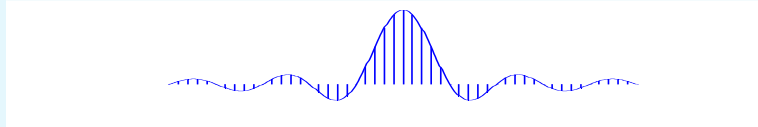


Fourier transforms.14

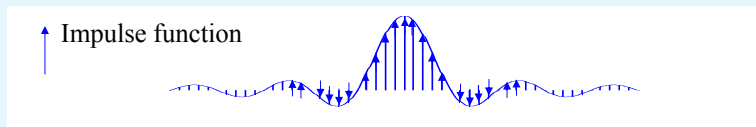
## Spectral density function $F(\omega)$

### Example

- Consider a rectangular pulse train
- The line spectrum of the Fourier series of the signal is



The spectral density of the signal is



Fourier transforms.15

## Existence of the Fourier transform

We may ignore the question of the existence of the Fourier transform of a time function when it is an accurately specified description of a physically realizable signal. In other words, physical realizability is a sufficient condition for the existence of a Fourier transform.

Fourier transforms.16



## Parseval's theorem for energy signals

Using the Parseval's theorem, we can find the energy of a signal in either the time domain or the frequency domain.

$$E = \int_{-\infty}^{\infty} |f(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(\omega)|^2 d\omega$$

Example

Energy contained in the frequency band  $\omega_1 < \omega < \omega_2$  of a real-valued signal is

$$\begin{aligned} & \frac{1}{2\pi} \left[ \int_{-\omega_2}^{-\omega_1} |F(\omega)|^2 d\omega + \int_{\omega_1}^{\omega_2} |F(\omega)|^2 d\omega \right] \\ &= \frac{1}{\pi} \int_{\omega_1}^{\omega_2} |F(\omega)|^2 d\omega \end{aligned}$$

Fourier transforms.17

## Fourier transforms of some signals

**Impulse function  $\delta(t)$**

The Fourier transform of a unit impulse  $\delta(t)$  is

$$\mathfrak{F}\{\delta(t)\} = \int_{-\infty}^{\infty} \delta(t)e^{-j\omega t} dt = e^{j0} = 1 \quad \because \int_{-\infty}^{\infty} \delta(t)f(t)dt = f(0)$$

It shows that an impulse function has a uniform spectral density over the entire frequency spectrum. In practice, a narrow pulse in time domain has a very wide bandwidth in frequency domain.

Example

If we increase the transmission rate of digital signal, a wider frequency bandwidth is needed.

Fourier transforms.18

## Fourier transforms of some signals

### Complex exponential function

The Fourier transform of a complex exponential function is

$$\mathfrak{F}\{e^{\pm j\omega_o t}\} = \int_{-\infty}^{\infty} e^{\pm j\omega_o t} e^{-j\omega t} dt = ?$$

On the other hand, the inverse Fourier transform of  $\delta(\omega \pm \omega_o)$  is

$$\mathfrak{F}^{-1}\{\delta(\omega \pm \omega_o)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \delta(\omega \pm \omega_o) e^{j\omega t} dt = \frac{1}{2\pi} e^{\pm j\omega_o t}$$
$$\therefore \mathfrak{F}\{e^{\pm j\omega_o t}\} = \mathfrak{F}\{\mathfrak{F}^{-1}\{\delta(\omega \pm \omega_o)\}\} = 2\pi\delta(\omega \pm \omega_o)$$

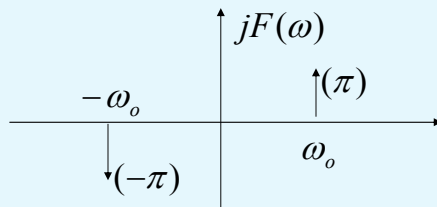
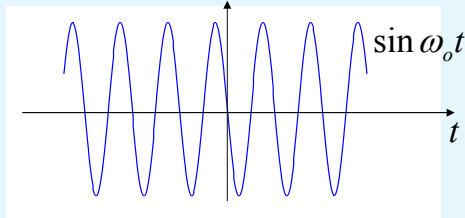
### Sinusoidal signals

The sinusoidal signals can be written as

$$\cos x = \frac{e^{jx} + e^{-jx}}{2} \qquad \sin x = \frac{e^{jx} - e^{-jx}}{2j}$$

The Fourier transform of these signals are

$$\mathfrak{F}\{\cos \omega_o t\} = \mathfrak{F}\left\{\frac{e^{j\omega_o t} + e^{-j\omega_o t}}{2}\right\} = \pi\delta(\omega - \omega_o) + \pi\delta(\omega + \omega_o)$$
$$\mathfrak{F}\{\sin \omega_o t\} = \mathfrak{F}\left\{\frac{e^{j\omega_o t} - e^{-j\omega_o t}}{2j}\right\} = \frac{\pi}{j}\delta(\omega - \omega_o) - \frac{\pi}{j}\delta(\omega + \omega_o)$$



Fourier transforms.21

### Periodic signal

A periodic signal  $f(t)$  can be represented by its exponential Fourier series.  $f(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_o t}$  where  $\omega_o = 2\pi / T$

$$\begin{aligned} \text{The Fourier transform is } \mathfrak{F}\{f(t)\} &= \mathfrak{F}\left\{\sum_{n=-\infty}^{\infty} F_n e^{jn\omega_o t}\right\} \\ &= \sum_{n=-\infty}^{\infty} F_n \mathfrak{F}\{e^{jn\omega_o t}\} \\ &= 2\pi \sum_{n=-\infty}^{\infty} F_n \delta(\omega - n\omega_o) \end{aligned}$$

Thus the spectral density of a periodic signal consists of a set of impulses located at the harmonic frequencies of the signal. The area of each impulse is  $2\pi$  times the values of its corresponding coefficient in the exponential Fourier series.

Fourier transforms.22

## Some properties of the Fourier transform

### Linearity (superposition)

The Fourier transform is a linear operation based on the properties of integration and therefore superposition applies.

$$\mathfrak{F}\{af(t) + bg(t)\} = aF(\omega) + bG(\omega)$$

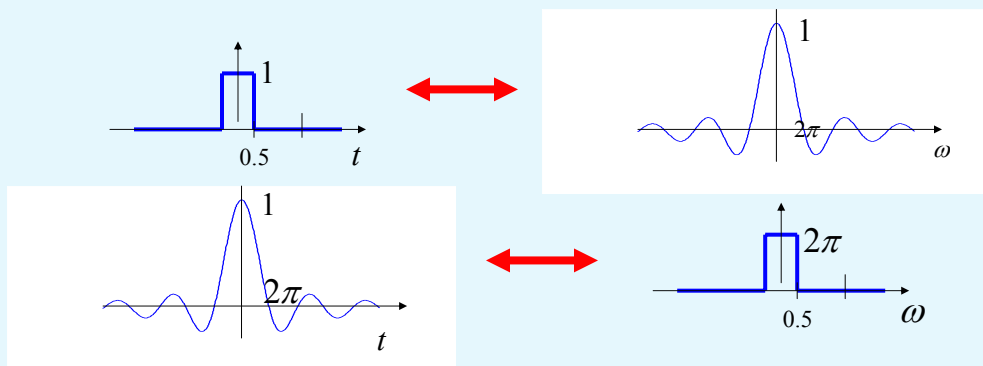
Fourier transforms.23

## Some properties of the Fourier transform

### Duality

If  $\mathfrak{F}\{f(t)\} = F(\omega)$ , then  $\mathfrak{F}\{F(t)\} = 2\pi f(-\omega)$

Example



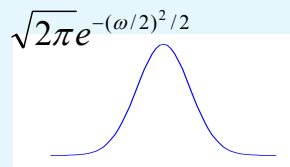
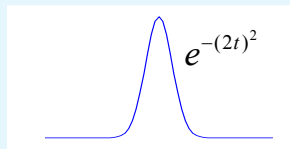
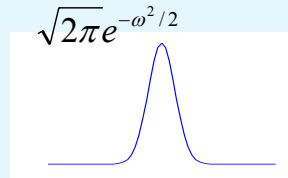
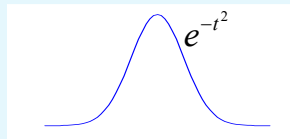
Fourier transforms.24

## Coordinate scaling

The expansion or compression of a time waveform affects the spectral density of the waveform. For a real-valued scaling constant  $\alpha$  and any signal  $f(t)$

$$\mathfrak{T}\{f(\alpha t)\} = \frac{1}{|\alpha|} F\left(\frac{\omega}{\alpha}\right)$$

Example



Fourier transforms.25

## Time shifting (delay)

$$\mathfrak{T}\{f(t - t_0)\} = F(\omega)e^{-j\omega t_0}$$

If a signal is delayed in time by  $t_0$ , its magnitude spectral density remains unchanged and a negative phase  $-\omega t_0$  is added to each frequency component.

Fourier transforms.26

## Frequency shifting (Modulation)

$$\mathfrak{F}\{f(t)e^{j\omega_0 t}\} = F(\omega - \omega_0)$$

Therefore multiplying a time function by  $e^{j\omega_0 t}$  causes its spectral density to be translated in frequency by  $\omega_0$ .

Example

$$\mathfrak{F}\{f(t)\cos\omega_0 t\} = \frac{1}{2}[F(\omega + \omega_0) + F(\omega - \omega_0)]$$



Fourier transforms.27

## Differentiation

$$\mathfrak{F}\left\{\frac{d}{dt}f(t)\right\} = j\omega F(\omega)$$

Time differentiation enhances the high frequency components of a signal.

## Integration

$$\mathfrak{F}\left\{\int_{-\infty}^t f(\tau)d\tau\right\} = \frac{1}{j\omega}F(\omega) + \pi F(0)\delta(\omega) \quad \text{where } F(0) = \int_{-\infty}^{\infty} f(t)dt$$

Integration in time suppresses the high-frequency components of a signal.

Fourier transforms.28