

LIMIT POINT -: let E be the subset of metric space M . A point $x \in M$ is said to be a limit point of E if there is $\{x_n\}_{n=1}^{\infty}$ of points of E which converges to x .

CLOSED SET -: let E be the subset of metric space M . E is said to be closed if $E = \bar{E}$ where \bar{E} is the subset containing all the limit points of E

Theorem -: let E be a subset of the metric space M . Then the point $x \in M$ is a limit point of E if and only if every open ball $B[x, r]$ about x contains at least one point of E

Proof -: necessary part-: let us suppose x is a limit point of E

There is a sequence $\{x_n\}_{n=1}^{\infty}$ of points of E such that $\rho(x_n, x) < r$

$$\Rightarrow \{x_n\}_{n=1}^{\infty} \in B[x, r]$$

Sufficient part -: conversely suppose every $B[x, r]$ contains a point of E and we have to prove that x is a limit point of E .

For $n \in \mathbb{N}$ the open ball $B\left[x, \frac{1}{n}\right]$ contains a point of $\{x_n\}_{n=1}^{\infty}$

Then $\{x_n\}_{n=1}^{\infty}$ is convergent to x because $\rho(x_n, x) < \frac{1}{n}$

$$\lim_{n \rightarrow \infty} (x_n) = x \quad \left(\text{as } n \rightarrow \infty \quad \frac{1}{n} \rightarrow 0\right)$$

Hence x is limit point of E

Theorem -: for any metric space M , both M and \emptyset are closed.

Q-: prove that for any metric space M the set $\{x\} \subset M$ is closed

Proof -: let us consider $\{x_n\}_{n=1}^{\infty}$ be the sequence of points the single term set $\{x\} \subset M$.

Now, $\{x_n\}_{n=1}^{\infty}$ can be written as $\{x, x, \dots\}$

$$\therefore \lim_{n \rightarrow \infty} (x_n) = x \quad x \in \{x\}$$

$\{x\}$ contains all its limit points.

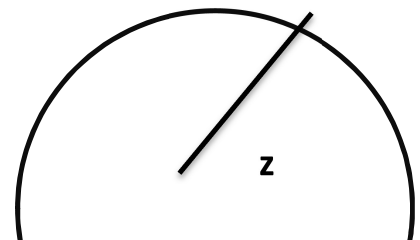
Hence $\{x\}$ is closed

Theorem -: if E is any subset of a metric space M then \bar{E} is closed i.e. $\bar{\bar{E}} = \bar{E}$

Proof -: let $x \in \bar{\bar{E}}$

x is a limit point of \bar{E} therefore the open ball, About x having radius r contains at least one element of \bar{E}

at least one element of \bar{E}



let that one element be y . i.e. $\rho(x, y) < r$

$y \in \bar{E}$

$\Rightarrow y$ is a limit point of E

\exists an element Z of $E \in B[y, t]$

Let $\rho(x, y) = s$ such that $t < r - s$

Now $\rho(x, z) \leq \rho(x, y) + \rho(y, z)$

$$= s + r - s$$

$$= r \quad \Rightarrow \rho(x, z) < r$$

$\therefore z \in \bar{E}$

Hence $z \in B[x, r]$

Thus $B[x, r]$ contains a points of E

Hence x is a limit point of E

$\therefore x \in \bar{E}$

Theorem -: let M be a metric space then both M and \emptyset are closed

Proof -: we know any single term set in any metric space M is closed and M can be written as $M = \bigcup_{i=1}^{\infty} s_i$ where s_i are single term set .

Hence M is closed

\emptyset doesn't contain any element

Hence \emptyset is closed

Theorem -: if f_1 and f_2 are closed subsets of the metric space M then $f_1 \cup f_2$ is also closed

Proof -: Given f_1 and f_2 are closed subsets of the metric space M .

Let $x \in \overline{f_1 \cup f_2}$.

I.e. x is a limit point of $f_1 \cup f_2$

Let $\{x_n\}_{n=1}^{\infty}$ is be a sequence of points of $f_1 \cup f_2$ converging to x .

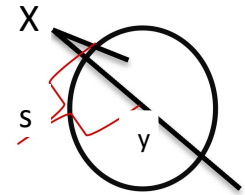
Let us consider subsequence of $\{x_n\}_{n=1}^{\infty}$

We know that subsequence of $\{x_n\}_{n=1}^{\infty}$ consisting the points of f_1 or f_2 also convergent to the same limit points x .

Given f_1 and f_2 are closed.

x is a limit point of f_1 or f_2 also

$\therefore x \in f_1$ or $x \in f_2$



$\Rightarrow x \in f_1 \cup f_2 \quad \therefore \overline{f_1 \cup f_2} \subset f_1 \cup f_2$ Hence $f_1 \cup f_2$ is closed .

Prove that union of infinite number of closed sets is not closed.

Or

Give examples of closed set whose infinite union is not closed.

Theorem -: if f is the family of closed sets in any metric space then $\bigcap_{F \in f} F$ is closed.

Proof -: let $G = \bigcap_{F \in f} F$

Let $x \in \bar{G}$

i.e. x is a limit point of G

\Rightarrow Then there is a open ball $B[x, r]$ about x containing at least one element of G .

Let it be y

$\Rightarrow y \in G$ or $y \in \bigcap_{F \in f} F$

y is an element of all closed sets $f \in F$

i.e. $y \in \forall f = \bar{F}$

x is a limit point of every $f \in F$

$\Rightarrow x \in \bar{F} = \bar{f} \in F$

$\Rightarrow x \in \bigcap_{F \in f} F$

$\Rightarrow x \in F \Rightarrow \bar{G} \subset G$

Hence G is closed .

Theorem -: if G is open subset of the metric space M then $G' = M - G$ is closed and vice-versa .

Proof -: let us suppose G is open subset of the metric space .

Let $x \in G$

\Rightarrow There is an open ball $B[x, r]$ about x containing G .

\Rightarrow The open $B[x, r]$ doesn't contain any point of G'

Hence x cannot be a limit point of G'

$\therefore G$ doesn't contain any limit point of G' .

Hence G' contains all its limit point, so G' is closed.

Let us suppose H is closed subset of the metric space M and we have to prove that the open ball $B[y, \delta]$ doesn't contain any element of H .

Then $y \in H$ which is a contradiction that $y \in H'$

Hence, there is an open ball $B[y, \delta]$ entirely lying inside H' .

Hence H is open.

Q-: using above then prove that $F_1 \cup F_2$ is closed, when F_1, F_2 closed subsets of the metric space are M .

Proof -: it is given F_1 and F_2 are closed

$\Rightarrow F_1'$ and F_2' are open subsets of M

$\Rightarrow F_1' \cap F_2'$ are open in M .

$\Rightarrow F_1' \cap F_2' = (F_1 \cup F_2)'$ Which is open in M

$\Rightarrow F_1 \cup F_2$ is closed (\because its complement is open)

Theorem-: let $\langle M_1, \rho_1 \rangle$ and $\langle M_2, \rho_2 \rangle$ be metric spaces and let $f: M_1 \rightarrow M_2$ then f is continuous on M_1 if and only if $f^{-1}(G)$ is closed subset of M_1 whenever G is closed subset of M_2 .

Proof -: **necessary part -:**

Let us suppose f is continuous on M_1

Given G is subset of M_2 , which is closed

$\Rightarrow G'$ is open in M_2

$\Rightarrow f^{-1}(G')$ is open subset of M_1

$$G \cup G' = M_2$$

We know, $f^{-1}(G') \cup f^{-1}(G) = M_1$

$\Rightarrow f^{-1}(G)$ is closed subset of M_1

Sufficient part -: Given $f^{-1}(G')$ is closed subset of M_1 and

$$G \text{ is closed in } M_2$$

We have to prove that f is continuous on M_1

G is closed in M_2

$\Rightarrow G'$ is open in M_2

Given $f^{-1}(G')$ is closed in M_1

$\therefore f^{-1}(G')$ is open in M_1

$\therefore f^{-1}(G')$ is open in M_1 whenever G' is open in M_2

$\therefore f$ is continuous