



Digital Signal & Image Processing

Lecture-2

Dr Prasanthi Rathnala

Department of ECE



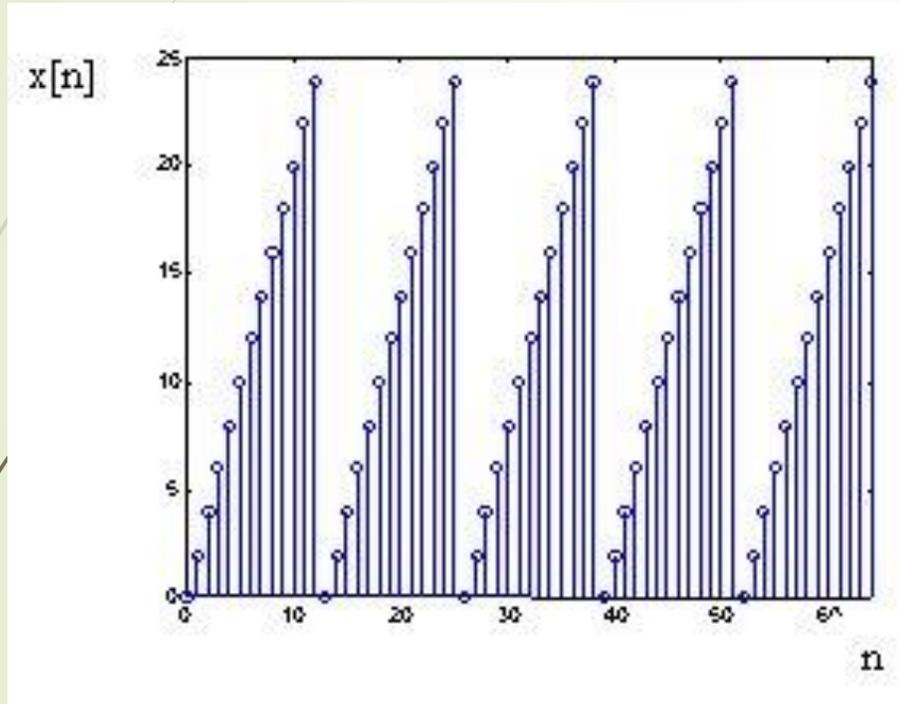
Overview

- Signal Spectrum
 - Periodic vs. Non-Periodic Signal
 - Spectra of Non-Periodic Signals
 - Properties of DTFT
 - Spectra of Periodic Signals
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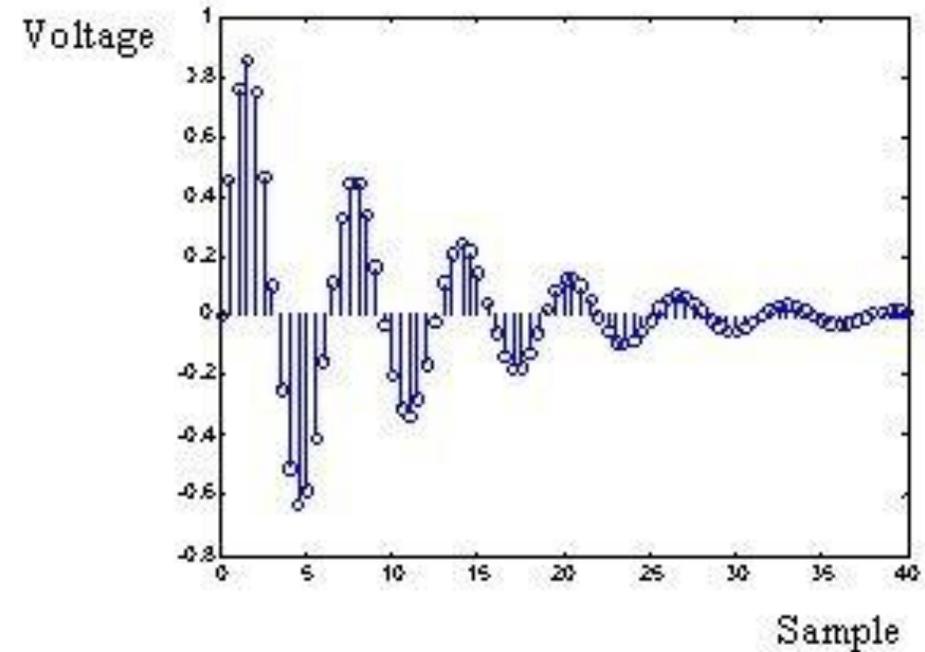
Signal Spectrum

- ▶ The **spectrum** of a signal describes its frequency content.
- ▶ For example, a sinusoid contains a single frequency, while white noise contains all frequencies.
- ▶ The tool used to calculate an accurate spectrum depends on the nature of the signal i.e., (Periodic or Non-Periodic)

Periodic vs. Non-Periodic Signal



- Periodic signals are those that repeat at regular intervals for all time.



- Non-periodic signals do not repeat at regular intervals.

Spectra of Non-Periodic Signals

- ▶ If the signal is non-periodic, the discrete time Fourier transform (DTFT) is used.
- ▶ The DTFT for a non-periodic signal gives signal's spectrum as

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-jn\Omega}$$

- ▶ The DTFT spectrum $X(\Omega)$ is a complex number and may be expressed as

$$X(\Omega) = |X(\Omega)|e^{j\theta(\Omega)}$$

Properties of DTFT

- The spectrum for both non-periodic and periodic signals furnishes a magnitude spectrum and a phase spectrum.
- The Magnitude Spectrum- relates to the size or amplitude of the components at each frequency.
- The Phase Spectrum- gives the phase relationships between the components at different frequencies.
- For non-periodic signals, $X(\Omega) = |X(\Omega)| e^{j\theta(\Omega)}$
- The magnitude spectrum is even and the phase spectrum is odd.
- Both magnitude and phase spectra are continuous, smooth and periodic with period 2π .

Properties of DTFT

- ▶ The **magnitude spectrum** may be plotted as
 $|X(\Omega)|$ versus digital frequency Ω
or
 $|X(f)|$ versus analog frequency f
- ▶ The **phase spectrum** may be plotted as
 $\theta(\Omega)$ versus digital frequency Ω
or
 $\theta(f)$ versus analog frequency f

Spectra of Non-Periodic Signals

- ▶ The calculation of the DTFT requires all samples of the non-periodic signal.
- ▶ When signal has infinite number of non-zero samples that decrease in size, the DTFT may be approximated by truncating the signal where its amplitude drops below some suitably low threshold.
- ▶ It is easier to interpret the spectra by converting the digital frequencies into analog frequencies.

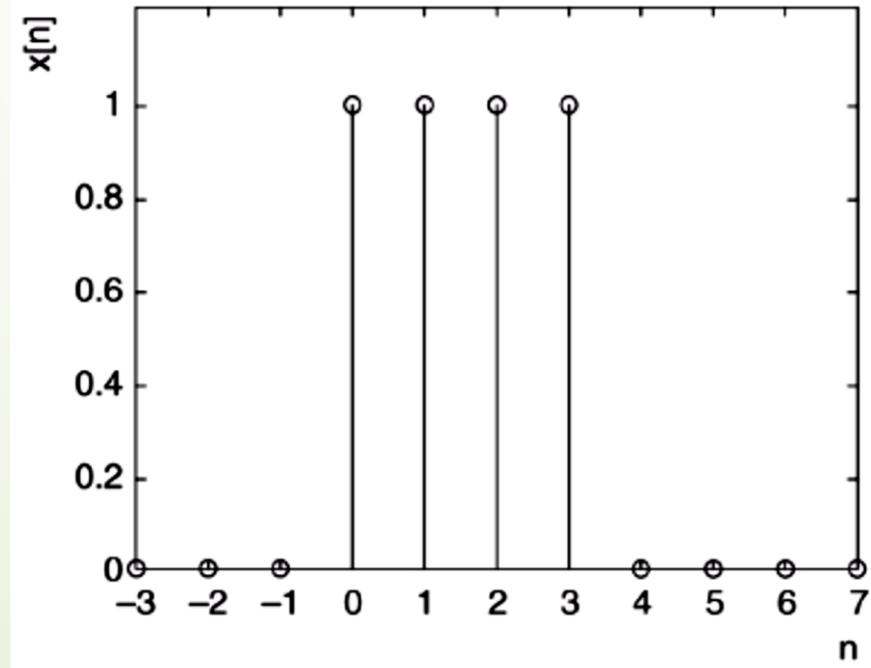
$$f = \Omega \frac{f_s}{2\pi}$$

Spectra of Non-Periodic Signals

Example 1: Find the magnitude and phase spectra for the rectangular pulse $x[n] = u[n] - u[n-4]$ as function of Ω . Plot linear gains and phases in radians.

Solution

► The rectangular pulse is plotted in following figure.



Spectra of Non-Periodic Signals

- ▶ The spectrum may be computed as

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-jn\Omega} = 1 + e^{-j\Omega} + e^{-j2\Omega} + e^{-j3\Omega}$$

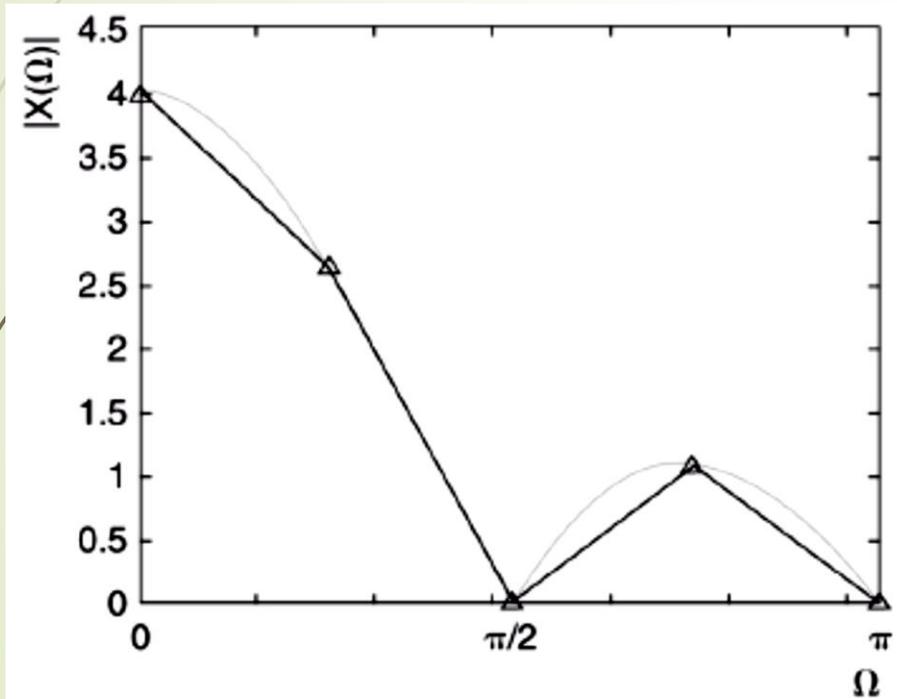
- ▶ we can compute the spectrum by substituting the values of Ω
- ▶ The computation method is analogous to that used in the DTFT.

$$\begin{aligned} X(-\pi) &= 0.00 \angle 0 \\ X\left(-\frac{3\pi}{4}\right) &= 1.0824 \angle 0.3927 \\ &\vdots \\ X\left(\frac{11\pi}{4}\right) &= 1.0824 \angle -0.3927 \\ X(3\pi) &= 0.00 \angle 0 \end{aligned}$$

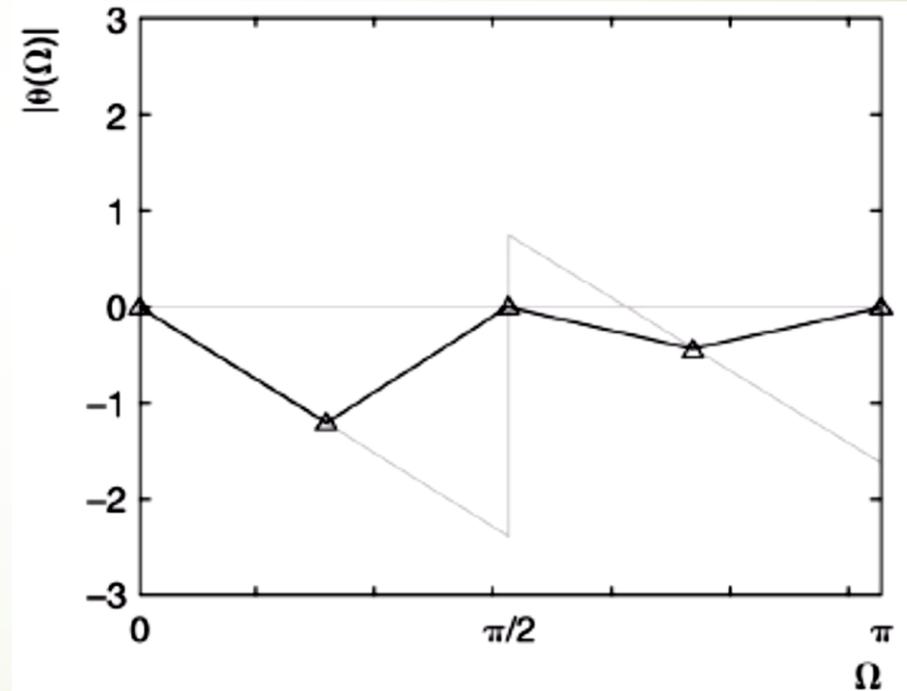
Ω	$X(\Omega)$	$ X(\Omega) $	$\theta(\Omega)$ (rads)
$-\pi$	$1 - 1 \left \frac{\pi}{4} + 1 \right \left 2\pi + 1 \right \left 3\pi \right $	0.0000	0.0000
$-\frac{3\pi}{4}$	$1 - 1 \left \frac{3\pi}{4} + 1 \right \left \frac{3\pi}{2} + 1 \right \left \frac{9\pi}{4} \right $	1.0824	0.3927
$-\frac{\pi}{2}$	$1 + 1 \left \frac{\pi}{2} + 1 \right \left \pi + 1 \right \left \frac{3\pi}{2} \right $	0.0000	0.0000
$-\frac{\pi}{4}$	$1 - 1 \left \frac{\pi}{4} + 1 \right \left \frac{\pi}{2} + 1 \right \left \frac{3\pi}{4} \right $	2.6131	1.1781
0	$1 + 1 \left 0 + 1 \right \left 0 + 1 \right \left 0 \right $	4.0000	0.0000
$\frac{\pi}{4}$	$1 - 1 \left -\frac{\pi}{4} + 1 \right \left -\frac{\pi}{2} + 1 \right \left -\frac{3\pi}{4} \right $	2.6131	-1.1781
$\frac{\pi}{2}$	$1 - 1 \left -\frac{\pi}{2} + 1 \right \left -\pi + 1 \right \left -\frac{3\pi}{2} \right $	0.0000	0.0000
$\frac{3\pi}{4}$	$1 + 1 \left -\frac{3\pi}{4} + 1 \right \left -\frac{3\pi}{2} + 1 \right \left -\frac{9\pi}{4} \right $	1.0824	-0.3927
π	$1 + 1 \left -\pi + 1 \right \left -2\pi + 1 \right \left -3\pi \right $	0.0000	0.0000
$\frac{5\pi}{4}$	$1 + 1 \left -\frac{5\pi}{4} + 1 \right \left -\frac{5\pi}{2} + 1 \right \left -\frac{15\pi}{4} \right $	1.0824	0.3927
$\frac{3\pi}{2}$	$1 + 1 \left -\frac{3\pi}{2} + 1 \right \left -3\pi + 1 \right \left -\frac{9\pi}{2} \right $	0.0000	0.0000
$\frac{7\pi}{4}$	$1 + 1 \left -\frac{7\pi}{4} + 1 \right \left -\frac{7\pi}{2} + 1 \right \left -\frac{21\pi}{4} \right $	2.6131	1.1781
2π	$1 + 1 \left -2\pi + 1 \right \left -4\pi + 1 \right \left -6\pi \right $	4.0000	0.0000
$\frac{9\pi}{4}$	$1 + 1 \left -\frac{9\pi}{4} + 1 \right \left -\frac{9\pi}{2} + 1 \right \left -\frac{27\pi}{4} \right $	2.6131	-1.1781
$\frac{5\pi}{2}$	$1 + 1 \left -\frac{5\pi}{2} + 1 \right \left -5\pi + 1 \right \left -\frac{15\pi}{2} \right $	0.0000	0.0000
$\frac{11\pi}{4}$	$1 + 1 \left -\frac{11\pi}{4} + 1 \right \left -\frac{11\pi}{2} + 1 \right \left -\frac{33\pi}{4} \right $	1.0824	-0.3927
3π	$1 + 1 \left -3\pi + 1 \right \left -6\pi + 1 \right \left -9\pi \right $	0.0000	0.0000

Spectra of Non-Periodic Signals

The magnitude and phase spectra are plotted for the range $0 \leq \Omega \leq \pi$



(a) Magnitude Spectrum

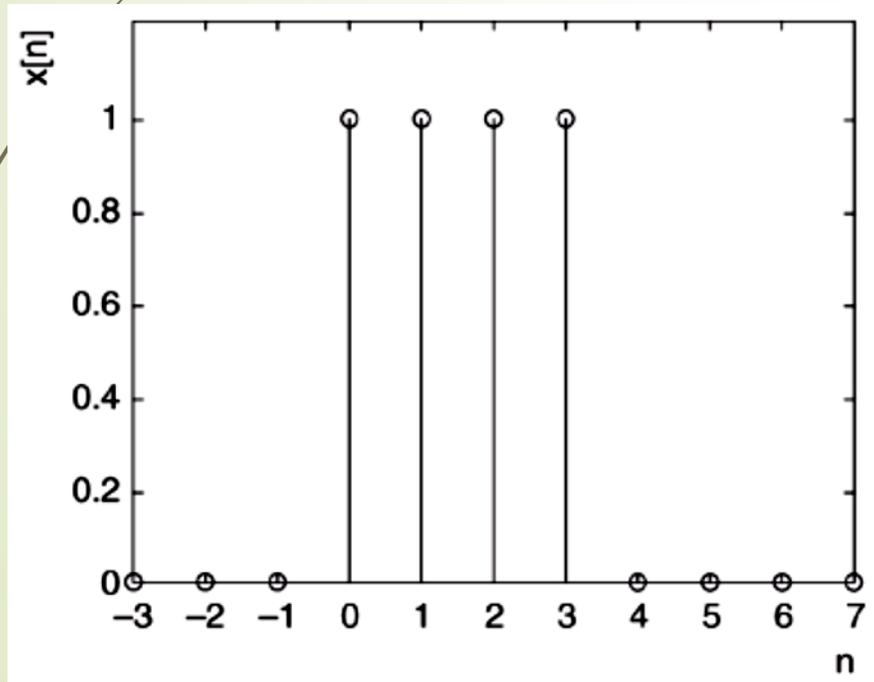


(b) Phase Spectrum

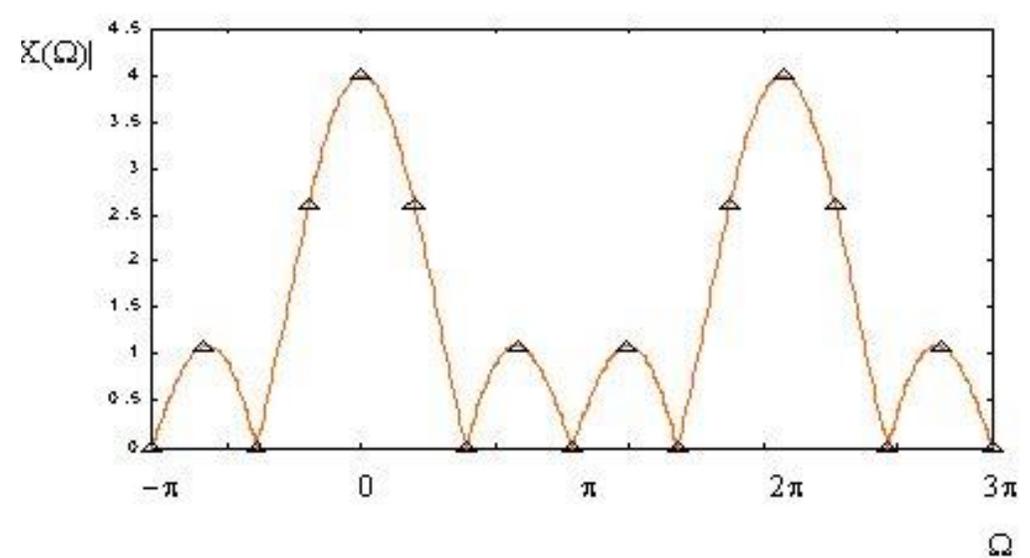
Spectra of Non-Periodic Signals

- Magnitude spectrum has a shape that is characteristic for all rectangular pulses, called a **sinc function**

Rectangular Pulse Signal



Magnitude Spectrum



Spectra of Non-Periodic Signals

Example 2: Find the magnitude and phase spectra for the signal $x[n] = (0.1)^n u[n]$, sampled at 15kHz. Plot the magnitude in dB and phase in degrees.

Solution

$$x[n] = \{1 \quad 0.1 \quad 0.01 \quad 0.001 \quad 0.0001 \quad \dots\}$$

- ▶ The signal is
- ▶ Since the sample amplitudes drop off quickly, the first three samples are sufficient to obtain good approximation to the DTFT spectrum.

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-jn\Omega} = 1 + 0.1e^{-j\Omega} + 0.01e^{-j2\Omega}$$

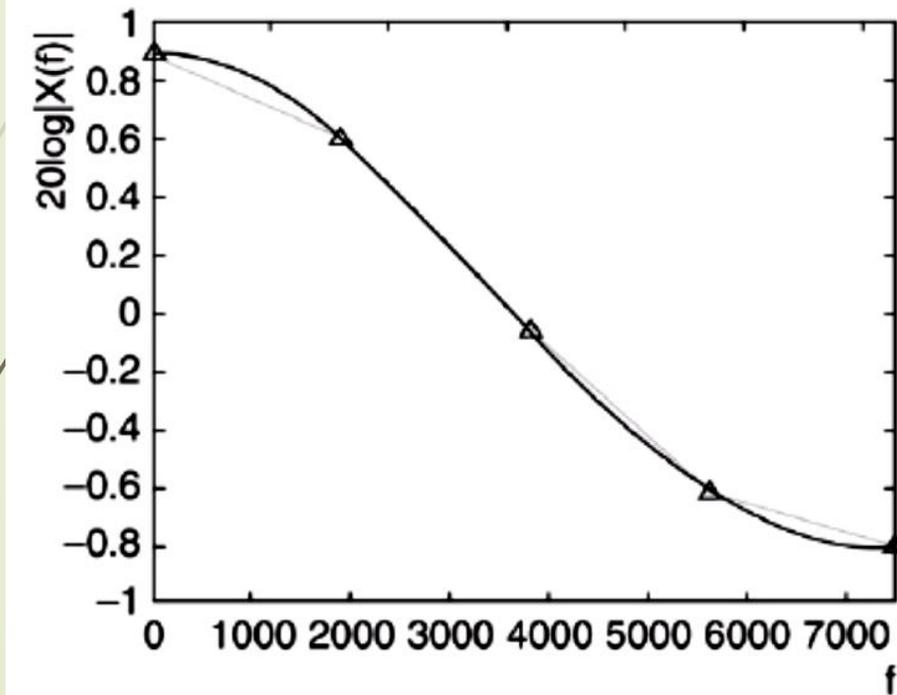
Spectra of Non-Periodic Signals

TABLE 8.3
Coefficients for Example 8.2

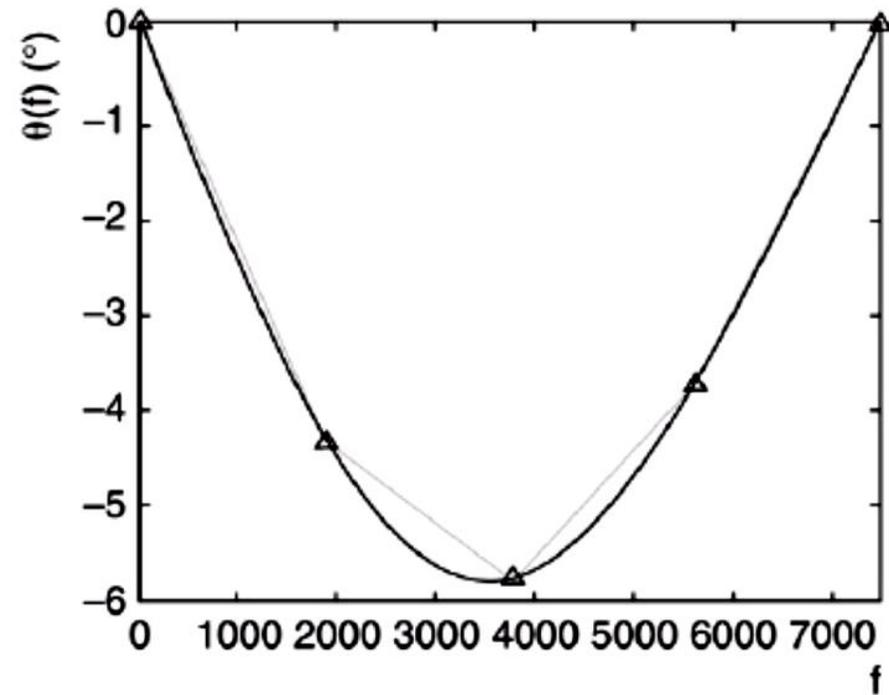
n	$X(n)$	$ X(n) $	$\theta(n)$ (radians)	$20 \log_{10} X(n) $ (dB)	$\theta(n)$ ($^{\circ}$)	f (Hz)
0	$1 - 0.1 0 + 0.01 0$	1.1100	0.0000	0.9065	0.0000	0
4	$1 + 0.1 \left[\frac{\pi + 0.01}{4} - \frac{\pi}{2} \right]$	1.0737	-0.0752	0.6181	-4.3108	1875
8	$1 - 0.1 \left[\frac{\pi + 0.01}{2} - \pi \right]$	0.9950	-0.1007	-0.0432	-5.7679	3750
12	$1 + 0.1 \left[\frac{3\pi + 0.01}{4} - \frac{3\pi}{2} \right]$	0.9313	-0.0652	-0.6185	-3.7378	5625
16	$1 + 0.1 \left[\pi + 0.01 \right] - 2\pi$	0.9100	0.0000	-0.8192	0.0000	7500

Spectra of Non-Periodic Signals

Magnitude Spectrum

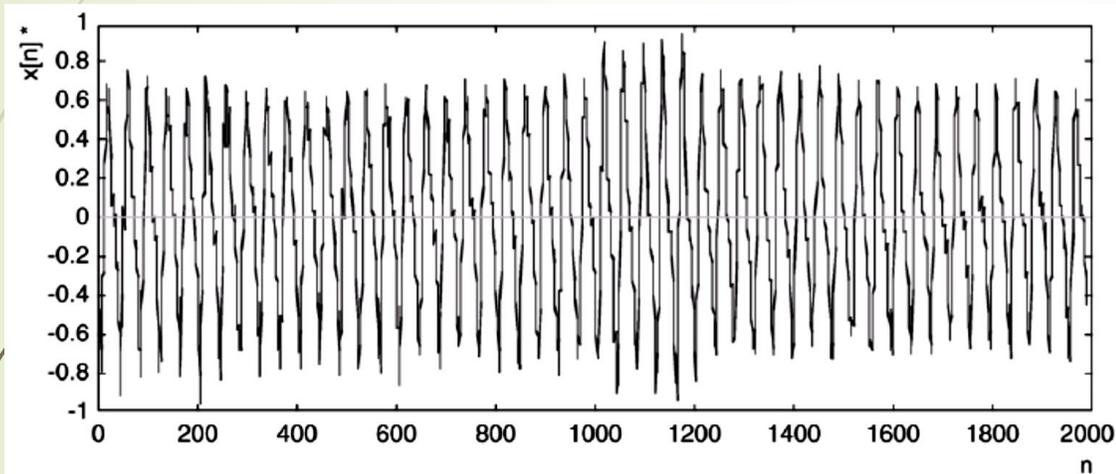


Phase Spectrum

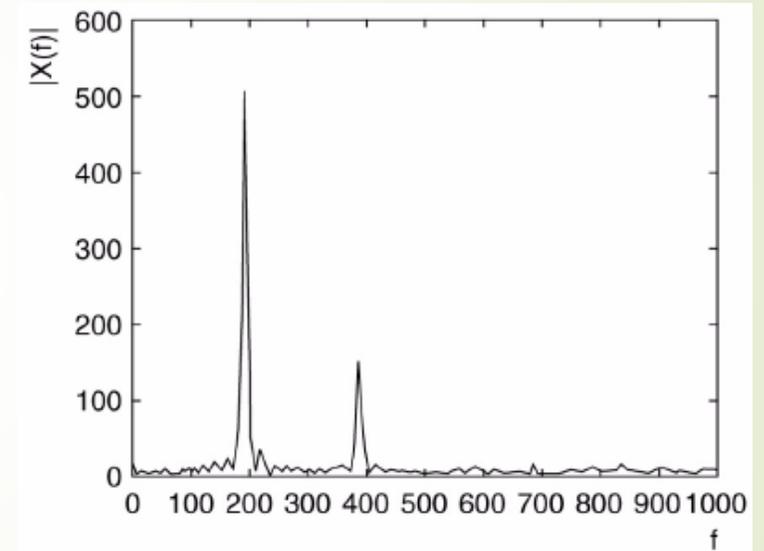


Spectra of Non-Periodic Signals

Example 3: A piece of the spoken vowel “eee” sampled at 8 kHz, and their spectrum are shown in the figures. What are the main frequency components of $x[n]$?



* For clarity, the envelope rather than the individual samples of the signal is plotted.

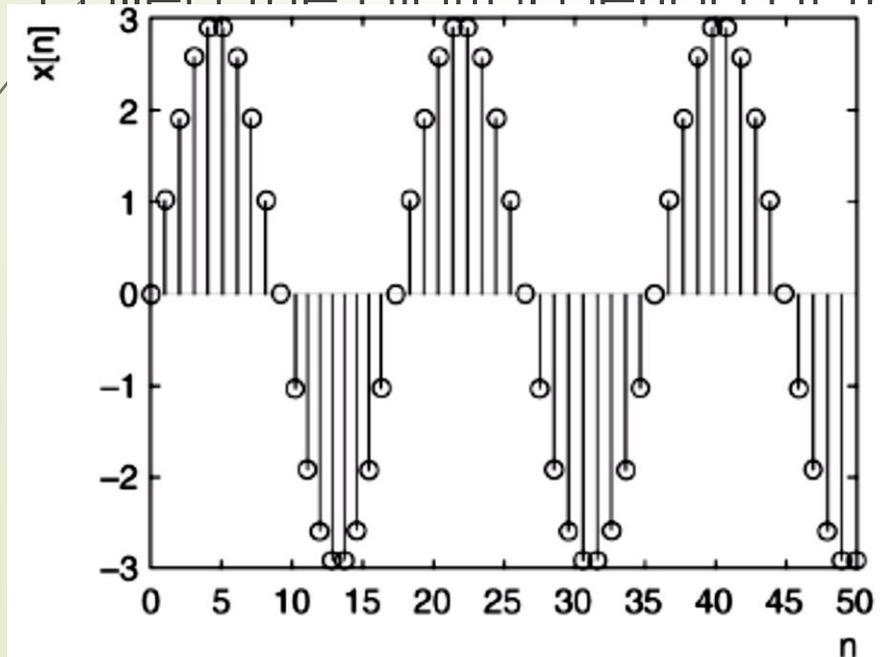


Solution

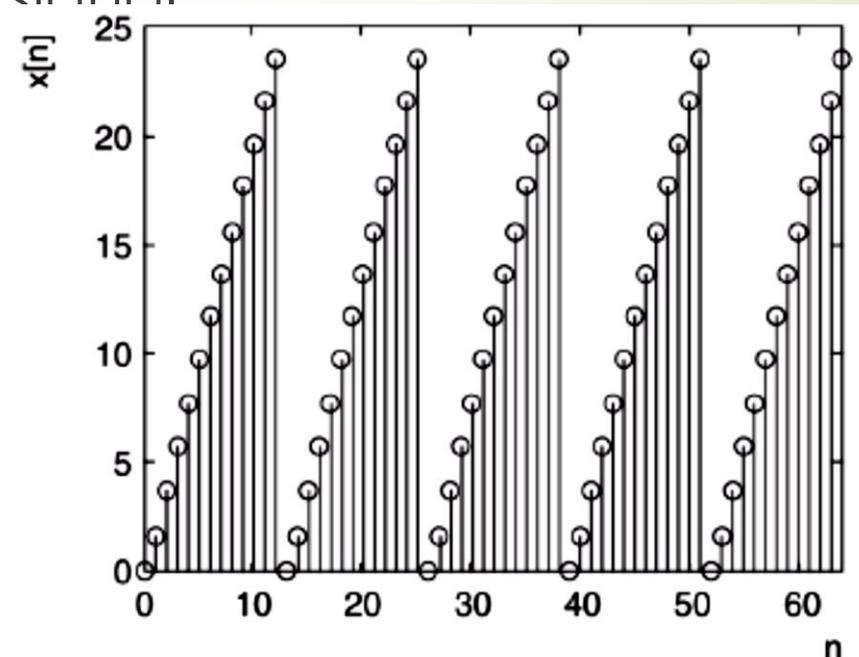
- The vowel “eee” sound is quite regular but not perfectly periodic.
- As magnitude spectrum shows the vowel “eee” consists almost exclusively of 200 Hz and 400 Hz frequency components.

Spectra of Periodic Signals

- ▶ **Periodic signals** are those that repeat at regular intervals for all time.
- ▶ The number of samples that occur in each interval is called the digital period of the signal



(a) Sinusoidal Signal



(b) Sawtooth Signal

Spectra of Periodic Signals

- ▶ In periodic signals, the same sequence repeats over all time, the DTFT is not an appropriate tool for calculating the spectrum.
- ▶ The infinite sum that is part of the DTFT would give an infinite result.
- ▶ The tool needed to find the spectrum of a periodic signals is the Discrete Fourier Series (DFS).

Spectra of Periodic Signals

- ▶ If the signal is non-periodic, the discrete time Fourier transform (DTFT) is used:

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-jn\Omega}$$

- ▶ If the signal is periodic, the discrete Fourier series (DFS) is used:

$$C_k = \sum_{n=0}^{N-1} x[n]e^{-j2\pi\frac{k}{N}n}$$

Spectra of Periodic Signals

- ▶ According to Fourier theory, every periodic signal can be expressed as the sum of sines and cosines or compactly, as the sum of complex exponentials.
- ▶ The Fourier series representation for a periodic signal $x[n]$ with period N is

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} c_k e^{j2\pi \frac{k}{N}n}$$

Where

n is the sample number

k is the coefficient number

$1/N$ is the scaling factor to recover $x[n]$ from its Fourier expansion

Spectra of Periodic Signals

- ▶ The Fourier coefficient c_k are calculated from the signal samples as

$$c_k = \sum_{n=0}^{N-1} x[n] e^{-j2\pi \frac{k}{N} n}$$

- ▶ Since $x[n]$ has a period of N samples therefore only N samples of the signal need to be used to find the coefficient c_k for all k .

Spectra of Periodic Signals

- ▶ The Fourier coefficient \mathbf{c}_k are complex numbers and may be written in the polar form as

$$\mathbf{c}_k = |\mathbf{c}_k|e^{j\varphi_k}$$

- ▶ The magnitude spectrum is plotted as $|\mathbf{c}_k|$ versus \mathbf{k}
- ▶ The phase spectrum is plotted as φ_k versus \mathbf{k}
- ▶ The DFS is **periodic** with period N
- ▶ The magnitude spectrum of DFS is always **Even**
- ▶ The phase spectrum of DFS is **Odd**
- ▶ Both magnitude and phase spectra for periodic signals are **line function**, with line spacing \mathbf{f}_s/N Hz, and contributions only at DC and harmonic frequencies.

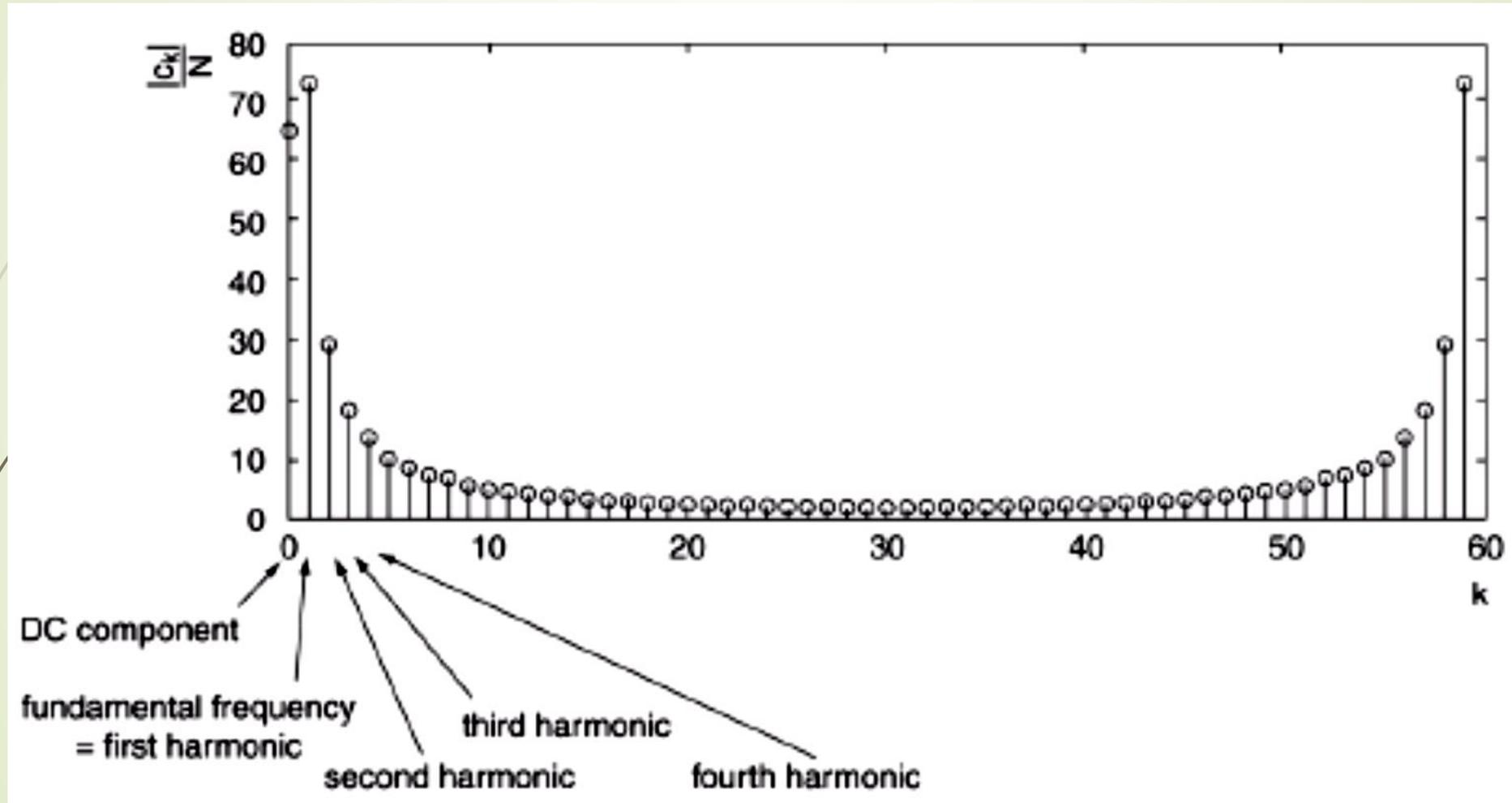
Spectra of Periodic Signals

- ▶ The DFS index k corresponds to the analog frequency

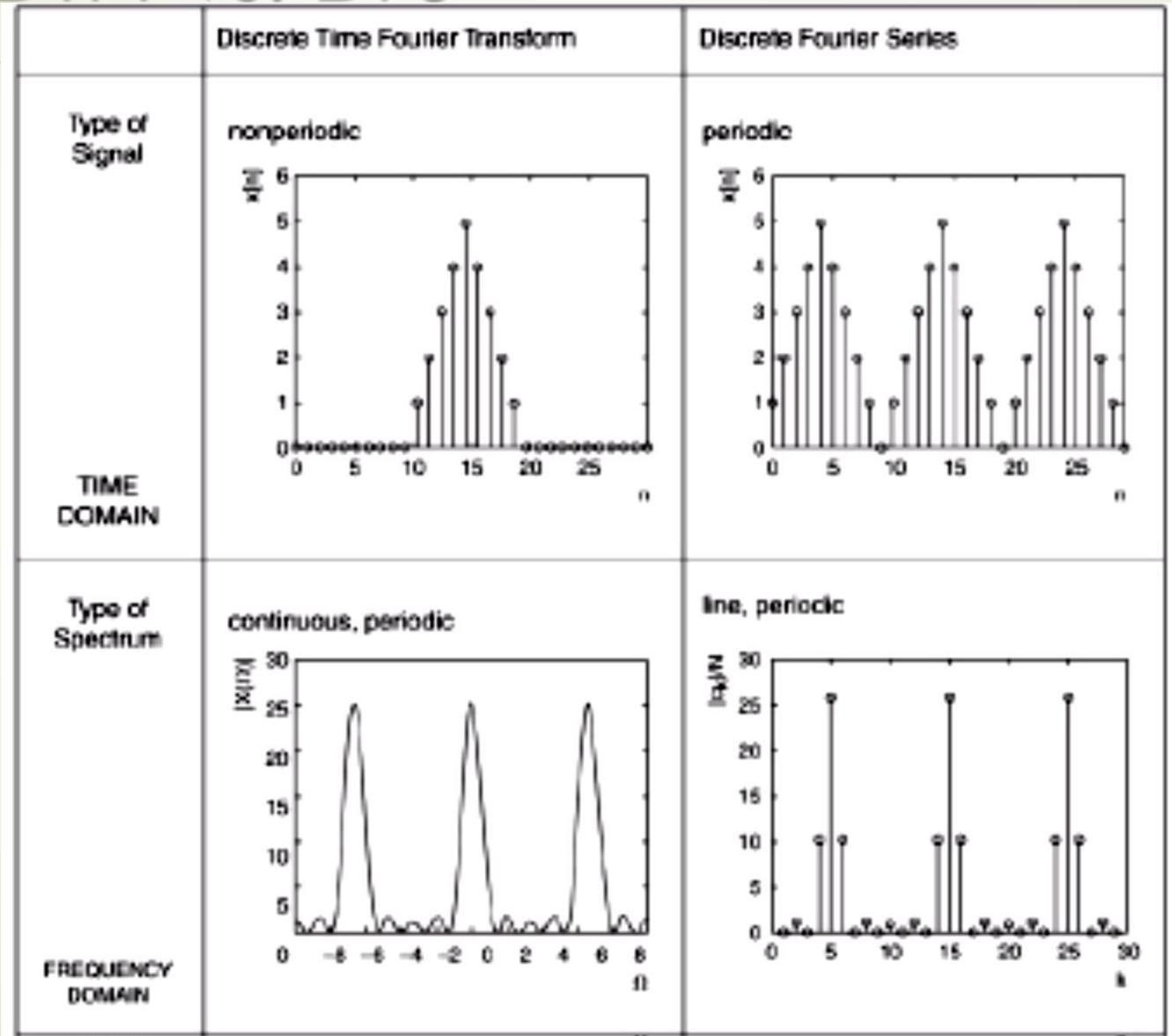
$$f = k \frac{f_s}{N}$$

- ▶ The $k = 0$ coefficient gives the DC component of the signal
- ▶ The $k = 1$ coefficient gives the Fundamental frequency f_s/N or first harmonic of the signal
- ▶ The reciprocal of the fundamental frequency f_s/N gives the one full cycle of the signal in time domain that is NT_s , where T_s is the sampling interval.
- ▶ Harmonics are the integer multiple of the fundamental frequency
- ▶ The $k > 1$ coefficient gives the higher harmonic of the signal

Spectra of Periodic Signals



DTFT vs. DFS



Complex Numbers

- **Rectangular Form:** $x + jy$
- **Polar Form:** $r\angle\theta$
- **Euler Form:** $e^{j\theta} = \cos\theta + j\sin\theta$
- **Order Pair Form:** $(\cos\theta, \sin\theta)$

Rectangular to Polar Conversion

- $r = \sqrt{x^2 + y^2}$
- $\theta = \tan^{-1} \frac{y}{x}$

Polar to Rectangular Conversion

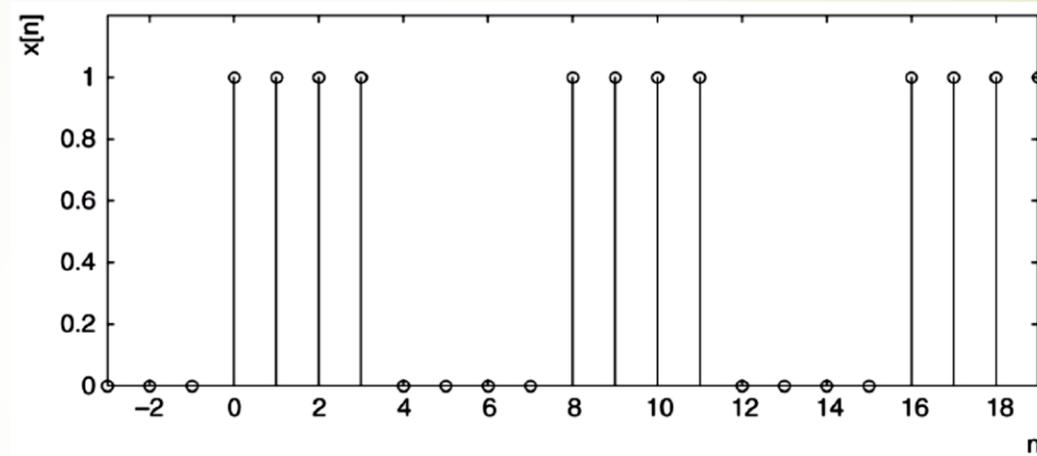
- $x = r\cos\theta$
- $y = r\sin\theta$

Note:

- for Addition (+) and Subtraction (-) use the Rectangular form
- for Multiplication (\times) and Division ($/$) use the Polar form

Spectra of Periodic Signals

Example 4: Find the magnitude and phase spectra for the periodic signal shown in the figure.



Solution

► The signal is periodic with period $N = 8$.

$$c_k = \sum_{n=0}^{N-1} x[n] e^{-j2\pi \frac{k}{N} n}$$

$$c_k = \sum_{n=0}^{8-1} x[n] e^{-j2\pi \frac{k}{8} n}$$

Spectra of Periodic Signals

$$c_k = 1 + e^{-j\pi\frac{k}{4}} + e^{-j\pi\frac{k}{2}} + e^{-j3\pi\frac{k}{4}}$$

When $k = 0$ $c_0 = 1 + e^{-j\pi\frac{0}{4}} + e^{-j\pi\frac{0}{2}} + e^{-j3\pi\frac{0}{4}} = 1 + 1 + 1 + 1 = 4$

$$c_0 = 1 + 1 + 1 + 1 = 4$$

$$c_0 = |4| \angle 0 \text{ rad}$$

$$\frac{|c_0|}{8} = \frac{|4|}{8} = 0.5$$

Spectra of Periodic Signals

$$c_k = 1 + e^{-j\pi\frac{k}{4}} + e^{-j\pi\frac{k}{2}} + e^{-j3\pi\frac{k}{4}}$$

When $k = 1$ $c_1 = 1 + e^{-j\pi\frac{1}{4}} + e^{-j\pi\frac{1}{2}} + e^{-j3\pi\frac{1}{4}}$

$$c_1 = 1 + [\cos(\pi/4) - j\sin(\pi/4)] + [\cos(\pi/2) - j\sin(\pi/2)] + [\cos(3\pi/4) - j\sin(3\pi/4)]$$

$$c_1 = 1 + [0.707 - j0.707] + [0 - j] + [-0.707 - j0.707]$$

$$c_1 = 1 - j2.414$$

$$c_1 = |2.6129| \angle -1.1781 \text{ rad}$$

$$\frac{|c_1|}{8} = \frac{|2.6129|}{8} = 0.3266$$

Note: compute the other values of C_k for $k = 2, 3, 4, 5, 6,$ and 7 by following the above procedure.

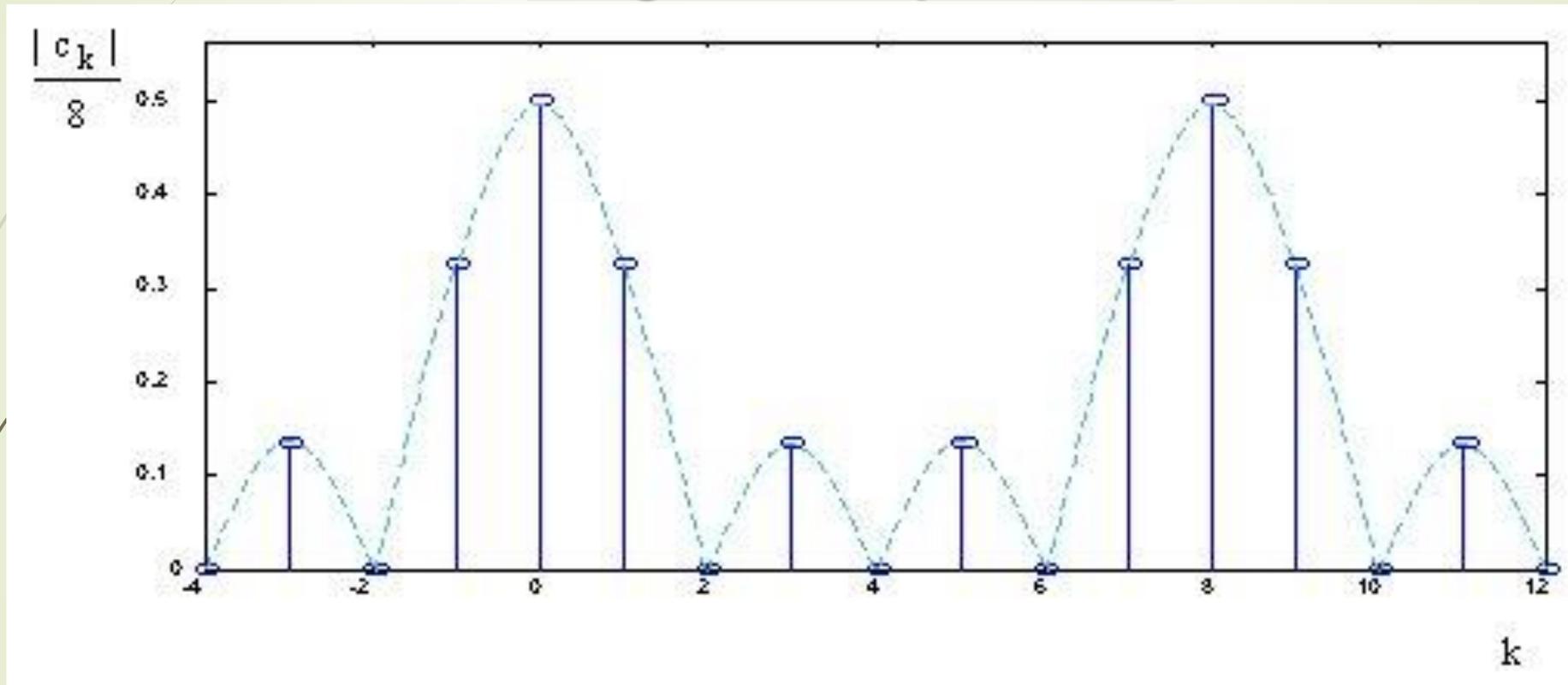
Spectra of Periodic Signals

$$c_k = 1 + e^{-j\pi\frac{k}{4}} + e^{-j\pi\frac{k}{2}} + e^{-j3\pi\frac{k}{4}}$$

k	c_k	$ c_k /8$	ϕ_k
0	$1 + 1 _0 + 1 _0 + 1 _0$	0.5000	0.0
1	$1 + 1 _{-\frac{\pi}{4}} + 1 _{-\frac{\pi}{2}} + 1 _{-\frac{3\pi}{4}}$	0.3266	-1.1781
2	$1 + 1 _{-\frac{\pi}{2}} + 1 _{-\pi} + 1 _{-\frac{3\pi}{2}}$	0.0	0.0
3	$1 + 1 _{-\frac{3\pi}{4}} + 1 _{-\frac{3\pi}{2}} + 1 _{-\frac{3\pi}{2}}$	0.1353	-0.3927
4	$1 + 1 _{-\pi} + 1 _{-2\pi} + 1 _{-3\pi}$	0.0	0.0
5	$1 + 1 _{-\frac{5\pi}{4}} + 1 _{-\frac{5\pi}{2}} + 1 _{-\frac{15\pi}{4}}$	0.1353	0.3927
6	$1 + 1 _{-\frac{3\pi}{2}} + 1 _{-3\pi} + 1 _{-\frac{9\pi}{2}}$	0.0	0.0
7	$1 + 1 _{-\frac{7\pi}{4}} + 1 _{-\frac{7\pi}{2}} + 1 _{-\frac{21\pi}{4}}$	0.3266	1.1781

Spectra of Periodic Signals

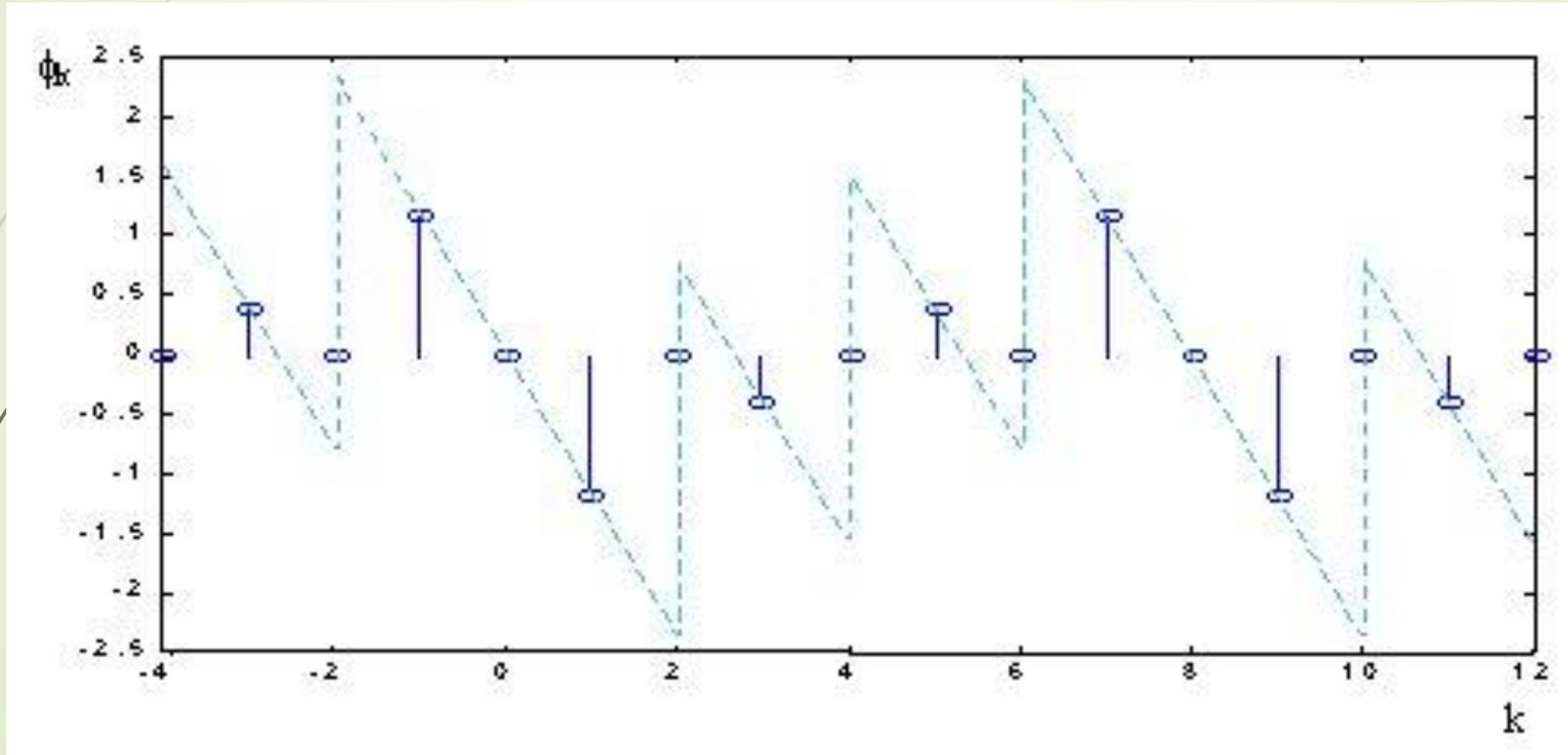
Magnitude Spectrum



- The dashed line in the magnitude spectrum for this square wave shows that its envelope has the shape of the absolute value of a **sinc function**.
- All square and rectangular periodic signals have this

Spectra of Periodic Signals

Phase Spectrum

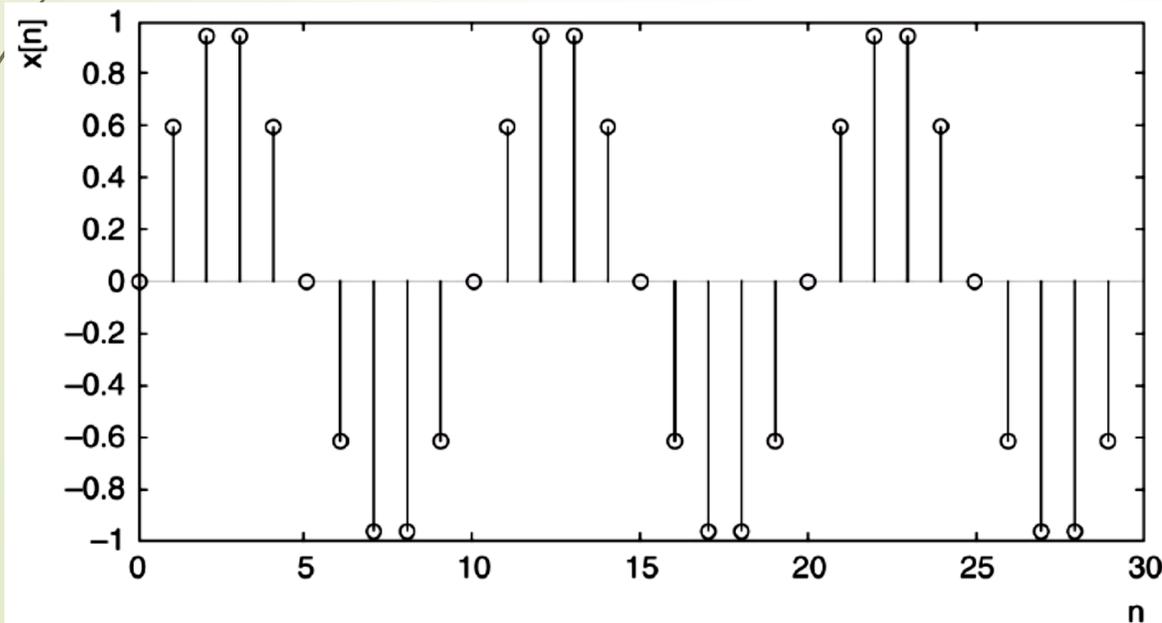


Spectra of Periodic Signals

Example 5: Find the magnitude and phase spectra for the periodic signal $x[n] = \sin(n\pi/5)$ with sampling rate is 1 kHz.

Solution

- The sample values are listed in the Table.
- The signal is plotted in the figure.
- The signal is periodic with period $N = 10$



n	$x[n]$
0	0.0000
1	0.5878
2	0.9511
3	0.9511
4	0.5878
5	0.0000
6	-0.5878
7	-0.9511
8	-0.9511
9	-0.5878

Spectra of Periodic Signals

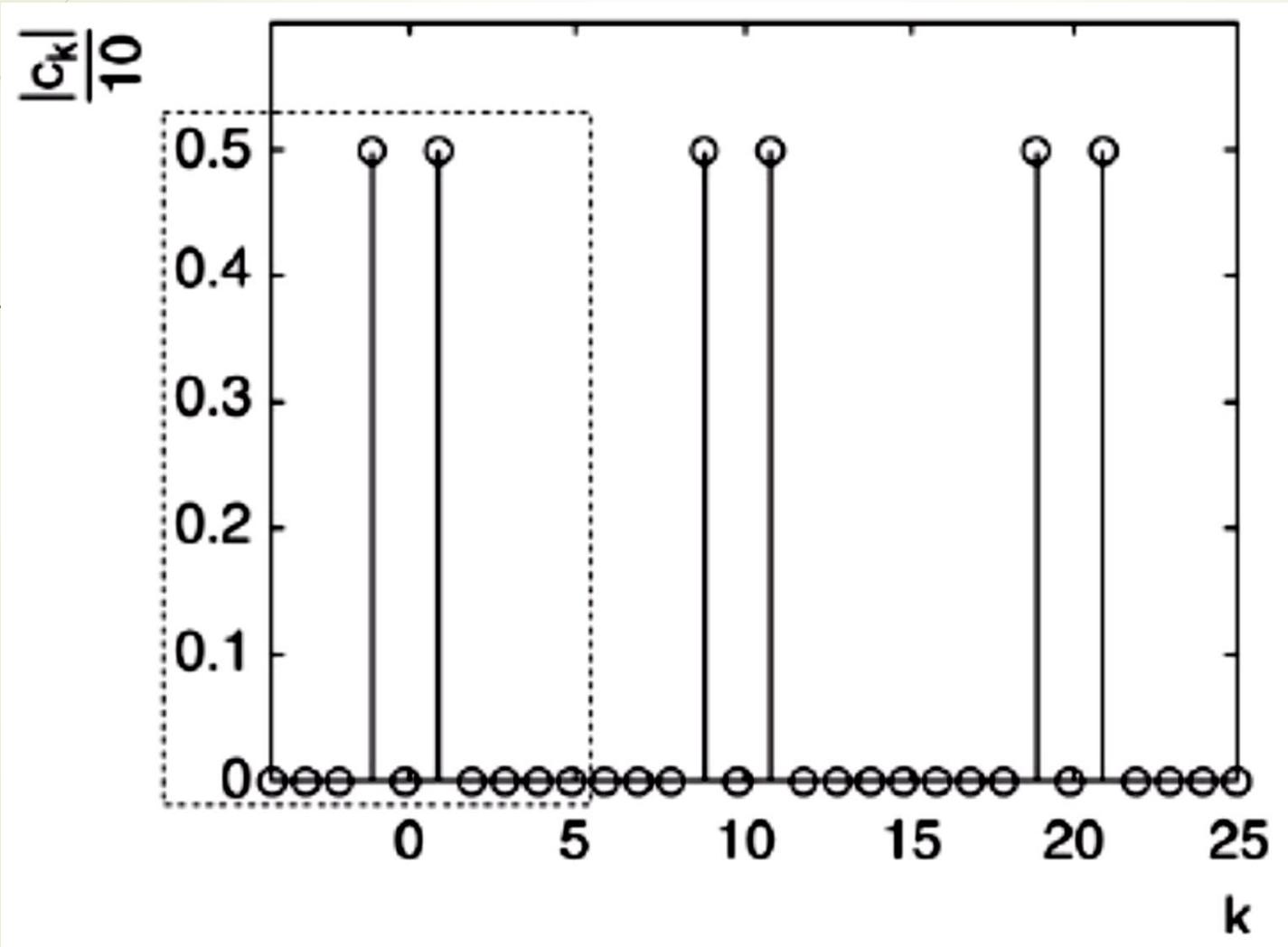
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$$c_k = \sum_{n=0}^{10-1} x[n]e^{-j2\pi\frac{k}{10}n}$$

k	c_k	$ c_k /V$	ϕ_k
0	0.0000	0.0000	0.0000
1	$-j5.0000$	0.50000	-1.5708
2	0.0000	0.0000	0.0000
3	0.0000	0.0000	0.0000
4	0.0000	0.0000	0.0000
5	0.0000	0.0000	0.0000
6	0.0000	0.0000	0.0000
7	0.0000	0.0000	0.0000
8	0.0000	0.0000	0.0000
9	$j5.0000$	0.5000	1.5708

Spectra of Periodic Signals

Magnitude Spectrum



Spectra of Periodic Signals

Phase Spectrum

