

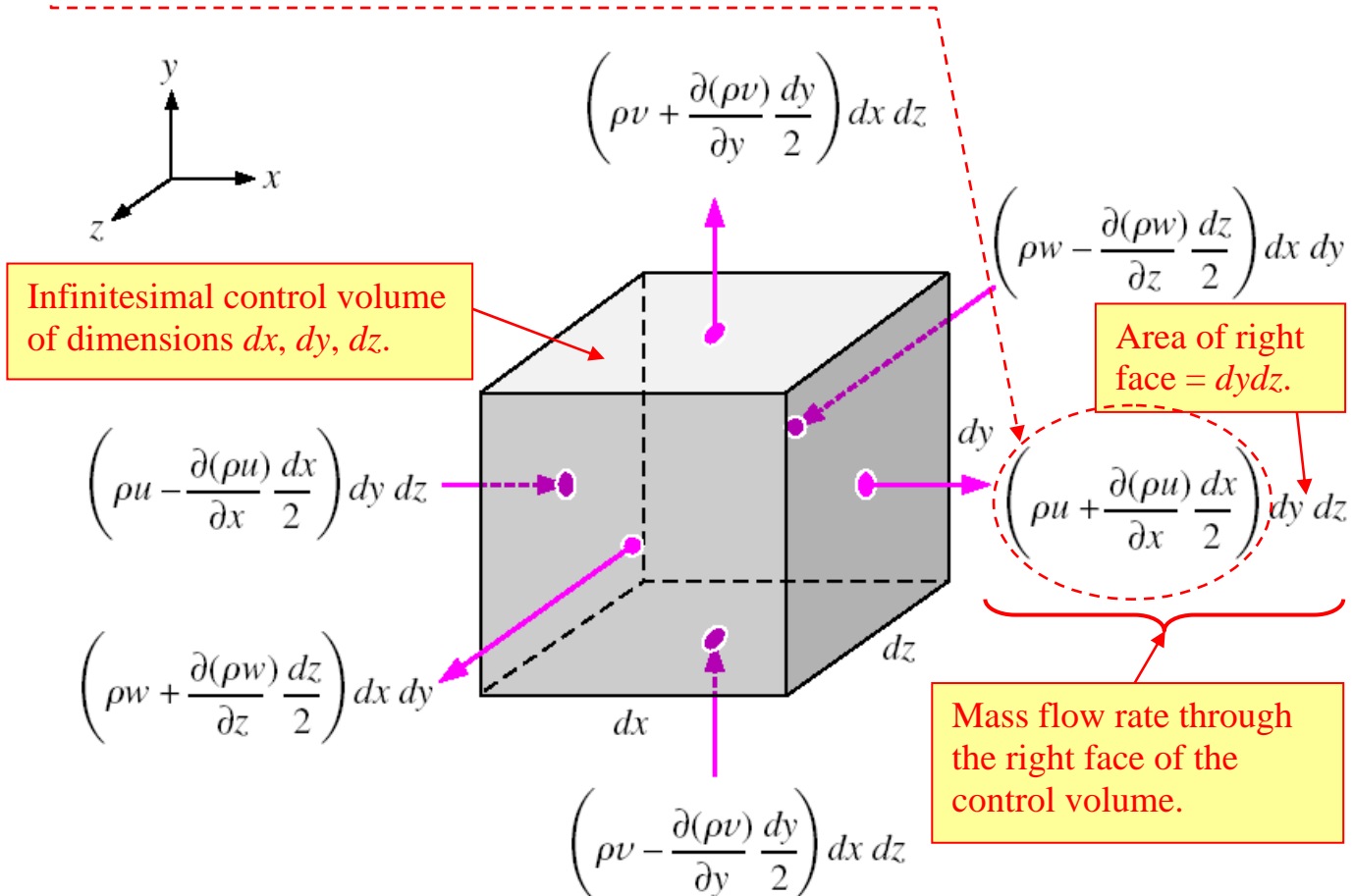
Derivation of the Continuity Equation (Section 9-2, Çengel and Cimbala)

We summarize the second derivation in the text – the one that uses a *differential control volume*. First, we approximate the mass flow rate into or out of each of the six surfaces of the control volume, using *Taylor series expansions* around the center point, where the velocity components and density are u , v , w , and ρ . For example, at the right face,

Ignore terms higher than order dx .

$$(\rho u)_{\text{center of right face}} = \rho u + \frac{\partial(\rho u)}{\partial x} \frac{dx}{2} + \frac{1}{2!} \frac{\partial^2(\rho u)}{\partial x^2} \left(\frac{dx}{2}\right)^2 + \dots \quad (9-6)$$

The mass flow rate through each face is equal to ρ times the normal component of velocity through the face times the area of the face. We show the mass flow rate through all six faces in the diagram below (Figure 9-5 in the text):



Next, we add up all the mass flow rates through all six faces of the control volume in order to generate the general (unsteady, incompressible) *continuity equation*:

Net mass flow rate into CV:

all the *positive* mass flow rates (into CV)

$$\sum_{\text{in}} \dot{m} \cong \underbrace{\left(\rho u - \frac{\partial(\rho u)}{\partial x} \frac{dx}{2} \right) dy dz}_{\text{left face}} + \underbrace{\left(\rho v - \frac{\partial(\rho v)}{\partial y} \frac{dy}{2} \right) dx dz}_{\text{bottom face}} + \underbrace{\left(\rho w - \frac{\partial(\rho w)}{\partial z} \frac{dz}{2} \right) dx dy}_{\text{rear face}}$$

Net mass flow rate out of CV:

all the *negative* mass flow rates (out of CV)

$$\sum_{\text{out}} \dot{m} \cong \underbrace{\left(\rho u + \frac{\partial(\rho u)}{\partial x} \frac{dx}{2} \right) dy dz}_{\text{right face}} + \underbrace{\left(\rho v + \frac{\partial(\rho v)}{\partial y} \frac{dy}{2} \right) dx dz}_{\text{top face}} + \underbrace{\left(\rho w + \frac{\partial(\rho w)}{\partial z} \frac{dz}{2} \right) dx dy}_{\text{front face}}$$

We plug these into the integral conservation of mass equation for our control volume:

$$\int_{\text{CV}} \frac{\partial \rho}{\partial t} dV = \sum_{\text{in}} \dot{m} - \sum_{\text{out}} \dot{m} \quad (9-2)$$

This term is approximated at the center of the tiny control volume, i.e.,

$$\int_{\text{CV}} \frac{\partial \rho}{\partial t} dV \cong \frac{\partial \rho}{\partial t} dx dy dz$$

The conservation of mass equation (Eq. 9-2) thus becomes

$$\frac{\partial \rho}{\partial t} dx dy dz = - \frac{\partial(\rho u)}{\partial x} dx dy dz - \frac{\partial(\rho v)}{\partial y} dx dy dz - \frac{\partial(\rho w)}{\partial z} dx dy dz$$

Dividing through by the volume of the control volume, $dx dy dz$, yields

Continuity equation in Cartesian coordinates:

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0 \quad (9-8)$$

Finally, we apply the definition of the **divergence** of a vector, i.e.,

$$\vec{\nabla} \cdot \vec{G} = \frac{\partial G_x}{\partial x} + \frac{\partial G_y}{\partial y} + \frac{\partial G_z}{\partial z} \quad \text{where } \vec{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \text{ and } \vec{G} = (G_x, G_y, G_z)$$

Letting $\vec{G} = \rho \vec{V}$ in the above equation, where $\vec{V} = (u, v, w)$, Eq. 9-8 is re-written as

$$\text{Continuity equation:} \quad \frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{V}) = 0 \quad (9-5)$$