

## Solutions of higher order linear differential equations with variable coefficients

### Equations reducible to linear differential equations with constant coefficients

An equation of the form

$$a_0 x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + a_2 x^{n-2} \frac{d^{n-2} y}{dx^{n-2}} + \dots + a_{n-1} x \frac{dy}{dx} + a_n y = Q(x) \quad \text{-----(1)}$$

Where  $a_0, a_1, a_2, \dots, a_{n-1}, a_n$  are constants with  $a_0 \neq 0$ , is called Cauchy's homogeneous linear differential equation. It is also known as Euler- Cauchy linear differential equation or Cauchy's Euler equation.

This can be reduced to a linear differential equation with constant coefficients by substituting  $x = e^t$  which implies  $t = \log x$ ,  $\frac{dt}{dx} = \frac{1}{x}$

$$\text{Now } \frac{dy}{dx} = \frac{dy}{dt} \frac{dt}{dx} = \frac{dy}{dt} \frac{1}{x}$$

$$x \frac{dy}{dx} = \frac{dy}{dt}$$

$$xDy = \theta y, \quad \text{where } \theta = \frac{d}{dt}, D = \frac{d}{dx}$$

$$\text{Similarly we get } x^2 \frac{d^2 y}{dx^2} = x^2 D^2 y = \theta(\theta - 1)y$$

$$x^3 \frac{d^3 y}{dx^3} = x^3 D^3 y = \theta(\theta - 2)(\theta - 1)y$$

...and so on

Substituting these values in (1), the equation reduces to differential equation with constant coefficients in  $y$  and  $t$  which can be solved by using the methods discussed earlier.

**Remember:**  $x = e^t$  or  $t = \log x$  and  $\theta = \frac{d}{dt}$ ,  $x Dy = \theta y$ ,  $x^2 D^2 y = \theta(\theta - 1)y$ ,  
 $x^3 D^3 y = \theta(\theta - 2)(\theta - 1)y \dots$

(V Imp) Problem .1 : Solve the differential Equation  $x^2 \frac{d^2y}{dx^2} - x \frac{dy}{dx} + y = \log x$

Solution : The given diff eqn is  $x^2 \frac{d^2y}{dx^2} - x \frac{dy}{dx} + y = \log x$

which is Cauchy- Euler homogeneous linear differential equation

It can be written as  $x^2 D^2 y - x D y + y = \log x$ ----(1)

Put  $x = e^t$  or  $t = \log x$  and  $\theta = \frac{d}{dt}$ ,  $x D y = \theta y$ ,  $x^2 D^2 y = \theta(\theta - 1)y$  in (1)

We get  $[\theta(\theta - 1)y - \theta y + y] = t$

$[\theta^2 - \theta - \theta + 1]y = t$

$[\theta^2 - 2\theta + 1]y = t$ -----(2)

This is differential equation with constant coefficients.

Its Auxiliary equation is  $m^2 - 2m + 1 = 0$

$(m - 1)^2 = 0$ ,  $m = 1, 1$  roots are real and equal.

Its complementary function is  $y_c(t) = (c_1 + c_2 t) (e)^t$  ....(3)

Now Particular integral is  $y_p(t) = \frac{1}{[\theta^2 - 2\theta + 1]} t$

$$P.I = \frac{1}{[\theta - 1]^2} t$$

$$= [1 - \theta]^{-2} t$$

$$= [1 + 2\theta + 3\theta^2 + \dots] t$$

$$= [1 + 2\theta] t, [\theta] t = \frac{d}{dt} (t) = 1$$

$$= t + 2$$

Therefore P.I =  $y_p(t) = t + 2$  ... (4)

The solution of (2)  $y(t) = y_c(t) + y_p(t)$

From (3) and (4), we get  $y(t) = (c_1 + c_2 t) (e)^t + t + 2$  --(5)

To get the solution of (1) replace  $t$  by  $\log x$ ,  $e^t$  by  $x$

From (5) ,  $y(x) = (c_1 + c_2 \log x)x + \log x + 2$  is the solution of

$$x^2 \frac{d^2y}{dx^2} - x \frac{dy}{dx} + y = \log x$$

(V Imp) Problem: 2 . Solve the differential Equation

$$x^2 \frac{d^2y}{dx^2} - 4x \frac{dy}{dx} + 6y = x^2$$

Solution : The given diff eqn is  $x^2 \frac{d^2y}{dx^2} - 4x \frac{dy}{dx} + 6y = x^2$

which is Cauchy- Euler homogeneous linear differential equation

$$\text{It can be written as } x^2 D^2 y - 4x Dy + 6y = x^2 \text{----(1)}$$

Put  $x = e^t$  or  $t = \log x$  and  $\theta = \frac{d}{dt}$ ,  $x Dy = \theta y$ ,  $x^2 D^2 y = \theta(\theta - 1)y$  in (1)

$$\text{We get } [\theta(\theta - 1)y - 4\theta y + 6y] = e^{2t}$$

$$[\theta^2 - \theta - 4\theta + 6]y = e^{2t}$$

$$[\theta^2 - 5\theta + 6]y = e^{2t} \text{-----(2)}$$

This is differential equation with constant coefficients.

$$\text{Its Auxiliary equation is } m^2 - 5m + 6 = 0$$

$(m-2)(m-3)=0$ ,  $m=2,3$  roots are real and distinct

Its complementary function is  $y_c(t) = c_1 e^{2t} + c_2 e^{3t} \dots(3)$

Now Particular integral is  $y_p(t) = \frac{1}{[\theta^2 - 5\theta + 6]} e^{2t}$

$$\text{P.I} = \frac{1}{[\theta^2 - 5\theta + 6]} e^{2t} \quad \text{Put } \theta = 2$$

since 2 is a root we get  $[\theta^2 - 5\theta + 6] = [2^2 - 5.2 + 6] = 0$

So that  $\text{P.I} = t \cdot \frac{1}{f'(\theta)} e^{2t} = t \cdot \frac{1}{[2\theta - 5]} e^{2t}$  , put  $\theta = 2$

$$\text{P.I} = t \cdot \frac{1}{[2.2 - 5]} e^{2t} = -t e^{2t} ,$$

$$\text{Therefore P.I} = y_p(t) = -te^{2t} \dots(4)$$

$$\text{The solution of (2) } y(t) = y_c(t) + y_p(t)$$

$$\text{From (3) and (4), we get } y(t) = c_1e^{2t} + c_2e^{3t} - te^{2t} \dots(5)$$

To get the solution of (1) replace t by log x,  $e^t$  by x,  $e^{2t} = x^2, e^{3t} = x^3$

From (5),  $y(x) = c_1x^2 + c_2x^3 - (\log x)x^2$  is the solution of

$$x^2 \frac{d^2y}{dx^2} - 4x \frac{dy}{dx} + 6y = x^2$$

Problem.3: Solve the differential Equation

$$x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + y = \log x \sin(\log x)$$

Problem.4. Solve the differential Equation

$$x^2 \frac{d^2y}{dx^2} - x \frac{dy}{dx} + y = 0$$

V. V. V. Imp Problem: 5. Solve the differential Equation

$$x^3 \frac{d^3y}{dx^3} + 2x^2 \frac{d^2y}{dx^2} + 2y = 10(x + \frac{1}{x})$$

## General Method

**RHS Q(x) contains other than  $ee^{ax}$ ,  $\sin bx$ ,  $\cos bx$ ,  $x^m$**

1. Solve  $\frac{d^2y}{dx^2} + a^2y = \sec ax$

Solution: Given equation in the standard form is  $(D^2 + a^2)y = \sec ax$

Which is a non-homogeneous and the homogeneous equation is

$$(D^2 + a^2)y = 0$$

$$\text{A.E. is } m^2 + a^2 = 0, m = \pm ia$$

$$\therefore \text{C.F.} = c_1 \cos ax + c_2 \sin ax$$

$$\text{P.I.} = \frac{1}{D^2 + a^2} \sec ax = \frac{1}{2ia} \left[ \frac{1}{D - ia} \sec ax - \frac{1}{D + ia} \sec ax \right]$$

$$\begin{aligned} \text{Now consider } \frac{1}{D - ia} \sec ax &= e^{iax} \int \sec ax e^{-iax} dx \\ &= e^{iax} \int \frac{\cos ax - i \sin ax}{\cos ax} dx \\ &= e^{iax} \int (1 - i \tan ax) dx = e^{iax} \left( x + \frac{i}{a} \log \cos ax \right) \end{aligned}$$

$$\text{Changing } i \text{ to } -i, \text{ we have } \frac{1}{D + ia} \sec ax = e^{-iax} \left( x + \frac{i}{a} \log \cos ax \right)$$

$$\begin{aligned} \text{P.I.} &= \frac{1}{2ia} \left[ e^{iax} \left( x + \frac{i}{a} \log \cos ax \right) - e^{-iax} \left( x + \frac{i}{a} \log \cos ax \right) \right] \\ &= \frac{x}{a} \left[ \frac{e^{iax} - e^{-iax}}{2i} \right] + \frac{1}{a^2} \log \cos ax \left[ \frac{e^{iax} + e^{-iax}}{2} \right] \\ &= \frac{x}{a} \sin ax + \frac{1}{a} \log (\cos ax) \cos ax \end{aligned}$$

Hence the general solution is

$$y = c_1 \cos ax + c_2 \sin ax + \frac{x}{a} \sin ax + \frac{1}{a} \cos ax \log (\cos ax)$$

Find the solution of  $\frac{d^2 y}{dx^2} - 2 \frac{dy}{dx} + 4y = e^x \cos x$

**Solution:** Given equation is  $\frac{d^2 y}{dx^2} - 2 \frac{dy}{dx} + 4y = e^x \cos x$

Given equation is non-homogeneous equation, whose homogeneous equation is,

$$(D^2 - 2D + 4)y = 0$$

A.E. is,

$$m^2 - 2m + 4 = 0$$

$$m = 1 \pm i\sqrt{3}$$

The complementary function is,

$$\text{C.F.} = e^x (c_1 \cos \sqrt{3}x + c_2 \sin \sqrt{3}x)$$

Writing the formula for particular integral

The particular integral is,

$$\begin{aligned} \text{P.I.} &= \frac{1}{f(D)} e^x \cos x \\ &= \frac{1}{D^2 - 2D + 4} e^x \cos x \end{aligned}$$

Put  $D = D + a = D + 1$

$$\begin{aligned} \text{P.I.} &= e^x \frac{1}{D^2 + 3} \cos x \\ &= e^x \frac{1}{-1 + 3} \cos x \\ &= \frac{e^x}{2} \cos x \end{aligned}$$

Simplification of Particular Integral

The general solution is,

$$y = \text{C.F.} + \text{I.F.}$$

$$y = e^x (c_1 \cos \sqrt{3}x + c_2 \sin \sqrt{3}x) + \frac{e^x}{2} \cos x$$

2. Solve  $\frac{d^2 y}{dx^2} + a^2 y = \sec ax$

Solution: Given equation in the standard form is  $(D^2 + a^2)y = \sec ax$

Which is a non-homogeneous and the homogeneous equation is

$$(D^2 + a^2)y = 0$$

A.E. is  $m^2 + a^2 = 0, m = \pm ia$

$\therefore$  C.F. =  $c_1 \cos ax + c_2 \sin ax$

$$\text{P.I.} = \frac{1}{D^2 + a^2} \sec ax = \frac{1}{2ia} \left[ \frac{1}{D - ia} \sec ax - \frac{1}{D + ia} \sec ax \right]$$

$$\begin{aligned} \text{Now consider } \frac{1}{D - ia} \sec ax &= e^{iax} \int \sec ax e^{-iax} dx \\ &= e^{iax} \int \frac{\cos ax - i \sin ax}{\cos ax} dx \\ &= e^{iax} \int (1 - i \tan ax) dx = e^{iax} \left( x + \frac{i}{a} \log \cos ax \right) \end{aligned}$$

Changing  $i$  to  $-i$ , we have  $\frac{1}{D + ia} \sec ax = e^{-iax} \left( x + \frac{i}{a} \log \cos ax \right)$

$$\text{P.I.} = \frac{1}{2ia} \left[ e^{iax} \left( x + \frac{i}{a} \log \cos ax \right) - e^{-iax} \left( x + \frac{i}{a} \log \cos ax \right) \right]$$

$$\begin{aligned}
&= \frac{x}{a} \left[ \frac{e^{iax} - e^{-iax}}{2i} \right] + \frac{1}{a^2} \log \cos ax \left[ \frac{e^{iax} + e^{-iax}}{2} \right] \\
&= \frac{x}{a} \sin ax + \frac{1}{a} \log (\cos ax) \cos ax
\end{aligned}$$

Hence the general solution is

$$y = c_1 \cos ax + c_2 \sin ax + \frac{x}{a} \sin ax + \frac{1}{a} \cos ax \log (\cos ax)$$

**3. Solve  $(D+1)^2 y = x \cos x$**

Solution: Given  $(D+1)^2 y = x \cos x$

$$\text{A.E. is } (m+1)^2 = 0, m = -1, -1$$

$$\text{C.F.} = (c_1 + c_2 x) e^{-x}$$

$$\text{P.I.} = \frac{1}{(D+1)^2} x \cos x$$

$$= \left[ x - \frac{1}{f(D)} f'(D) \right] \frac{1}{f(D)} \cos x$$

$$= \left[ x - \frac{2(D+1)}{(D+1)^2} \right] \frac{1}{-1^2 + 2D + 1} \cos x$$

$$= \left[ x - \frac{2}{D+1} \right] \frac{\sin x}{2}$$

$$= \frac{x}{2} \sin x - \frac{D-1}{D^2-1} \sin x$$

$$= \frac{x}{2} \sin x + \frac{\cos x - \sin x}{2}$$

Hence the general solution is  $y = (c_1 + c_2 x) e^{-x} + \frac{x}{2} \sin x + \frac{\cos x - \sin x}{2}$

**4. Solve  $(D^3 - D^2 - 6D)y = 1 + x^2$**

Solution: Given equation is  $(D^3 - D^2 - 6D)y = 1 + x^2$

$$\text{A.E. is } m^3 - m^2 - 6m = 0, \quad m(m^2 - m - 6) = 0$$

$$\Rightarrow m(m-3)(m+2)=0,$$

$$\therefore m = 0, 3, -2$$

$$\therefore \text{C.F.} = y_c = c_1 e^{0x} + c_2 e^{3x} + c_3 e^{-2x}$$

$$\begin{aligned} \text{P.I.} &= \frac{1+x^2}{D^3 - D^2 - 6D} = -\frac{1}{6D} \frac{1+x^2}{\left(1 + \frac{D-D^2}{6}\right)} \\ &= -\frac{1}{6D} \left(1 + \frac{D-D^2}{6} + \left(\frac{D-D^2}{6}\right)^2 - \dots\right) (1+x^2) \\ &= \frac{-1}{6} \cdot \frac{1}{D} \left[1 - \left(\frac{D-D^2}{6}\right) + \left(\frac{D-D^2}{6}\right)^2 - \dots\right] (1+x^2) \\ &= \frac{-1}{6} \cdot \frac{1}{D} \left[1 - \frac{D}{6} + \frac{D^2}{6} + \frac{D^2 + D^4 - 2D^3}{36} + \dots\right] (1+x^2) \\ &= \frac{-1}{6} \cdot \left[\frac{1}{D} - \frac{1}{6} + \frac{D}{6} + \frac{D}{36} - \frac{2D^2}{36}\right] (1+x^2) \text{ (ignoring the terms containing } D^3, D^4, \dots) \\ &= \frac{-1}{6} \cdot \left[x + \frac{x^3}{3} - \frac{1}{6} - \frac{1}{6}x^2 + \frac{2x}{6} + \frac{2x}{36} - \frac{2}{36}\right] (2) \\ &= \frac{-1}{6} \cdot \left[\frac{25x}{18} - \frac{x^2}{6} + \frac{x^3}{3} - \frac{1}{6} - \frac{1}{9}\right] \end{aligned}$$

General solution is  $y = y_c + y_p$

$$y = c_1 + c_2 e^{3x} + c_3 e^{-2x} - \frac{1}{6} \cdot \left[\frac{25x}{18} - \frac{x^2}{6} + \frac{x^3}{3} - \frac{1}{6} - \frac{1}{9}\right].$$

5. Solve the differential equation  $(D^3 + 1)y = \cos(2x - 1)$

Solution: Given differential equation is  $(D^3 + 1)y = \cos(2x - 1)$

$$\text{A.E. is } m^3 + 1 = 0$$

$$\Rightarrow (m+1)(m^2 - m + 1) = 0$$

$$\Rightarrow m = -1, \frac{1 \pm i\sqrt{3}}{2}$$

$$\therefore \text{C.F.} = c_1 e^{-x} + e^{x/2} \left[ c_2 \cos \frac{\sqrt{3}x}{2} + c_3 \sin \frac{\sqrt{3}x}{2} \right]$$

$$\text{P.I.} = \frac{1}{D^3 + 1} \cos(2x - 1)$$

$$\text{Putting } D^2 = -4$$

$$\text{P.I.} = \frac{1}{1 - 4D} \cos(2x - 1) = \frac{1 + 4D}{1 - 16D^2} [\cos(2x - 1)]$$

$$\text{Again putting } D^2 = -4$$

$$\text{P.I.} = \frac{1}{65} [\cos(2x - 1) - 8 \sin(2x - 1)]$$

General Solution is  $y = \text{C.F.} + \text{P.I.}$

$$= c_1 e^{-x} + e^{x/2} \left[ c_2 \cos \frac{\sqrt{3}x}{2} + c_3 \sin \frac{\sqrt{3}x}{2} \right] + \frac{1}{65} [\cos(2x - 1) - 8 \sin(2x - 1)]$$

6 .Find the complete solution of

$$(D^2 + 5D - 6)y = \sin 4x \sin x$$

Solution: Given equation is  $(D^2 + 5D - 6)y = \sin 4x \sin x$

$$\text{A.E. is } m^2 + 5m - 6 = 0, m = -6, 1$$

The roots are real and distinct .

$\therefore$  C.F. is  $y_c = c_1 e^{-6x} + c_2 e^x$

$$\begin{aligned}
 \text{P.I. is } y_p &= \frac{\sin 4x \sin x}{D^2 + 5D - 6} = \frac{1}{2} \cdot \frac{2 \sin 4x \sin x}{D^2 + 5D - 6} \\
 &= \frac{1}{2} \left[ \frac{\cos 3x - \cos 5x}{D^2 + 5D - 6} \right] = \frac{1}{2} \left[ \frac{\cos 3x}{D^2 + 5D - 6} - \frac{\cos 5x}{D^2 + 5D - 6} \right] \\
 y_{p_1} &= \frac{1}{2} \left[ \frac{\cos 3x}{D^2 + 5D - 6} \right] \quad (\text{put } D^2 = -9) \\
 &= \frac{1}{2} \left[ \frac{\cos 3x}{5D - 15} \right] = \frac{1}{10} \left[ \frac{\cos 3x}{D - 3} \right] = \frac{1}{10} \left[ \frac{(D+3)\cos 3x}{D^2 - 9} \right] \quad (\text{put } D^2 = -9) \\
 &= \frac{1}{10} \left[ \frac{-3 \sin 3x + 3 \cos 3x}{-18} \right] = \frac{-\sin 3x + \cos 3x}{-60} \\
 &= \frac{\sin 3x - \cos 3x}{60} \\
 y_{p_2} &= -\frac{1}{2} \cdot \frac{\cos 5x}{D^2 + 5D - 6} \quad (\text{put } D^2 = -25) \\
 &= -\frac{1}{2} \cdot \frac{\cos 5x}{-25 + 5D - 6} = -\frac{1}{2} \cdot \frac{\cos 5x}{5D - 31} \\
 &= \frac{-25 \sin 5x + 31 \cos 5x}{2(-625 - 961)} \\
 &= \frac{25 \sin 5x - 31 \cos 5x}{3172}
 \end{aligned}$$

$$\text{General solution is } y = c_1 e^{-6x} + c_2 e^x + \frac{\sin 3x - \cos 3x}{60} + \frac{25 \sin 5x - 31 \cos 5x}{3172}$$

1. Solve  $(D^3 - 3D^2 + 4)y = (1 + e^{-x})^3$

Solution: Given  $(D^3 - 3D^2 + 4)y = (1 + e^{-x})^3$

A.E. is  $m^3 - 3m^2 + 4 = 0$ ,

i.e.  $(m+1)(m-2)^2 = 0$ ,  $m = -1, 2, 2$ ,

C.F. =  $y = c_1 e^{-x} + (c_2 + c_3 x) e^{2x}$

$$\text{Now P.I is } y_p = \frac{1 + e^{-3x} + 3e^{-2x} + 3e^{-x}}{D^3 - 3D^2 + 4}$$

$$= \frac{1}{D^3 - 3D^2 + 4} + \frac{e^{-3x}}{D^3 - 3D^2 + 4} + \frac{3e^{-2x}}{D^3 - 3D^2 + 4} + \frac{3e^{-x}}{D^3 - 3D^2 + 4}$$

$$= y_{p_1} + y_{p_2} + y_{p_3} + y_{p_4}$$

$$\text{Now } y_{p_1} = \frac{1}{D^3 - 3D^2 + 4} \quad \text{put } D = 0$$

$$= \frac{1}{4}$$

$$y_{p_2} = \frac{e^{-3x}}{D^3 - 3D^2 + 4} \quad (\text{put } D = -3)$$

$$= \frac{e^{-3x}}{-27 + -27 + 4} = \frac{e^{-3x}}{4}$$

$$y_{p_3} = \frac{3e^{-2x}}{D^3 - 3D^2 + 4} \quad (\text{put } D = -2)$$

$$= \frac{3e^{-2x}}{-8 - 12 + 4} = \frac{3e^{-2x}}{16}$$

$$y_{p_4} = \frac{3e^{-x}}{(D+1)(D^2 - 4D + 4)} \quad (\text{put } D = -1)$$

$$= \frac{xe^{-x}}{1 + 4 + 4} = \frac{xe^{-x}}{3}$$

$$\text{General solution is } y = c_1 e^{-x} + (c_2 + c_3 x) e^{2x} + \frac{1}{4} + \frac{e^{-3x}}{4} + \frac{3e^{-2x}}{16} + \frac{xe^{-x}}{3}$$

