

## ADVANCE ANALYSIS

## Lecture-1: BASIC CONCEPT

**SET-:** A set is a collection of distinct objects considered as an object in its own right. Example  
 -:  $S = \{a, b, c\}$

A set is a well-defined collection of objects some typical sets are-:

*N: The set of natural numbers*

*I: The set of integers*

*Q: The set of rational numbers*

*R: The set of real numbers*

**NEIGHBOURHOOD OF A POINT-:** A set  $N \subseteq R$  is called the neighbourhood of a point  $a$ , if there exists an open interval  $I$  containing  $a$  and contained in  $N$ , that,

$$a \in I \subseteq N.$$

**OPEN SETS-:** A set  $S$  is said to be open if it is a nbd of each of its points, that, for each  $x \in S$  there exists an open interval  $I_x$  such that

$$m_x \in I_x \subseteq S$$

Example-: The set  $R$  of real number is an open set.

**CLOSED SET-:** A set is said to be closed if each of its limit points is a member of the set. Thus  $S$  is closed iff  $S' \subseteq \bar{S}$  or  $\bar{S} = S$ .

Example-: The set  $R$  of real number is a closed set

Note: The set of real number is an example of both open and closed set.

**DIVERGENT SEQUENCES -:** A sequence is said to be divergent if it does not have finite limit.

**INTERIOR POINTS -:** A point  $x$  is an interior point of a set  $S$  if  $S$  is a nbd of  $x$ . In other words,  $x$  is an interior point of  $S$  if  $\exists$  an open interval  $]a, b[$  containing  $x$  and contained in  $S$ , that  $x \in ]a, b[ \subseteq S$

Thus a set is nbd of each of its interior points.

**LIMIT POINTS OF A SET-:** A real number  $\xi$  is a limit point of a set  $S (\subset R)$  if every nbd of  $\xi$  contains an infinite number of members of  $S$ . Or contains at least one member of  $S$ .

**SEQUENCES** -: A function whose domain is the set  $N$  of natural numbers and range a set of real numbers denoted symbolically as  $S: N \rightarrow R$ .

Example -:  $\{S_n\} = \{(-1)^n\}, n \in N$

**CONVERGENCE OF SEQUENCE** -: A sequences  $\{S_n\}$  is said to be converges to real number if for each  $\epsilon > 0$ , there exists a positive integer  $m$  such that  $|S_n - l| < \epsilon$ , for all  $n \geq m$ .

$$S_n \rightarrow l \text{ as } n \rightarrow \infty \text{ or } \lim_{n \rightarrow \infty} S_n = l$$

Example -:  $f(x) = x^2$

$$\lim_{x \rightarrow a} x^2 = a^2 \text{ (Limiting value)}$$

$$f(a) = a^2 \text{ (Functional value)}$$

❖ If functional value = limiting value then continuity.

**BOUNDS OF A SEQUENCE** -: A sequences  $\{S_n\}$  is said to be bonded if  $\exists k, K$  such that,

$$k \leq S_n \leq K, \forall n \in N$$

**MONOTONIC SEQUENCE** -: if a sequences is either increasing or decreasing then we call it monotonic.

## MODUEL -1

## Lecture-2: LIMITS AND METRIC SPACES

**LIMITS:-** A function  $f(x)$  is said to have limit  $l \in R$  at  $x = a$ , if  $\delta > 0, \exists \epsilon > 0$  corresponding to  $\delta$  such that  $|f(x) - l| < \epsilon$ , when  $|x - a| < \delta$ . Or we may write  $\lim_{x \rightarrow a} f(x) = l$ .

**Example -:** Show that  $\lim_{x \rightarrow 3} x^2 + 2x = 15$

Ans- Given  $f(x) = x^2 + 2x$ ,  $l = 15$ ,  $a = 3$

$x$  approaches to 3

i.e.  $|x - 3| < \delta$  (for some  $\delta > 0$ )

or  $|x - 3| < \delta < 1$  or  $|x - 3| < 1$

$\Rightarrow -1 < |x - 3| < \delta$

$\Rightarrow 2 < x < 4$  or  $7 < x + 5 < 9$

i.e.  $x \in [2, 4]$

$$\begin{aligned} \text{Now, } |x^2 + 2x - 15| &= |(x - 3)(x + 5)| \\ &= |x - 3||x + 5| < 9\delta \end{aligned}$$

As we can write  $(x + 5) < 9$  as  $(x + 5) \in [7, 9]$

$\therefore |x^2 + 2x - 15| < 9\delta$

Let us choose  $9\delta = \epsilon$

$\Rightarrow |x^2 + 2x - 15| < \epsilon$  (for choosing  $\delta = \frac{\epsilon}{9}$ ).

**Example -:** show that  $\lim_{x \rightarrow 1} \sqrt{x + 3} = 2$

Proof -: given  $f(x) = \sqrt{x + 3}$ ,  $l = 2$ ,  $a = 1$

$x$  approaches to 1

$|x - 1| < \delta$  (for some  $\delta > 0$ )

Or  $|x - 1| < \delta < 1$

Or  $-1 < |x - 1| < 1$

Or  $0 < x < 2$ ;  $\sqrt{3} + 2 < \sqrt{x + 3} + 2 < \sqrt{5} + 2$

i.e.  $x \in ]0, 2[$

$$\begin{aligned} \text{now, } |\sqrt{x+3} - 2| &= \frac{|\sqrt{x+3}-2||\sqrt{x+3}+2|}{|\sqrt{x+3}+2|} \\ &= \frac{|(x+3)-4|}{|\sqrt{x+3}+2|} \\ &= \frac{|x-1|}{|\sqrt{x+3}+2|} < \frac{\delta}{\sqrt{5}+2} < \epsilon \end{aligned}$$

For  $\delta = \min(1, \sqrt{5} + 2)\epsilon$

**Example-: show that  $\lim_{x \rightarrow -2} x^2 + 3x = -2$**

Proof-: given  $f(x) = x^2 + 3x, l = -2, a = -2$

*x approaches to -2*

$$|x + 2| < \delta \quad (\text{for some } \delta > 0)$$

$$\text{Or } |x + 2| < \delta < 1$$

$$\text{Or } -1 < |x + 2| < 1$$

$$\text{Or } -3 < x < -1$$

i.e.  $x \in ]-3, -1[$

$$\begin{aligned} \text{Now, } |x^2 + 3x + 2| &= |(x + 1)(x + 2)| \\ &= |x + 1||x + 2| < \delta \end{aligned}$$

As we can write  $(x + 1) < 1$  ( as  $(x + 1) \in ]-3, -1[$  )

$$\therefore |x^2 + 3x + 2| < \delta$$

Let us choose  $\delta = \min(1, \epsilon)$

$$\Rightarrow |x^2 + 3x + 2| < \epsilon \quad (\text{for choosing } \delta = \epsilon).$$

**Example -: show that  $\lim_{x \rightarrow 1} \frac{x^2-1}{x-1} = 2$**

Proof -: given  $f(x) = \sqrt{x+3}, l = 2, a = 1$

*x approaches to 1*

i.e.  $|x - 1| < \delta$  (for some  $\delta > 0$ )

Or  $|x - 1| < \delta < 1$

Or  $-1 < |x - 1| < 1$  or  $1 < |x + 1| < 3$

Or  $0 \leq x \leq 2$

i.e.  $x \in [0,2]$

Now,  $\left| \frac{x^2-1}{x-1} \right| = \left| \frac{(x-1)(x+1)}{x-1} \right| = |x + 1|$

We can write  $(x + 3) < 3$  [ as  $(x + 1) \in (1,3)$  ]

$= |x + 1| < \delta$

$\therefore \left| \frac{x^2-1}{x-1} - 2 \right| = |x + 1 - 2| = |x + 1| < \delta$

Consider  $\epsilon = \delta$  and choosing  $\delta = \min(1, \epsilon)$

$\left| \frac{x^2-1}{x-1} - 2 \right| < \epsilon$  when  $|x - 1| < \delta$ .

**Example -:** show that  $\lim_{x \rightarrow 0} \sqrt{4 - x} = 2$ .

Proof -: Given  $f(x) = \sqrt{x + 3}$ ,  $l = 2, a = 0$

*x approaches to 0*

$|x - 0| < \delta$  (for some  $\delta > 0$ )

Or  $|x| < \delta < 1$

Or  $-1 < |x - 1| < 1$

i.e.  $x \in ]-1,1[$

$$\begin{aligned} \therefore \left| \sqrt{4 + x} - 2 \right| &= \frac{|\sqrt{4-x}-2||\sqrt{4-x}+2|}{|\sqrt{4-x}+2|} \\ &= \frac{|(4-x)-4|}{|\sqrt{4-x}+2|} \\ &= \frac{|-x|}{|\sqrt{4-x}+2|} < \frac{\delta}{\sqrt{3}+2} < \epsilon \end{aligned}$$

$\sqrt{5} + 2 < \sqrt{4 - x} + 2 < \sqrt{3} + 2$

Choosing  $\delta = \min(1, (\sqrt{3} + 2)\epsilon)$

Or  $\left| \sqrt{4 + x} - 2 \right| < \epsilon$

For  $\delta = (\sqrt{3} + 2)\epsilon$ .

**THEOREM-: IF  $\lim_{x \rightarrow a} f(x) = L$  and  $\lim_{x \rightarrow a} g(x) = M$  then**

1.  $\lim_{x \rightarrow a} f(x) + g(x) = L + M$
2.  $\lim_{x \rightarrow a} f(x) - g(x) = L - M$
3.  $\lim_{x \rightarrow a} f(x)g(x) = LM$
4.  $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{L}{M}$  (provided  $M \neq 0$ )

Proof -: Given  $\lim_{x \rightarrow a} f(x) = L$  and  $\lim_{x \rightarrow a} g(x) = M$

For some  $\delta_1 > 0$ ,  $\exists \epsilon > 0$  such that  $|f(x) - L| < \frac{\epsilon}{2}$

(When  $|x - a| < \delta_1$ )

Similarly for  $\delta_2 > 0$ ,  $\exists \epsilon > 0$  such that  $|g(x) - M| < \frac{\epsilon}{2}$

(When  $|x - a| < \delta_2$ )

1. Now  $|f(x) + g(x) - (L + M)|$   
 $= |f(x) - L + g(x) - M|$   
 $\leq |f(x) - L| + |g(x) - M|$  (as we know  $|x + y| \leq |x| + |y|$ )

Choosing  $\delta = \min(\delta_1, \delta_2)$ , we get

$$|f(x) + g(x) - (L + M)| < \frac{\epsilon}{2} + \frac{\epsilon}{2}, \text{ when } |x - a| < \delta$$

$$\therefore |f(x) + g(x) - (L + M)| < \epsilon, \quad |x - a| < \delta$$

$$\text{Or } \lim_{x \rightarrow a} f(x) + g(x) = L + M$$

2. Now  $|f(x) - g(x) - (L - M)|$   
 $= |f(x) - L - g(x) + M|$   
 $\leq |f(x) - L| + |M - g(x)|$  (as we know  $|x + y| \leq |x| + |y|$ )

Choosing  $\delta = \min(\delta_1, \delta_2)$ , we get

$$|f(x) - g(x) - (L - M)| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2}, \text{ when } |x - a| < \delta$$

$$\therefore |f(x) + g(x) - (L + M)| < \epsilon, \quad |x - a| < \delta$$

$$\text{Or } \lim_{x \rightarrow a} f(x) - g(x) = L - M$$

**3.  $\lim_{x \rightarrow a} f(x) = L$  and  $\lim_{x \rightarrow a} g(x) = M$**

I.e. for  $\delta > 0$ ,  $\exists \epsilon > 0$   $|f(x) - L| < K_1\epsilon$  ( $0 < |x - a| < \delta_1$ )

Similarly  $|g(x) - M| < K_2\epsilon$  ( $0 < |x - a| < \delta_2$ )

$$\text{Where } K_1 = \frac{1}{2(1+|M|)}, \quad K_2 = \frac{1}{2(1+|L|)}$$

$$\text{Or } |g(x) - M| < 1$$

$$\text{Now, } |g(x)| = |g(x) - M + M| \leq |g(x) - M| + |M| < 1 + |M|$$

$$\text{For } \delta = \min(\delta_1, \delta_2)$$

$$\text{Again } |f(x)g(x) - LM|$$

$$= |f(x)g(x) - Lg(x) + Lg(x) - LM|$$

$$= |g(x)(f(x) - L) + L(g(x) - M)|$$

$$\leq |g(x)(f(x) - L)| + |L(g(x) - M)|$$

$$< |g(x)||f(x) - L| + |L||g(x) - M|$$

$$< 1 + |M|K_1\epsilon + 1 + |L|K_2\epsilon$$

$$= \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

i.e. for  $|f(x)g(x) - LM| < \epsilon$  when ( $0 < |x - a| < \delta$ )

4. Before proving this we shall prove that

**Q:- prove that  $\lim_{x \rightarrow a} \frac{1}{g(x)} = \frac{1}{M}$  if  $\lim_{x \rightarrow a} g(x) = M$**

Proof-: Given  $\lim_{x \rightarrow a} g(x) = M$

for  $\delta > 0$ ,  $\exists \epsilon > 0$  such that

$$|g(x) - M| < K\epsilon$$

$$\text{Or } |g(x) - M| < 1$$

$$\text{Now, } |M| = |M - g(x) + g(x)|$$

$$\leq |M - g(x)| + |g(x)|$$

$$\text{or } |M| \leq 1 + |g(x)|$$

$$|g(x)| \geq |M| - 1$$

$$\text{Now } \left| \frac{1}{g(x)} - \frac{1}{M} \right| = \left| \frac{M - g(x)}{Mg(x)} \right|$$

$$\leq \frac{M-g(x)}{|M||g(x)|}$$

$$\leq \frac{K\epsilon}{|M|(|M|-1)}$$

Now choosing  $K = |M|(|M| - 1)$

We get  $\left| \frac{1}{g(x)} - \frac{1}{M} \right| = \epsilon$  for  $(0 < |x - a| < \delta)$

Again given  $\lim_{x \rightarrow a} f(x) = L$  and  $\lim_{x \rightarrow a} g(x) = M$

We know  $\lim_{x \rightarrow a} f(x)h(x) = LN$  (for  $\lim_{x \rightarrow a} h(x) = N$ )

And  $\lim_{x \rightarrow a} \frac{1}{g(x)} = \frac{1}{M}$

Taking  $h(x) = \frac{1}{g(x)}$ , we get

$$\lim_{x \rightarrow a} f(x)g(x) = \lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{L}{M} \quad \text{where } N = \frac{1}{M}, M \neq 0$$

**REAL VALUED FUNCTION -:** Any function of the form  $f: A \rightarrow R$  is said to be a real valued function .

**NON INCREASING FUNCTION-:** If  $f$  is real valued function on any interval  $J \subset R$ , 'f' is said to be non-increasing function on  $J$  if  $f(x) \geq f(y)$  when  $x < y$ ,  $x, y \in J$

**NON DECREASING FUNCTION-:** If  $f$  is real valued function on any interval  $J \subset R$ , 'f' is said to be non-decreasing function on  $J$  if  $f(x) \leq f(y)$  when  $x < y$ ,  $x, y \in J$

**MONOTONE FUNCTION -:** A function  $f$  is said to be monotone if it is either non-increasing or non-decreasing.

**Theorem -:** Let  $f$  be a non-decreasing function on the bounded interval  $(a, b)$ , if  $f$  is bounded above on  $(a, b)$ , then  $\lim_{x \rightarrow b^-} f(x)$  exist. Also iff  $f$  is bounded below on  $(a, b)$  then  $\lim_{x \rightarrow a^+} f(x)$  exists.

Proof -: let us considered the function  $f$  is bounded above and

Non-decreasing in the interval  $(a, b)$ , therefore  $\exists$  a suprimum, let it be  $M$ , i.e.  $M =$  least upper bound  $f(x)$ ,  $x \in (a, b)$ , for  $\epsilon > 0$ , the number  $M - \epsilon$  is not an upper bound of  $f(x)$  for  $x \in (a, b)$

let  $y \in (a, b)$  suc that

$$f(y) > M - \epsilon \text{ where } , y = b - \delta$$

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$$f(b - \delta) > M - \epsilon$$

$$\therefore f(x) > M - \epsilon \quad \text{for } b - \delta < x < b$$

$\lim_{x \rightarrow b^-} f(x)$  exists

Similarly if  $f$  is bounded above and the function is non-decreasing on  $(a, b)$  then we can show that  $\lim_{x \rightarrow b^-} f(x)$  exists.

**Theorem-:** let  $f$  be a non-increasing function on the bounded open interval  $(a, b)$  . then If  $\lim_{x \rightarrow b^-} f(x)$  exists while , if  $f$  is bounded below on  $(a, b)$  then  $\lim_{x \rightarrow a^+} f(x)$  exists .

**COROLLARY -:** if  $f$  is monotonic function on the open interval  $(a, b)$  and if  $c \in (a, b)$  then  $\lim_{x \rightarrow c^+} f(x)$  and  $\lim_{x \rightarrow c^-} f(x)$  both exists .

Proof -:let us consider  $f$  is non-decreasing for  $\delta > 0$  ,  $\exists$  an bounded open interval  $(c - \delta, c + \delta) \subset (a, b)$  then the function  $f$  is bonded above in  $(c - \delta, c)$  by the upper bound  $f(c)$  and hence  $\lim_{x \rightarrow c^-} f(x)$  exists .

Similarly it is called bounded below in  $(c, c + \delta)$  by the lower bound  $f(c)$  .

Hence  $\lim_{x \rightarrow c^-} f(x)$  exists .

**MODULOUS -: $|0|$**

$$|x| > 0 \quad (\because x \neq 0)$$

$$|x| = |-x|$$

$$|x + y| \leq |x| + |y|$$

## METRIC SPACE

**Metric space -:** let  $M$  be any set , a metric for  $M$  is a function  $\rho$  with the domain  $M \times M$  and range contained in  $[0, \infty)$  such that

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- i.  $\rho(x, x) = 0, (x \in M)$
- ii.  $\rho(x, y) > 0, (x \neq y, x, y \in M)$
- iii.  $\rho(x, y) = \rho(y, x) (x, y \in M)$
- iv.  $\rho(x, y) \leq \rho(x, z) + \rho(y, z) (x, y, z \in M)$

**Q-: show that the set of real numbers is a metric space with the metric ' $\rho$ ' defined by**  
 $\rho(x, y) = |x - y|$

Or  $\langle R, \rho \rangle$  is a metric space  $f(x, y) = |x - y|$  (*absolute metric*)

- i.  $\rho(x, x) = |x - x| = 0, (x \in M)$
- ii.  $\rho(x, y) = |x - y| > 0, (x \neq y, x, y \in M)$
- iii.  $\rho(x, y) = |x - y| = |y - x| = \rho(y, x) (\forall x, y \in R)$
- iv. for  $x, y, z \in R$

$$\begin{aligned}\rho(x, y) &= |x - y| = |x - z + z - y| \\ &\leq |x - z| + |z - y| \\ &= \rho(x, z) + \rho(y, z)\end{aligned}$$

**Q-: show that set of real numbers be the discrete metric is a metric space.**

- i.  $d(x, x) = 0, (x \in M)$
- ii.  $d(x, y) = 1 > 0, (x \neq y, x, y \in R)$
- iii.  $d(x, y) = 1 = d(y, x) (x, y \in R)$
- iv. for  $x, y, z \in R$

$$\rho(x, y) = 1 < 2 = 1 + 1 = d(x, z) + d(z, y)$$

**NOTE-: DISCRETE METRIC -:**

$$d(x, x) = 0$$

$$d(x, y) = 1$$

$$\begin{aligned}d(x, y) &= 0 \text{ if } x = y \\ &= 1 \text{ if } x \neq y\end{aligned}$$

**Q -: show that if  $\rho$  is a metric for set  $M$  then so is  $2\rho$**

Ans -: Given  $\rho$  is a metric for the set .

- $\rho(x, x) = 0, (x \in M)$   
 $\Rightarrow 2\rho(x, x) = 0$

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- $\rho(x, y) > 0$  ,  $(x \neq y, x, y \in M)$   
 $\Rightarrow 2\rho(x, y) > 0$
- $\rho(x, y) = \rho(y, x)$   $(x, y \in M)$   
 $\Rightarrow 2\rho(x, y) = 2\rho(y, x)$
- $\rho(x, y) \leq \rho(x, z) + \rho(y, z)$   $(x, y, z \in M)$   
 $\Rightarrow 2\rho(x, y) \leq 2\rho(x, z) + 2\rho(y, z)$

**Q:- show that if  $\rho$  and  $\sigma$  are both metrics for a set  $M$  then  $\rho + \sigma$  is also a metric for the set  $M$ .**

Ans -: Given both  $\rho$  and  $\sigma$  are both metrics for set  $M$

- $\rho(x, x) = 0, \sigma(x, x) = 0$   $(x \in M)$   
 $\Rightarrow \rho(x, x) + \sigma(x, x) = 0$
  - $\rho(x, y) > 0, \sigma(x, y) > 0$  ,  $(x, y \in M)$   
 $\Rightarrow \rho(x, y) + \sigma(x, y) > 0$
  - $\rho(x, y) = \rho(y, x), \sigma(x, y) = \sigma(y, x)$   $(x, y \in M)$   
 $\Rightarrow \rho(x, y) + \sigma(x, y) = \rho(y, x) + \sigma(y, x)$
- For  $x, y, z \in R$
- $\rho(x, y) \leq \rho(x, z) + \rho(y, z)$   
 $\sigma(x, y) \leq \sigma(x, z) + \sigma(z, y)$

$$\Rightarrow \rho(x, y) + \sigma(x, y) \leq (\rho(x, z) + \sigma(x, z)) + (\rho(z, y) + \sigma(z, y))$$

**Theorem -: let  $f$  be non-increasing function on the bounded  $(a, b)$ .if  $f$  is bounded above on  $(a, b)$  then  $\lim_{x \rightarrow b^-} f(x)$  exists, while if  $f$  is bounded below then  $\lim_{x \rightarrow a^+} f(x)$  exists.**

Proof -: consider  $f$  is non-increasing and bounded above on  $(a, b)$

$$\text{Let } M = g.l.b f(x) , x \in (a, b)$$

For  $\epsilon > 0$  ,  $M + \epsilon$  is not a bound

$$\text{Let } y = b - \delta \text{ such that } f(y) < M + \epsilon$$

$$\text{Or } f(x) > m , \forall x \in (a, b)$$

$$\text{i.e. } m < f(x) < m + \epsilon , b - \delta < x < b$$

$$\text{Or } |f(x) - m| < \epsilon , \text{ when } b - \delta < x < b$$

$\lim_{x \rightarrow b^-} f(x)$  exists.

**THE CLASS  $l^2$  :-** The class  $l^2$  is the class of all sequences

$$\{S_n\}_{n=1}^{\infty} = S \quad \text{such} \quad \text{that} \quad \sum_{n=1}^{\infty} S_n^2 < \infty$$

**NOTE :- (SCWARZ'S INEQUALITY) :-** if  $S = \{S_n\}_{n=1}^{\infty}$  and

$t = \{t_n\}_{n=1}^{\infty}$  are in  $l^2$  then  $\sum_{n=1}^{\infty} S_n t_n$  is also absolutely convergent and  $|\sum_{n=1}^{\infty} S_n t_n| \leq |\sum_{n=1}^{\infty} S_n| |\sum_{n=1}^{\infty} t_n|$ .

**(MINKOWSKI INEQUALITY)-:** if  $S = \{S_n\}_{n=1}^{\infty}$  and  $t = \{t_n\}_{n=1}^{\infty}$  are in  $l^2$  then such that  $t = \{t_n + S_n\}_{n=1}^{\infty}$  is in  $l^2$  and

$$\left[ \sum_{n=1}^{\infty} (S_n + t_n)^2 \right]^{\frac{1}{2}} \leq \left[ \sum_{n=1}^{\infty} S_n^2 \right]^{\frac{1}{2}} + \left[ \sum_{n=1}^{\infty} t_n^2 \right]^{\frac{1}{2}}$$

**Q:-**For  $n \in I$  if  $x = (x_1, x_2 \dots \dots x_n)$  and  $y = (y_1, y_2 \dots \dots y_n)$  are two ordered  $n$  tuples of real numbers define  $\rho(x, y) = [\sum_{k=1}^{\infty} (x_k - y_k)^2]^{\frac{1}{2}}$  then show that  $\langle R^n, \rho \rangle$  is a metric space .

Ans :-  $\rho(x, x) = [\sum_{k=1}^{\infty} (x_k - x_k)^2]^{\frac{1}{2}} = 0$

$$\rho(x, y) = [\sum_{k=1}^{\infty} (x_k - y_k)^2]^{\frac{1}{2}} > 0$$

$$\rho(x, y) = [\sum_{k=1}^{\infty} (x_k - y_k)^2]^{\frac{1}{2}}$$

$$[\sum_{k=1}^{\infty} (y_k - x_k)^2]^{\frac{1}{2}} = \rho(y, x)$$

for  $x, y, z \in R^n$ , where  $x = (x_1, x_2 \dots \dots x_n)$  and  $y = (y_1, y_2 \dots \dots y_n)$

$$z = (z_1, z_2 \dots \dots z_n)$$

Let  $\rho(x, z) = [\sum_{k=1}^{\infty} (x_k - z_k)^2]^{\frac{1}{2}}$

Taking  $a_k = x_k - z_k$  ,  $b_k = z_k - y_k$

$$\rho(x, z) = [\sum_{k=1}^{\infty} a_k^2]^{\frac{1}{2}} \quad \rho(z, y) = [\sum_{k=1}^{\infty} b_k^2]^{\frac{1}{2}}$$

$$\rho(x, y) = [\sum_{k=1}^{\infty} (a_k + b_k)^2]^{\frac{1}{2}}$$

$$\leq [\sum_{n=1}^{\infty} (a_k)^2]^{\frac{1}{2}} + [\sum_{n=1}^{\infty} b_k^2]^{\frac{1}{2}}$$

(Using minkowskis's inequality)

$$= \rho(x, z) + \rho(z, y)$$

$$\therefore \rho(x, y) \leq \rho(x, z) + \rho(z, y)$$

**PROVE THAT  $\langle l^2, \rho \rangle$  IS A MATRICE SPACE, WHERE  $\rho$  IS DEFINED AS  $\rho(x, y)$  defined as  $\rho(x, y) = \|x - y\|_2$ .**

Proof -:for  $x, y, z \in l^2$

- $\rho(x, x) = \|x - x\|_2 = 0$
  - $\rho(x, y) = \|x - y\|_2 > 0$  (if  $x \neq y$ )
  - $\rho(x, y) = \|x - y\|_2 = \|(-1)(y - x)\|_2$   
 $\quad \quad \quad |-1|\|y - x\|_2 = \|y - x\|_2 = \rho(y, x)$
  - $\rho(x, y) = \|x - y\|_2 = \|x - z + z - y\|_2$   
 $\quad \quad \quad \leq \|x - z\|_2 + \|z - y\|_2$   
 $\quad \quad \quad = \rho(x, z) + \rho(z, y)$
- $\therefore \rho(x, y) \leq \rho(x, z) + \rho(z, y)$

**NOTE-:** The NORM for a sequence in  $l^2$  has following properties -:

- $\|s\|_2 \geq 0 \quad s \in l^2$
- $\|s\|_2 = 0 \quad \text{iff } s = 0 \in l^2$
- $\|cs\|_2 = |c|\|s\|_2$  for  $s \in l^2, c \in R$  where  $c$  is constant
- $\|s + t\|_2 \leq \|s\|_2 + \|t\|_2$  (for  $s, t \in l^2$ )

**(THE CLASS  $l^\infty$ ):-** The class  $l^\infty$  denote a set of all bounded sequence of real numbers.

**Q-:** show that  $\langle l^\infty, \rho \rangle$  is a metric space where  $\rho$  is defined as  $\rho(x, y) = \sup_{1 \leq n \leq \infty} |x_n - y_n|$

Proof -: for  $x, y, z \in l^\infty$

- $\rho(x, x) = \sup_{1 \leq n \leq \infty} |x_n - x_n| = 0$
- $\rho(x, y) = \sup_{1 \leq n \leq \infty} |x_n - y_n| > 0$
- $\rho(x, y) = \sup_{1 \leq n \leq \infty} |x_n - y_n| = \sup_{1 \leq n \leq \infty} |y_n - x_n| = \rho(y, x)$

$$\begin{aligned}
 \bullet \quad \rho(x, y) &= \limsup_{1 \leq n \leq \infty} |x_n - y_n| = \limsup_{1 \leq n \leq \infty} |x_n - z_n + z_n - y_n| \\
 &\leq \limsup_{1 \leq n \leq \infty} |x_n - z_n| + \limsup_{1 \leq n \leq \infty} |z_n - y_n| \\
 &= \rho(x, z) + \rho(z, y)
 \end{aligned}$$

Q:- For  $P(x_1, y_1)$  and  $Q(x_2, y_2)$  defined  $\rho(P, Q) = |x_1 - x_2| + |y_1 - y_2|$  show that  $\rho$  is metric for the set of ordered pair of real numbers further if  $\tau(P, Q) = \max(|x_1 - x_2|, |y_1 - y_2|)$  show that  $\tau$  is a metric for the same of ordered pairs of real number.

Proof -: for  $P, Q, R \in M$  ( $M$  is the set of real number)

- $\rho(P, P) = |x_1 - x_1| + |y_1 - y_1| = 0$
- $\rho(P, Q) = |x_1 - x_2| + |y_1 - y_2| > 0$
- $\rho(P, Q) = |x_1 - x_2| + |y_1 - y_2| = |x_2 - x_1| + |y_2 - y_1|$   
 $= |y_2 - y_1| + |x_2 - x_1| = \rho(Q, P)$
- $\rho(P, Q) = |x_1 - x_2| + |y_1 - y_2|$   
 $= |x_1 - x_2 + x_3 - x_3| + |y_1 - y_2 + y_3 - y_3|$   
 $\leq |x_1 - x_3| + |x_3 - x_2| + |y_1 - y_3| + |y_3 - y_2|$

$$\begin{aligned}
 &= (|x_1 - x_3| + |y_1 - y_3|) + (|x_3 - x_2| + |y_3 - y_2|) \\
 &= \rho(P, R) + \rho(R, P)
 \end{aligned}$$

$$\rho(P, Q) \leq \rho(P, R) + \rho(R, P)$$

Again for  $P, Q, R \in M$

- $\tau(P, Q) = \max(|x_1 - x_2|, |y_1 - y_2|) = 0$
- $\tau(P, Q) = \max(|x_1 - x_2|, |y_1 - y_2|) > 0$   $x_1 \neq x_2, y_1 \neq y_2$
- $\tau(P, Q) = \max(|x_1 - x_2|, |y_1 - y_2|)$   
 $= \max(|x_2 - x_1|, |y_2 - y_1|)$   
 $= \max(|y_2 - y_1|, |x_2 - x_1|)$   
 $= \tau(P, Q)$
- $\tau(P, Q) = \max(|x_1 - x_2|, |y_1 - y_2|)$   
 $= \max(|x_1 - x_3 + x_3 - x_2|, |y_1 - y_3 + y_3 - y_2|)$   
 $\leq \max(|x_1 - x_3|, |y_1 - y_3|) + \max(|x_3 - x_2|, |y_3 - y_2|)$   
 $= \tau(P, R) + \tau(R, Q)$

**Definition -:** suppose  $\langle M, \rho \rangle$  are matrices spaces where  $a \in M_1, \rho \in M_2$ . Consider the function  $f$  whose domain is  $M_1$  and the range is contained in  $M_2$ , then we say  $f(x)$  approaches to  $L$  when  $x$  approaches  $a$  that if for given  $\epsilon > 0 \exists \delta > 0$  such that  $\rho_2(f(x), L) < \epsilon$  when  $\rho_1(x, a) < \delta$

$$\lim_{x \rightarrow a} f(x) = l$$

**Q-:** let  $f: l^2 \rightarrow R$  be defined as follows if  $x = \{x_n\}_{n=1}^{\infty} \in l^2, f(x) = x_1$  (i.e. image under  $f$  of any sequence in  $l^2$  is the first term of the sequence). let  $a = \{a_n\}_{n=1}^{\infty}$  be fixed element in  $l^2$  prove that  $\lim_{x \rightarrow a} f(x) = a_1$ .

**Proof -:** for  $\epsilon > 0$  we have to find  $\delta > 0$  so that  $\rho_2(f(x), a_1) < \epsilon$   $\rho_1(x, a) < \delta$  we must show that

$$|f(x) - a_1| < \epsilon \text{ Where } \|x - a\|_2 < \delta$$

$$\begin{aligned} \|x - a\|_2 &= [\sum_{n=1}^{\infty} (x_n - a_n)^2]^{\frac{1}{2}} \\ &\geq [x_1 - a_1]^2 \geq [x_1 - a_1] \end{aligned}$$

Now,  $|x_1 - a_1| \leq \|x - a\|_2 < \delta$

Choosing  $\epsilon = \delta$ , we get

$$|f(x) - a_1| = |x_1 - a_1| < \epsilon = \delta$$

$$\lim_{x \rightarrow a} f(x) = a_1 \text{ proved}$$

**Definition -:** let  $\langle M, \rho \rangle$  be matrix space and let  $\{s_n\}_{n=1}^{\infty}$  be a sequence of points in  $M$  then  $\{s_n\}_{n=1}^{\infty}$  is said to be convergent if  $\rho(s_n, L) < \epsilon, n \geq N$  or  $\lim_{x \rightarrow a} s_n = L, n \geq N$  in  $M$

**Definition -:** let  $\langle M, \rho \rangle$  be matrix space and let  $\{s_n\}_{n=1}^{\infty}$  be a sequence of points of  $M$  then  $\{s_n\}_{n=1}^{\infty}$  is said to be Cauchy sequence if  $\epsilon > 0, \exists N \in I$  such that  $\rho(s_m, s_n) < \epsilon, m, n \geq N$

**Theorem-:** let  $\langle R, \rho \rangle$  be a metric space  $\{s_n\}_{n=1}^{\infty}$  is a sequence of points in  $M$  then  $\{s_n\}_{n=1}^{\infty}$  is Cauchy sequence (for some  $\epsilon > 0$ ) there is a number  $l$  such that  $\lim_{x \rightarrow a} s_n = l \quad n \geq N$

Proof -: Given  $\langle R, \rho \rangle$  is a metric space.

$\{s_n\}_{n=1}^{\infty}$  is sequence of points in  $M$   $\{s_n\}_{n=1}^{\infty}$  is convergent

For some  $\epsilon > 0$  there is number  $l$  such that

$$\lim_{x \rightarrow a} s_n = l \quad n \geq N$$

$$\text{i.e. } \rho(s_n, l) < \frac{\epsilon}{2} \quad n \geq N$$

$$|s_n - L| < \frac{\epsilon}{2} \quad n \geq N$$

We can also write  $\rho(s_m, l) < \frac{\epsilon}{2}$  for  $m \geq N$

$$\begin{aligned} \rho(s_n, s_m) &= |s_n - s_m| \\ &= |s_n - L + L - s_m| \\ &\leq |s_n - L| + |L - s_m| \\ &= |s_n - L| + |s_m - L| \\ &\frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon, \text{ for } n, m \geq N \end{aligned}$$

$$\Rightarrow \rho(s_n, s_m) \leq \epsilon \quad \text{where } n, m \geq N$$

**Q-:** show that the sequence of points in any metric space cannot converges to the distinct limit.

## CHAPTER -2

### CONTINUOUS FUNCTION ON A METRIC SPACE

**Continuous-:** A function  $f$  is said to be continuous if  $a \in R'$  if  $\lim_{x \rightarrow a} f(x) = f(a)$

Q-: check the function is continuous or not at the point  $x = 0$

$$\text{Solution -: } f(x) = \frac{\sin}{x}$$

**ADVANCE ANALYSIS**

$$f(a) = \frac{\sin a}{a}, f(0) = \frac{\sin 0}{0} = \frac{0}{0}$$

$$\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$$

∴ Limiting value ≠ functional value

Hence the given function is not continuous

$$\text{Q-: } f(x) = \begin{cases} \frac{\sin x}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases} \text{ is continuous}$$

$$\text{Q-: } f(x) = x^2 + 2x \text{ at } x = 3$$

$$\text{Ans -: } f(x) = (3)^2 + 2(3) = 9 + 6 = 15$$

$$\lim_{x \rightarrow 3} f(x) = \lim_{x \rightarrow 3} x^2 + 2x = 15$$

∴ Limiting value ≠ functional value

Hence the given function is not continuous

**Theorem -:** if the real valued function  $f$  and  $g$  are continuous at  $a \in \mathbb{R}^1$  then so are  $f + g, f - g, fg, f/g$  ( $\because g(a) \neq 0$ )

**Theorem -:** if the  $f$  and  $g$  real valued function are  $f$  is continuous at  $a$ , if  $g$  is continuous at  $f(a)$  then  $g \circ f$  (composite function of  $g$  and  $f$ ) is continuous at  $a$ .

Proof -: Given  $f$  is continuous

$$\text{i.e. } \lim_{x \rightarrow a} f(x) = f(a)$$

for  $\epsilon > 0, \exists \delta > 0$  such that

$$|f(x) - f(a)| < \epsilon \text{ when } |x - a| < \delta \rightarrow (1)$$

## ADVANCE ANALYSIS

Again given  $g$  is continuous at  $f(a)$

$$\lim_{f(x) \rightarrow f(a)} g(f(x)) = g(f(a))$$

Or  $\lim_{y \rightarrow a} g(y) = g(b)$

for  $n > 0, \exists \delta > 0$  such that

$$|g(y) - g(b)| < n \text{ when } |y - b| < \epsilon \rightarrow (2)$$

Or  $|g(f(x)) - g(f(a))| < n$ , when  $|y - b| < \epsilon$

$g \circ f$  is continuous at  $a \in \mathbb{R}^1$  (using 1)

**Q:- prove that  $cf$  is continuous at ' $a$ ' if  $f$  is continuous at  $a$  and  $c \in \mathbb{R}$**

Proof -: Given  $f$  is continuous at  $a$

$$\text{i.e. } \lim_{x \rightarrow a} f(x) = f(a) \rightarrow (1)$$

and we have to prove that  $cf$  is continuous at  $a$

$$c \lim_{x \rightarrow a} f(x) = cf(a) \quad (\text{multiply } c \text{ both sides of (1)})$$

$$\text{or } \lim_{x \rightarrow a} cf(x) = cf(a)$$

Hence  $cf$  is continuous at  $a$

**Q:- the real valued function  $f$  is continuous at  $a \in \mathbb{R}^1$  if and only if given  $\epsilon > 0, \exists \delta > 0$  such that  $|f(x) - f(a)| < \epsilon$  when  $|x - a| < \delta$**

Proof -: necessary part -:

Given  $f$  is continuous at  $a \in \mathbb{R}$

## ADVANCE ANALYSIS

$\lim_{x \rightarrow a} f(x) = f(a)$  for  $\epsilon > 0, \exists \delta > 0$  such that

$$|f(x) - f(a)| < \epsilon \text{ when } |x - a| < \delta$$

Sufficient part -:

$$\text{Given } |f(x) - f(a)| < \epsilon \text{ when } |x - a| < \delta$$

We have to prove that  $f(x)$  is continuous at  $a$

$\therefore$  limits of the function  $f(x)$  exists at  $a$  and is equal to  $f(a)$

$$\text{So, } \lim_{x \rightarrow a} f(x) = f(a) \quad f \text{ is continuous at } a \in R^1$$

**Open ball -:** if  $a \in R^1$  and  $r > 0$  then the set of all  $x \in R^1$  whose distance to  $a$  is less than  $r$ . The set is said to be the open ball i.e.  $B[a, r] = \{x \in R^1 \mid |x - a| < r\}$  that the bounded open interval  $(a - r, a + r)$

**Definition -:** (continuous) -: A function  $f$  is said to be continuous at  $a$  if and only if for given  $\epsilon > 0, \exists \delta > 0$  such that

$$f(x) \in B[f(a), \epsilon], \text{ when } x \in B[a, \delta]$$

**Theorem-:** The real valued function  $f$  is (said to be) continuous at  $a \in R^1$  if and only if the inverse image under  $f$  of any open ball  $B[f(a), \epsilon]$  about  $f(a)$  contains an open ball  $B[a, \delta]$  about  $a$ . that for given  $\epsilon > 0, \exists \delta > 0$  such that  $f^{-1}[B[f(a), \epsilon]] \supset B[a, \delta]$

Proof -: necessary part -:

Let us assume  $f$  is continuous at  $a \in R^1$

For given  $\epsilon > 0, \delta > 0$  such that  $f(x) \in B[f(a), \epsilon]$  when  $x \in B[a, \delta]$

Let  $U = B[f(a), \epsilon]$

$$f(x) \in U, \text{ when } x \in B[a, \delta]$$

$$f^{-1}(f(x) \in f^{-1}(u)), \quad \forall x \in B[a, \delta]$$

Or

$$B[a, \delta] \subset f^{-1}(u)$$

$$x \in f^{-1}(u) \quad \forall x \in B[a, \delta]$$

$$B[a, \delta] \subset f^{-1}[B[f(a), \epsilon]]$$

## ADVANCE ANALYSIS

Sufficient part -: Conversely suppose  $B[a, b] \subset f^{-1}[B[f(a), \epsilon]]$  and we have proof that  $f$  is continuous at  $a \in R^1$ . let us assume  $f$  is not continuous at  $a \in R^1$ .

$$f(x) \notin B[f(a), \epsilon] \text{ when } , x \in B[a, \delta]$$

$$f^{-1}(f(x)) \notin f^{-1}[B[f(a), \epsilon]] \text{ when } , x \in B[a, \delta]$$

$$\therefore B[a, \delta] \not\subset f^{-1}[B[f(a), \epsilon]]$$

Therefore it is contradiction hence our assumption is wrong.

$\therefore f$  is continuous at  $a \in R^1$ .

**Theorem -:** The real valued function  $f$  is said continuous at  $R^1$  if and only if whenever  $\{x_n\}_{n=1}^{\infty}$  is a sequence of real number converging to  $a$  then the sequence  $\{f(x_n)\}_{n=1}^{\infty}$  converging to  $f(a)$  that  $f$  is continuous at  $a$  if and only if  $\lim_{x \rightarrow a} x_n = a$

$$\lim_{x \rightarrow a} f(x_n) = f(a)$$

**Proof -:** Necessary part -: let us assume that  $f$  is continuous at  $a \in R^1$  again, let  $\{x_n\}_{n=1}^{\infty}$  be any sequence of real number converging to  $a$   $\lim_{x \rightarrow a} x_n = a$  for  $\delta > 0, \exists N \in I^+$  such that

$$|x_n - a| < \delta, n \geq N$$

For  $\epsilon > 0 \exists \delta > 0$  such that

$$|f(x_n) - f(a)| < \epsilon \text{ when } |x_n - a| < \delta \text{ (as } f \text{ is continuous)}$$

$$f(x_n) \in [B[f(a), \epsilon]] \text{ when } |x_n - a| < \delta$$

$$\text{When } \lim_{x \rightarrow a} x_n = a \quad n \geq N \Rightarrow \lim_{x \rightarrow a} f(x_n) = f(a)$$

**Sufficient part -:** suppose  $\lim_{x \rightarrow a} f(x_n) = f(a) \quad n \geq N$

And we have to proof that  $f$  is continuous at .

Let us consider the contradicted statement that  $f$  is not continuous at  $a \in R^1$  .

The inverse image under  $f$  of the open ball  $B[f(a), \epsilon]$  doesnot contain the open ball  $B[a, \delta]$ .

Let us consider  $s = \frac{1}{n}, \forall n \in I^+$

## ADVANCE ANALYSIS

For each  $n \in I^+$  there is a point  $x_n \in B\left[a, \frac{1}{n}\right]$ .

Such that  $f(x_n) \notin B[f(a), \epsilon]$  when  $|x_n - a| < \frac{1}{n}$ .

$$\Rightarrow |f(x_n) - f(a)| \geq \epsilon$$

$$\text{Or } |f(x_n) - f(a)| \geq \epsilon$$

$$\lim_{x \rightarrow a} f(x) \neq f(a) \quad n \geq N$$

Which is a contradiction

### FUNCTIONS CONTINUOUS ON METRIC SPACE

**Definition-**: let  $\langle M, \rho \rangle$  be metric space. If  $a \in M$  and  $r > 0$  then  $B[a, r]$  is defined as the set of all points in  $M$  whose distance to  $M$  is less than  $r$  or  $\rho(a, x) < r$ .  
 $B[a, r] = \{x \in M \mid \rho(a, x) < r\}$  Then  $B[a, r]$  is said to be an open ball radius  $r$  about  $a$ .

**Definition -**: The function  $f$  is said to be continuous  $a \in M_1$  if  $\lim_{x \rightarrow a} f(x) = f(a)$

**Theorem -**: The function  $f$  is continuous at  $a \in M_1$  if and only if any one of the following condition holds.

- Given  $\epsilon > 0, \exists \delta > 0$  such that  $\rho_2(f(x), f(a)) < \epsilon$ , when  $\rho_1(x, a) < \delta$
- The inverse image under  $f$  of any  $B[f(a), \epsilon]$  about  $f(a)$  contains an open ball  $[a, \delta]$  about  $a$ .
- Whenever the sequence  $\{x_n\}_{n=1}^{\infty}$  of points in  $M_1$  converging to  $a$  then the sequence  $f\{x_n\}_{n=1}^{\infty}$  in  $M_2$  convergent to  $f(a)$ .

**Theorem -**: let  $\langle M_1, \rho_1 \rangle, \langle M_2, \rho_2 \rangle$  and  $\langle M_3, \rho_3 \rangle$  be metric space and let  $f: M_1 \rightarrow M_2, g: M_2 \rightarrow M_3$  if  $f$  is continuous at  $a \in M_1$  and  $g$  is continuous at  $f(a) \in M_2$  then  $g \circ f$  is continuous at  $a$ .

Proof:- Given  $f$  is continuous at  $a \in M$

$$\lim_{x \rightarrow a} f(x) = f(a)$$

Let  $\{x_n\}_{n=1}^{\infty}$  be a sequence of points in  $M_1$

For  $n > 0, \exists \rho > 0$  such that

## ADVANCE ANALYSIS

$$\rho_1 |f(x_n) - f(a)| < n \text{ when } |x_n - a| < \delta$$

$$\text{Or } \rho_2 |f(x_n), f(a)| < n \text{ when } \rho_1[x_n, a]$$

$$\text{Let } f\{x_n\}_{n=1}^{\infty} = y \text{ when } |x_n - a| < \delta$$

Again,  $g$  is continuous at  $f(a) = b \in M_2$

$$|g(y_n) - g(b)| < \epsilon \text{ when } |y_n - b| < n$$

$$\text{Or } |g(f(x_n)) - g(f(a))| < \epsilon \text{ when } |y_n - b| < n$$

$$\lim_{y_n \rightarrow f(a)} g(f(x_n)) = g(f(a))$$

$$\text{Or } \lim_{x \rightarrow a} g(f(x_n)) = g(f(a))$$

Hence  $g \circ f$  is continuous at.

**Definition** -: let  $M_1$  and  $M_2$  be metric space and let  $f$  is a function,  $f: M_1 \rightarrow M_2$  then  $f$  is said to be continuous from  $M_1$  into  $M_2$  if is of  $M_1$

**OPEN SET** -: let  $M$  be metric space .consider  $G$  be the open subset that  $\forall x \in G, \exists a$  is  $r > 0$  such that the entire open ball  $B[x, r]$  contained in  $G$ .

**Q-: show that  $B[a, 1] = \{a\}$  is in  $R_d$**

$$\text{Proof -: } B[a, 1] = \{x \in R_d \mid d(a, x) < 1\}$$

$$x \in R_d, d(a, x) = \begin{cases} 1 & \text{if } a \neq x \\ 0 & \text{if } a = x \end{cases}$$

$0 < 1$ , Hence  $a$  is the only one element in  $B[a, 1]$

**Q-: Single term set  $\{a\}$  is open in the metric space  $R_d$  whereas it is not open in  $R$ .**

1. Proof -: we know  $B[a, 1] = \{a\}$  in  $R_d$

$$\text{or } B[a, 1] \subset \{a\} \text{ in } R_d$$

Hence  $\{a\}$  is open set in  $R_d$

2.  $B[a, 1] = [1, -1]$  or the elements in the open interval  $[1 - 1]$

Hence,  $B[a, 1]$  is not subset of  $\{a\}$  in  $R$

$$B[a, 1] \not\subset \{a\}$$

Hence  $\{a\}$  is not open set in  $R$

## ADVANCE ANALYSIS

## ADVANCE ANALYSIS

**Theorem -:** for any metric space  $M$  both  $M$  and  $\emptyset$  are open.

Proof -: for any  $x \in M$  we can construct an open ball  $B[x, r]$ , which is subset of  $M$

Hence  $M$  is an open set

We know  $\emptyset$  does not contain any element.

Hence it automatically satisfies the condition of open ball .

$\therefore \emptyset$  is open

**Theorem -;** let  $F$  be any nonempty family of open subsets of a metric space  $M$ . Then  $\bigcup_{G \in F} G$  is also open subset of  $M$ .

Proof -: let  $H = \bigcup_{G \in F} G$

**Case-1-:** let  $H = \emptyset$  , as  $\emptyset$  has no element then automatically it satisfies all the condition of open set.

$\therefore$  It is open

**Case-2-:**  $H \neq \emptyset$

At least one element belongs to  $H$

i.e.  $x \in H$

$\Rightarrow x \in G$  (any)

We know  $G \in F$  that  $G$  is open

$\Rightarrow$  for  $r > 0$  ,  $\exists$  an open ball  $B[x, r] \subseteq G$

$\Rightarrow B[x, r] \subseteq G \subseteq H$

$\therefore B[x, r] \subseteq H$  hence  $H$  is open

**Theorem -:** Every subset of  $R_d$  is open

Proof -: let any  $G \subseteq R_d$

Case-1 if  $G$  is empty that  $G = \emptyset$

then automatically it satisfies all the condition of open set .

Hence  $G = \emptyset$  is open

Case-2 if  $G \neq \emptyset$  , let  $x \in G$

## ADVANCE ANALYSIS

## ADVANCE ANALYSIS

We know that the single term set in  $R_d$  is open

Therefore any non-empty set in  $G$  is also open

**Theorem -:** if  $G_1$  and  $G_2$  are open subsets of the metric space  $M_1$  then

$G_1 \cap G_2$  is also open.

Proof -: case-1-: if  $G_1 \cap G_2 = \emptyset$  then there is nothing to prove

Case-2 -: if  $G_1 \cap G_2 \neq \emptyset$

Let  $x \in G_1 \cap G_2$

$\Rightarrow x \in G_1$  and  $x \in G_2$

Given  $G_1$  and  $G_2$  are open

i.e.  $\exists B[x, r_1] \subseteq G_1$  and  $B[x, r_2] \subseteq G_2$

let  $r = \min(r_1, r_2)$  such that

$B[x, r] \subseteq G_1 \cap G_2$

$G_1 \cap G_2$  is open

**Theorem -:** Prove that  $\bigcap_{n=1}^{\infty} G_n$  is not open if all  $G_n$  are open

**Theorem -:** Every open subset  $G$  of  $R^1$  can be written as  $G = \bigcup I_n$  where  $I = I_1, I_2, \dots, \dots, I_n$  (finite number of intervals) which are mutually disjoint.

Proof -: if  $x \in G \subset R^1$

Given  $G$  is open set

$\therefore \exists$  an open ball  $B[x, r] \subseteq G$

Hence,  $B[x, r]$  is open interval in  $G \subset R^1$

Let  $I_x$  represents the largest open interval in  $G$  containing  $x$

i.e.  $I_x \subseteq G$

if  $x \in G$  and  $y \in G$  then

$I_x = I_y$  or  $I_x \neq I_y$

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$$I_x \neq I_y, I_x \cap I_y = \emptyset$$

Then  $I_x \cap I_y$  is another open interval contained in  $G$  and larger than  $I_x$ . Which contradicts the assumptions  $I_x$  is the largest open interval in  $G$  containing  $x$ . Since disjoint intervals cannot contain the same rational number and there are only countable many rational numbers. Hence there cannot be uncountable many disjoint open intervals. Therefore,  $G = \cup I_n$  where  $n$  is finite and countable.

**Theorem-:** let  $\langle M_1, \rho_1 \rangle$  and  $\langle M_2, \rho_2 \rangle$  be metric spaces and  $f: M_1 \rightarrow M_2$ . Then  $f$  is continuous on  $M_1$  if and only if  $f^{-1}(G)$  is open in  $M_1$  whenever  $G$  is open in  $M_2$

Proof -: necessary part -: given  $f$  is continuous  $M_1$  to  $M_2$  and  $G$  is open in  $M_2$ . our claim is to prove  $f^{-1}(G)$  is open in  $M_1$

Let  $x \in f^{-1}(G)$

Then we should prove that  $B[x, r]$  is an open ball contained in  $f^{-1}(G)$

Since  $x \in f^{-1}(G)$

$$\Rightarrow y = f(x) \in G$$

Given  $G$  is open

Hence there must be an open ball  $B[y, s]$  about  $y$  contained in  $G$

Given  $f$  is continuous

Hence  $f^{-1}[B[y, s]]$  contains the open interval  $B[x, r]$

$$f^{-1}[B[y, s]] \supseteq B[x, r]$$

$$f^{-1}(G) \supseteq B[x, r]$$

Hence  $f^{-1}(G)$  is open in  $M_1$

Sufficient part -:

Given  $G$  is open in  $M_2$  and  $f^{-1}(G)$  is open in  $M_1$

We have to prove that  $f$  is continuous

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Let  $B = B[f(a), \epsilon]$ ,  $B$  be any ball about  $f(a)$

$B$  is open in  $M_2$

Hence by assumption  $f^{-1}(B)$  is open in  $M_1$

Since  $a \in f^{-1}(B)$  and  $f^{-1}(B)$  is open in  $M_1$

Hence there must be an open ball about  $a$

i.e.,  $B[a, \delta] \subseteq f^{-1}(B)$

i.e.,  $B[a, \delta] \subseteq f^{-1}[B[f(a), \epsilon]]$

hence  $f$  is continuous

**LIMIT POINT** -: let  $E$  be the subset of metric space  $M$ . A point  $x \in M$  is said to be a limit point of  $E$  if there is  $\{x_n\}_{n=1}^{\infty}$  of points of  $E$  which converges to  $x$ .

**CLOSED SET** -: let  $E$  be the subset of metric space  $M$ .  $E$  is said to be closed if  $E = \bar{E}$  where  $\bar{E}$  is the subset containing all the limit points of  $E$

**Theorem** -: let  $E$  be a subset of the metric space  $M$ . Then the point  $x \in M$  is a limit point of  $E$  if and only if every open ball  $B[x, r]$  about  $x$  contains at least one point of  $E$

Proof -: necessary part -: let us suppose  $x$  is a limit point of  $E$

There is a sequence  $\{x_n\}_{n=1}^{\infty}$  of points of  $E$  such that  $\rho(x_n, x) < r$

$\Rightarrow \{x_n\}_{n=1}^{\infty} \in B[x, r]$

Sufficient part -: conversely suppose every  $B[x, r]$  contains a point of  $E$  and we have to prove that  $x$  is a limit point of  $E$ .

For  $n \in \mathbb{N}$  the open ball  $B\left[x, \frac{1}{n}\right]$  contains a point of  $\{x_n\}_{n=1}^{\infty}$

Then  $\{x_n\}_{n=1}^{\infty}$  is convergent to  $x$  because  $\rho(x_n, x) < \frac{1}{n}$

$$\lim_{n \rightarrow \infty} (x_n) = x \quad \left(\text{as } n \rightarrow \infty \quad \frac{1}{n} \rightarrow 0\right)$$

Hence  $x$  is limit point of  $E$

**Theorem** -: for any metric space  $M_1$  both  $M$  and  $\emptyset$  are closed.

Q -: prove that for any metric space  $M$  the set  $\{x\} \subset M$  is closed

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Proof -: let us consider  $\{x_n\}_{n=1}^{\infty}$  be the sequence of points the single term set  $\{x\} \subset M$  .

Now ,  $\{x_n\}_{n=1}^{\infty}$  can be written as  $\{x, x, \dots\}$

$$\therefore \lim_{x \rightarrow a} (x_n) = x \quad x \in \{x\}$$

$\{x\}$  contains all its limit points .

Hence  $\{x\}$  is closed

**Theorem** -: if  $E$  is any subset of a metric space  $M$  then  $\bar{E}$  is closed i.e.  $\overline{\bar{E}} = \bar{E}$

Proof -: let  $x \in \bar{E}$

$x$  is a limit point of  $\bar{E}$  therefore the open ball, About  $x$  having radius  $r$  contains at least one element of  $\bar{E}$

at least one element of  $\bar{E}$

let that one element be  $y$ . i.e.  $\rho(x, y) < r$

$$y \in \bar{E}$$

$\Rightarrow y$  is a limit point of  $E$

$\exists$  an element  $Z$  of  $E \in B[y, t]$

Let  $\rho(x, y) = s$  such that  $t < r - s$

Now ,  $\rho(x, z) \leq \rho(x, y) + \rho(y, z)$

$$= s + t$$

$$= s + r - s \quad \Rightarrow \rho(x, z) < r$$

$$\therefore z \in \bar{E}$$

Hence  $z \in B[x, r]$

Thus  $B[x, r]$  contains a points of  $E$

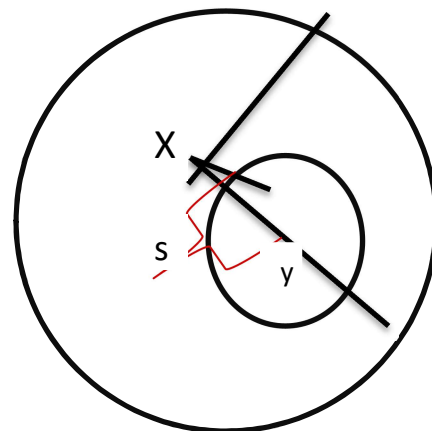
Hence  $x$  is a limit point of  $E$

$$\therefore x \in \bar{E}$$

**Theorem** -: let  $M$  be a metric space then both  $M$  and  $\emptyset$  are closed

Proof -: we know any single term set in any metric space  $M$  is closed and  $M$  can be written as  $M = \bigcup_{i=1}^{\infty} S_i$  where  $S_i$  are single term set .

Hence  $M$  is closed



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$\emptyset$  doesn't contain any element

Hence  $\emptyset$  is closed

**Theorem -:** if  $f_1$  and  $f_2$  are closed subsets of the metric space  $M$  then  $f_1 \cup f_2$  is also closed

Proof -: Given  $f_1$  and  $f_2$  are closed subsets of the metric space  $M$ .

Let  $x \in \overline{f_1 \cup f_2}$ .

I.e.  $x$  is a limit point of  $f_1 \cup f_2$

Let  $\{x_n\}_{n=1}^{\infty}$  is be a sequence of points of  $f_1 \cup f_2$  converging to  $x$ .

Let us consider subsequence of  $\{x_n\}_{n=1}^{\infty}$

We know that subsequence of  $\{x_n\}_{n=1}^{\infty}$  consisting the points of  $f_1$  or  $f_2$  also convergent to the same limit points  $x$ .

Given  $f_1$  and  $f_2$  are closed.

$x$  is a limit point of  $f_1$  or  $f_2$  also

$\therefore x \in f_1$  or  $x \in f_2$

$\Rightarrow x \in f_1 \cup f_2 \quad \therefore \overline{f_1 \cup f_2} \subset f_1 \cup f_2$  Hence  $f_1 \cup f_2$  is closed.

Prove that union of infinite number of closed sets is not closed.

Or

Give examples of closed set whose infinite union is not closed.

**Theorem -:** if  $f$  is the family of closed sets in any metric space then  $\bigcap_{F \in f} F$  is closed.

Proof -: let  $G = \bigcap_{F \in f} F$

Let  $x \in \bar{G}$

i.e.  $x$  is a limit point of  $G$

$\Rightarrow$  Then there is a open ball  $B[x, r]$  about  $x$  containing at least one element of  $G$ .

Let it be  $y$

$\Rightarrow y \in G$  or  $y \in \bigcap_{F \in f} F$

$y$  is an element of all closed sets  $f \in F$

i.e.  $y \in \forall f = \bar{F}$

$x$  is a limit point of every  $f \in F$

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$$\Rightarrow x \in F = \bar{f} \in F$$

$$\Rightarrow x \in \bigcap_{F \in \mathcal{f}} F$$

$$\Rightarrow x \in F \Rightarrow \bar{G} \subset G$$

Hence  $G$  is closed .

Theorem -: if  $G$  is open subset of the metric space  $M$  then  $G' = M - G$  is closed and vice-versa .

Proof -: let us suppose  $G$  is open subset of the metric space .

Let  $x \in G$

$\Rightarrow$  There is an open ball  $B[x, r]$  about  $x$  containing  $G$ .

$\Rightarrow$  The open  $B[x, r]$  doesn't contain any point of  $G'$

Hence  $x$  cannot be a limit point of  $G'$

$\therefore G$  doesn't contain any limit point of  $G'$ .

Hence  $G'$  contains all its limit point ,so  $G'$  is closed .

Let us suppose  $H$  is closed subset of the metric space  $M$  .and we have to prove that the open ball  $B[y, \delta]$  doesn't contain any element of  $H$  .

Then  $y \in H$  which is a contradiction that  $y \in H'$

Hence, there is an open ball  $B[y, \delta]$  entirely lying inside  $H'$ .

Hence  $H$  is open.

**Q-:** using above then prove that  $F_1 \cup F_2$  is closed, when  $F_1, F_2$  closed subsets of the metric space are  $M$ .

Proof -: it is given  $F_1$  and  $F_2$  are closed

$\Rightarrow F_1'$  and  $F_2'$  are open subsets of  $M$

$\Rightarrow F_1' \cap F_2'$  are open in of  $M$ .

$\Rightarrow F_1' \cap F_2' = (F_1 \cup F_2)$  Which is open in  $M$

$\Rightarrow F_1 \cup F_2$  is closed ( $\because$  its compliment is open )

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**Theorem-:** let  $\langle M_1, \rho_1 \rangle$  and  $\langle M_2, \rho_2 \rangle$  be metric spaces and let  $f: M_1 \rightarrow M_2$  then  $f$  is continuous on  $M_1$  if and only if  $f^{-1}(G)$  is closed subset of  $M_1$  whenever  $G$  is closed subset of  $M_2$ .

Proof -: **necessary part -:**

Let us suppose  $f$  is continuous on  $M_1$

Given  $G$  is subset of  $M_2$ , which is closed

$\Rightarrow G'$  is open in  $M_2$

$\Rightarrow f^{-1}(G')$  is open subset of  $M_1$

$$G \cup G' = M_2$$

We know,  $f^{-1}(G') \cup f^{-1}(G) = M_1$

$\Rightarrow f^{-1}(G')$  is closed subset of  $M_1$

**Sufficient part -:** Given  $f^{-1}(G')$  is closed subset of  $M_1$  and

$G$  is closed in  $M_2$

We have to proof that  $f$  is continuous on  $M_1$

$G$  is closed in  $M_2$

$\Rightarrow G'$  is open in  $M_2$

Given  $f^{-1}(G')$  is closed in  $M_1$

$\therefore f^{-1}(G')$  is open in  $M_1$

$\therefore f^{-1}(G')$  is open in  $M_1$  whenever  $G'$  is open  $M_2$

$\therefore f$  is continuous

**Cluster point and dense set**

**NOTE -:** if  $\langle M, \rho \rangle$  is any metric space and  $A \subset M$  then  $A$  is always open in the metric space  $\langle A, \rho \rangle$  even though  $A$  is not open in  $M$ .

**NOTATIONS -:** let  $A \subset M$  (where  $M$  is a metric space) and it  $a \in A$ ,

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$$B_A[a, r] = \{x \in A \mid \rho(a, x) < r\}$$

$$B_M[a, r] = \{x \in M \mid \rho(a, x) < r\}$$

Examples -(1)  $A = [0, 1] \subset R$

$$B_A[0, r] = \{x \in A \mid \rho(0, x) < r\}$$

$$B_M[0, r] = \{x \in M \mid \rho(0, x) < r\}$$

(2) Let  $\langle M, \rho \rangle = R$  and  $\langle A, \rho \rangle = [0, 1]$

$$\text{Then } B\left[0, \frac{1}{2}\right] \text{ in } R = \left(\frac{1}{2}, \frac{1}{2}\right)$$

$$B\left[0, \frac{1}{2}\right] \text{ in } A = \left[0, \frac{1}{2}\right)$$

**Theorem -:** let  $\langle M, \rho \rangle$  be a metric space and  $A$  be proper subset of  $M (A \subset M)$ . Then the subset  $G_A$  of  $A$  is an open sub set of  $\langle A, \rho \rangle$  if and only if there exists an open subset  $G_M$  of  $\langle M, \rho \rangle$  such that  $G_A = A \cap G_M$

**Proof -:** **necessary part -:** let us consider  $G_A$  be an open subset of  $G_A$

Given  $A$  is a proper subset of  $\langle M, \rho \rangle$

Let  $a \in G_A$

$\Rightarrow$  For  $r_a > 0$ ,  $\exists$  an open ball  $B_A[a, r]$  contained in  $G_A$ .

Define  $G_M = \bigcup_{a \in G_A} B_M[a, r_a]$

$\Rightarrow G_M$  is an open subset of the metric space  $M$ .

Now,  $G_M \cap A = G_A$

We have to proof that  $G_A$  is open in  $\langle A, \rho \rangle$

Given  $G_M$  is open .

$\Rightarrow$  There must be an open ball  $B_M[a, r] \subset G_M$

$\Rightarrow B_M[a, r] \cap A \subset G_M \cap A$

$\Rightarrow B_M[a, r] \subset G_A$

Hence  $G_A$  is open .

**Connected -:** let  $\langle M, \rho \rangle$  be metric space and let  $A$  be the subset of  $M$  .if  $A$  has either of the properties

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- It is impossible to find non-empty subset  $A_1, A_2$  of  $M$  such that  $A = A_1 \cup A_2$ ,  $\overline{A_1} \cap A_2 = \emptyset$ ,  $A_1 \cap \overline{A_2} = \emptyset$
- When  $\langle A, \rho \rangle$  is itself regarded as a metric space then there is no set except  $A$  and  $\emptyset$  which is both open and closed in  $\langle A, \rho \rangle$ , then  $A$  is said to be connected.

**Theorem-:** let  $\langle M, \rho \rangle$  be a metric space and let  $A$  be a subset of  $M$  then if  $A$  has either one of the following properties it has the other (above (a) and (b) properties).

Proof -: let us consider (a) holds and we have to prove that (b) also holds .i.e  $a \Rightarrow b$

i.e. if not let us suppose (b) does not hold

i.e. there would be a nonempty proper subset of  $A$ , i.e.  $A_1$  which is both open and closed in  $A$

let  $A_2 = A - A_1$

$\Rightarrow A_2$  would also be both open and closed in  $A$ .

If  $x \in M$  is a limit point of  $A_2$

i.e.  $x \in A_2$

$\Rightarrow x \in A_1$  ( if  $x$  is in  $A_1$  then  $x$  would be in  $A - A_2$ . So  $A_2$  is not a closed set )

$\therefore A_1 \cap \overline{A_2} = \emptyset$

Which is a contradiction

Similarly we can show that  $\overline{A_1} \cap A_2 = \emptyset$  which is a contradiction.

Considering (b) holds we will prove (a) holds

i.e.  $(b) \Rightarrow (a)$

let us suppose consider (a) doesn't hold

i.e. there would exist non-empty proper subsets  $A_1$  and  $A_2$  of  $M$  such that  $A = A_1 \cup A_2$

$$\overline{A_1} \cap A_2 = \emptyset$$

$$A_1 \cap \overline{A_2} = \emptyset$$

Let  $G = M - \overline{A_2}$  (we know if  $E$  is any subset of a metric space  $M$  then  $\overline{E}$  is closed)

$\overline{A_2}$  is closed  $G = M - \overline{A_2}$  is complement of  $\overline{A_2}$

Hence  $G$  is open

Given  $A_1 \cap \overline{A_2} = \emptyset$

$\Rightarrow A_1 \subset G$  ( $\because G = M - \overline{A_2}$ )

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$$\therefore G \cap A = A_1$$

Hence  $A_1$  is an open subset of  $\langle A, \rho \rangle$  ----- (1)

Similarly we can show that  $A_2$  is open subset of  $\langle A, \rho \rangle$

$$A_1 = A - A_2$$

Thus  $A_1$  is closed ----- (2)

Combining (1) and (2), we get

$A_1$  is both open and closed in  $\langle A, \rho \rangle$

Which is contradiction to our assumption that (b) holds .

$\therefore$  if (b) is true then (a) is true

**Theorem -:** The subset of  $A$  of  $R'$  is connected if and only if whenever  $a \in A, b \in A$  be  $A$  with the condition  $a < b$  be  $A$  with the condition  $a < b$  then  $c \in A$  for any  $c$  such that  $a < c < b$  .

**Proof -:** **necessary part -:** let us  $A \subset R'$  is connected and  $a \in A, b \in A$  with  $a < b$

Let us assume a number  $c \in R - A$  with the condition  $a < c < b$

Let  $A_1 = A \cap (-\infty, c)$  and  $A_2 = A \cap (c, -\infty)$

Then  $A = A_1 \cup A_2$

If  $x \in \overline{A_1}$

i.e.  $x$  is a limit point of a sequence of number of  $(-\infty, c)$

$$\Rightarrow x \leq c$$

$$\Rightarrow x \notin A_2$$

Hence  $\overline{A_1} \cap A_2 = \emptyset$

Similarly we can show that  $A_1 \cap \overline{A_2} = \emptyset$

$\exists A_1, A_2$  such that  $A = A_1 \cup A_2, \overline{A_1} \cap A_2 = \emptyset, A_1 \cap \overline{A_2} = \emptyset$

Hence  $A$  is not connected which is a contradiction to our assumption

$\therefore$  our assumption is wrong

i.e.  $c \in A$  whenever  $a < c < b$

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**sufficient part -:** let us suppose consider  $A$  is not connected

$\Rightarrow \exists$  nonempty set  $A_1$  and  $A_2$  such that  $A = A_1 \cup A_2$ ,  $\overline{A_1} \cap A_2 = \emptyset$ ,  $A_1 \cap \overline{A_2} = \emptyset$

Let us consider  $a_1 \in A_1$ ,  $a_2 \in A_2$

$$\Rightarrow a_1 \neq a_2$$

Let us assume  $a_1 < a_2$

Now we shall prove that  $(a_1, a_2) \not\subset A$

Let  $B$  be the set of all  $x \in A_1$  such that

$$a_1 \leq x \leq a_2$$

i.e.

$$B = A_1 \cap [a_1, a_2]$$

hence  $B$  is nonempty bounded set of real numbers. Hence  $B$  has the least upper bound i.e.  $\bar{a}$

Hence  $\bar{a} \in \overline{B}$

$$\Rightarrow B \subset A_1$$

$$\Rightarrow \overline{B} \subset \overline{A_1}$$

$$\Rightarrow \bar{a} \in \overline{A_1}$$

$$\Rightarrow \bar{a} \notin \overline{A_2} \quad \because \overline{A_1} \cap A_2 = \emptyset$$

But  $\bar{a} \in [a_1, a_2]$

$$\therefore a_1 < \bar{a} < a_2$$

But either  $\bar{a} \in A$  or  $\bar{a} \notin \overline{A_2}$

If  $\bar{a} \notin \overline{A_2}$  then  $\bar{a} \neq a_1$

$$\Rightarrow a_1 > \bar{a}$$

$$\therefore a_1 < \bar{a} < a_2 \text{ and } \bar{a} \notin A$$

$$\Rightarrow (a_1, a_2) \not\subset A \text{ ----- (1)}$$

If  $\bar{a} \in A$  then  $\bar{a} \in \overline{A_1}$  ( $\bar{a} \notin \overline{A_2}$ )

$\therefore \bar{a}$  is not there is a number  $c \notin A$  with the condition  $\bar{a} < c < a_2$

$$\Rightarrow (a_1, a_2) \not\subset A \text{ ----- (2)}$$

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Combining (1) and (2) we conclude that

$(a_1, a_2) \notin A$  whenever  $a_1 \in A_1, a_2 \in A_2$  with  $a_1 < a_2$ .

**Theorem -:** let  $f$  be continuous function from  $M_1$ , metric space  $M_1$  (domain of the function) is connected then the range of  $f$  is also connected.

Proof -: Given  $f: M_1 \rightarrow M_2$  is continuous

We have to prove that range of  $f$  etc

$A = f(x), \forall x \in M_1$  is also connected

Let us consider of not that  $A$  is not connected.

Let  $A \subset M_2$  which is not connected

$\Rightarrow \exists$  a non-empty proper subset of  $A$  which is both open and close in  $A$ .

$\therefore$  it is contradiction to our assumption that  $A$  is not connected.

$\therefore$  for a continuous function  $f$  from a metric space  $M_1$  into another metric space  $M_2$ , if  $M_1$  is connected then the range of the function is also connected.

**Definition -: (characteristics function )-:**

A function defined as  $\psi(x) = \begin{cases} 0 & \text{if } x \in M - A \\ 1 & \text{if } x \in A \end{cases}$  where  $A \subset M$  and  $M$  is metric space is known as characteristics function of  $A$ .

**Theorem -:** let  $M$  be a metric space .then  $M$  is connected if and only if every continuous characteristics function on  $M$  is constant.

**Proof -: necessary part-:** Let  $M$  be any metric space and  $\psi$  be the characteristics function of  $A \subset M$ .

$$\text{i.e. } \psi(x) = \begin{cases} 0 & \text{if } x \in M - A \\ 1 & \text{if } x \in A \end{cases} \quad \text{or } (x \notin A)$$

We know  $\{1\}$  is closed in the metric space  $M$ .

$\psi^{-1}(1) = A$  -----(1) is also a closed set in the metric space  $M$

$\therefore M - A$  is open in the metric space  $M$

$$\psi^{-1}(0) = M - A$$

Which is also closed in the metric space  $M$

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$\therefore M - (M - A) = A$  -----(2) is open in the metric space  $M$

Combining (1) and (2) we conclude that  $\exists$  a set  $A \subset M$  which is both open and closed .

Hence  $M$  is not connected

Which contradicts to our assumption that  $\exists$  a characteristics function

$$\psi(x) = \begin{cases} 0 & \text{if } x \in M - A \\ 1 & \text{if } x \in A \end{cases}$$

**Sufficient part -:** conversely suppose every continuous characteristics function in the metric space is constant.

Let  $A \subset M$

$$\psi(x) = 0 \text{ or } \psi(x) = 1 \quad \forall x \in A$$

And we have to prove that  $M$  is connected ,

If not ,

i.e. let  $M$  is connected

$\exists A \subset M$  such that  $A$  is both open and closed

$$\text{Let } \psi(A) = 1 \text{ or } \psi(A)^{-1} = A$$

Now  $\{1\}$  is both open and closed in  $M$

Which is not always true .

i.e. a contradiction

Hence  $M$  is connected.

**Theorem -:** if  $A_1$  and  $A_2$  are connected subsets of metric space  $M$  and if  $A_1 \cap A_2 = \emptyset$  , then  $A_1 \cup A_2$  is also connected

Proof -: let  $\psi$  be continuous characteristics function on  $A_1 \cup A_2$

$$\text{Let } x \in A_1 \cup A_2$$

i.e.  $x \in A_1$  or  $x \in A_2$  (as  $A_1 \cap A_2 = \emptyset$  )

$$x \in A_1 \Rightarrow \psi(x) = x_0$$

As  $A_1$  is connected

Similarly  $\psi(x) = x_0$  for  $x \in A_2$  (as  $A_2$  is connected )

$$\psi(x) = x_0 \text{ for } x \in A_1 \cup A_2$$

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Hence  $A_1 \cup A_2$  is connected .

Such that

**Definition -: (bounded) -:** (1) let  $\langle M, \rho \rangle$  be the metric space then the subset  $A$  of  $M$  is said to be bounded if there exist a positive number  $K$  such that the distance between  $\rho(x, y) < K, \forall x, y \in A$  .

(2) if  $A$  is bounded then there is a diameter of  $A$  is denoted by .

$$\text{diam } A = \text{l.u. b } \rho(x, y) \text{ for } x \in A, y \in A$$

(3) if  $A$  is bounded then distance of  $A$  is  $\text{diam } A = \infty$

(4) let  $\langle M, \rho \rangle$  be a metric space . the subset  $A$  of  $M$  is said to be totally bounded if given  $\epsilon > 0 \exists$  finite number of subsets  $A_1, A_2, \dots, \dots, A_n$  of  $M$  such that  $\text{diam } A_k < \epsilon$  (for  $k = 1, 2, \dots, n$ ) and  $A = \bigcup_{k=1}^n A_k$  .

**Theorem-:** if the subset  $A$  of the metric space  $\langle M, \rho \rangle$  is totally bounded then  $A$  is bounded .

Proof -: Given  $A$  is totally bounded.

$\Rightarrow \exists$  Non-empty subsets  $A_1, A_2, \dots, \dots, A_n$  of  $M$

Such that  $\text{diam } A_k < 1$  (for  $k = 1, 2, 3, \dots, n$ )

And  $A = \bigcup_{k=1}^n A_k$  .

Let  $a_k$  be any point in  $A_k$  .

$$\text{Let } D = \rho(a_1, a_2) + \rho(a_2, a_3) + \rho(a_3, a_4) + \dots + \rho(a_{n-1}, a_n)$$

Let  $x, y \in A$  , where  $x \in A_i, y \in A_j$  for some  $i$  and  $j$

Assume  $i \leq j$

$$\rho(x, y) \leq \rho(x, a_i) + \rho(a_i, a_{i+1}) + \rho(a_{i+1}, a_{i+2}) + \dots + \rho(a_{j-1}, a_j) + \rho(a_j, y)$$

Since  $\text{diam } A_k < 1$

Hence  $\rho(x, a_i) < 1$  and  $\rho(a_j, y) < 1$

$\Rightarrow \rho(x, y) \leq 1 + D = 1 \therefore \rho(x, y) \leq D + 2$  Hence  $A$  is bounded .

**Corollary -:** A subset  $A$  of the metric space  $R_d$  is totally bounded if and only if  $A$  contains only a finite number of points.

**Proof-: necessary part -:** (A contains only a finite number of points )

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Let  $A$  is a subsets of the metric space  $R_d$  is totally bounded .

$$diam A_k < 1 \text{ and } A = \cup_{k=1}^n A_k .$$

To proof , $A$  contains only a finite number of points

Let  $a_k$  be a points in  $A_k$

$$\text{Let } A_k = B[a_k, \epsilon]$$

$$\text{As } \rho(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases} \text{ in } R_d$$

And we know in  $R_d$  every open ball contains only one element

As there are only finite numbers of sets  $A_1, A_2, \dots \dots \dots A_n$

There are only finite number of points also .

**Sufficient part** -: Given  $A$  is subsets of  $R_d$  which contains only a finite number of points .

To prove that  $A$  is totally bounded .

Let there be only finite number of sets  $A_1, A_2, \dots \dots \dots A_n$

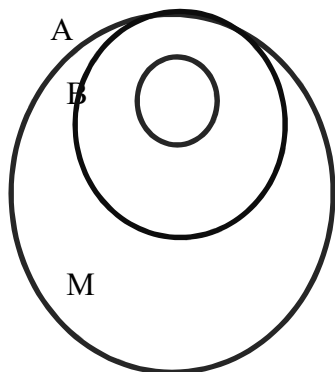
Then in  $R_d$  every open ball contains only one element

$$A_k = \{a_k\} = B[a_k, \epsilon]$$

$$\Rightarrow diam A_k < \epsilon \text{ And } A = \cup_{k=1}^n A_k .$$

Hence  $A$  is totally bounded

**Definition** -: let  $A$  be subsets of the metric space the subsets  $B$  of  $A$  is said to be  $\epsilon - dence$  where  $\epsilon > 0$  if for every  $x \in A , \exists y \in B$  such that  $(x, y) < \epsilon$  .



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i.e.  $B$  is  $\epsilon$ -dense in  $A$  if each point of  $A$  is within  $\epsilon$  distance from some point of  $B$

**Theorem-:** the subsets  $A$  of the metric space  $\langle M, \rho \rangle$  is totally bounded if and only if for every  $\epsilon > 0$ ,  $A$  contains a finite subset  $\{x_1, x_2, \dots, x_n\}$  which is  $\epsilon$ -dense in  $A$

Proof -: Necessary part -:

Given a subset  $A$  of  $M$  is totally bounded

fix  $\epsilon > 0$

$\text{diam } A_k < \epsilon$  And  $A = \bigcup_{k=1}^n A_k$ .

Let us assume  $A_k \neq \emptyset$  for  $(k = 1, 2, \dots, n)$

If  $a_k \in A_k$  ( $k = 1, 2, \dots, n$ )

$\Rightarrow$  the subset  $\{a_1, a_2, \dots, a_n\}$  is  $\epsilon$ -dense in  $A$

$\Rightarrow$  if  $A$  is totally bounded then  $A$  has finite number of  $\epsilon$ -dense set.

Sufficient part -:

Conversely suppose  $A$  has finite subset which are  $\epsilon$ -dense in  $A$

And we have to prove that  $A$  is totally bounded

Let  $x_i \in A_i$

Let  $A_i = B[x_i, \epsilon/n]$  covers totally  $A$

$\text{diam } A_i < \epsilon$  and  $A = \bigcup_{i=1}^n A_i$

Hence it is totally bounded.

**Theorem-:** let  $\langle M, \rho \rangle$  be metric space, the subset  $A$  of  $M$  is totally bounded, if and only if every sequence of points of  $A$  contains Cauchy sub-sequence.

Necessary part-: Give  $A \subset M$  is totally bounded.

We have to prove that every sequence of points  $A$  contains Cauchy

Sub-sequence.

Let  $\{x_n\}_{n=1}^{\infty}$  be a sequence of points of  $A$

$A$  is totally bounded.

i.e.  $A$  is covered by finite number of sub-cover whose  $\text{diam} < 1$

One of the set is  $A_1$  contains  $\{x_n\}$

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For initially many values of  $n_1$

Choose any  $n_1 \in I$  such that  $x_{n_1} \in A_1$

We know  $A_1$  is totally bounded.

Hence  $A_1$  can be covered by finite number subsets

Whose  $diam < \frac{1}{2}$

Let  $A_2$  be one of sub covers of  $A_1$  most contain  $\{x_n\}$

For infinity many values of  $n$

Let  $n_2$  be any integers  $> n_1$  such that  $x_{n_2} \in A_1$

Continuing in this way for  $n_2 \in I$  subsets  $A_k$  of  $A_{k-1}$

With  $diam A_k < \frac{1}{k}$  and the sub sequence  $x_{n_k} \in A_k$  of the sequence  $\{x_n\}_{n=1}^{\infty}$ .

Since  $x_{n_k}, x_{n_{k+1}}, x_{n_{k+2}} \dots$  all are in  $A_k$

And since  $diam A_k < \frac{1}{k}$

$\Rightarrow \{x_{n_k}\}_{k=1}^{\infty}$  is a Cauchy sequence of  $\{x_n\}_{n=1}^{\infty}$

Sufficient part -: Given every sequence of points of a contains a Cauchy sequence.

We have to prove that  $A$  is totally bounded

Let us assume  $A$  is not totally bounded.

$\exists \epsilon > 0$  such that  $A$  Contains no finite  $\epsilon - dense$  set .

If  $x_1 \in A$

$\{x_1\}$  is not  $\epsilon - dense$   $A$ .

$\therefore \exists x_1 \in A$  such that  $\rho(x_1, x_2) \geq \epsilon$

$\Rightarrow \rho(x_1, x_2) \geq \epsilon$

Continuing in this way we get

$\{x_n\}_{n=1}^{\infty}$  of points of  $A$  such that  $\rho(x_1, x_2) \geq \epsilon$

The sequence  $\{x_n\}_{n=1}^{\infty}$  no Cauchy sub- sequence

So which is a contradiction. Hence our assumption is wrong

$\therefore$  Hence  $A$  is totally bounded .

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**Complete metric space →**

A metric space  $(M, \rho)$  is said to be complete. If every Cauchy sequence of points in  $M$  converges a point in  $M$ .

**Example-** In metric  $R'$  every Cauchy sequence of points in  $R'$  converges to a point in  $R'$ . Hence the metric space  $R'$  is complete.

**Or show that  $l^2$  is complete.**

Proof – let  $s^{(1)}, s^{(2)}, \dots$  be Cauchy sequence of point in  $l^2$

We must show that  $s \in l^2$  such that  $s^n \rightarrow s$  as  $n \rightarrow \infty$

$\therefore s^{(n)}$  Itself is a sequence, thus let the  $k^{th}$  term of the sequence  $s^{(n)}$  will be  $s_k^{(n)}$

i.e.  $s^{(n)} = \left\{ s_k^{(n)} \right\}_{k=1}^{\infty}$

$$\|s^{(n)}\|_2 = \left[ \sum_{k=1}^n s_k^{(n)2} \right]^{\frac{1}{2}}$$

$$\Rightarrow \|s^{(n)}\|_2^2 = \sum_{k=1}^{\infty} s_k^{(n)2}$$

$\therefore s_1, s_2, s_3, \dots$  is a cauchy sequence in  $l^2$

For  $\epsilon > 0, \exists N \in I$

Such that  $\rho[s^{(n)}, s^{(m)}] < \epsilon (n, m \geq N)$ .

$$\Rightarrow \|s^{(n)}, s^{(m)}\|_2 < \epsilon (n, m \geq N) \dots \dots \dots (1)$$

For  $m \geq N$

$$\|s^{(n)}, s^{(m)}\|_2 < \epsilon$$

For  $n \geq N$

$$\|s^{(n)}\|_2 = \|s^{(n)} - s^{(N)} + s^{(N)}\|_2 < \epsilon + \|s^{(N)}\|_2$$

I.e. for some  $A > 0$

$$\|s^{(N)}\|_2 \leq A$$

Hence, for some fixed  $K \in I$  we have

$$\left| s_k^{(n)} - s_k^{(m)} \right| < \|s_k^{(n)} - s_k^{(m)}\|_2 < \epsilon$$

$\Rightarrow$  For, some fixed  $i$ ,

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A sequence,  $\{s_k^{(n)}\}_{k=1}^{\infty}$  is a Cauchy sequence in  $R'$ .

Let  $s$  denotes a sequence,  $\{s_k^{(n)}\}_{k=1}^{\infty}$

We have to show that,  $s \in l^2$

Now, from equation (2) we get,

$$\|s_k^{(n)}\|_2^2 = \sum_{k=1}^n s_k^{(n)2} \leq A^2$$

Hence any integer i.e.  $1 \in I$

$$\sum_{k=1}^l s_k^{(n)2} \leq A^2$$

$$\Rightarrow s = \{s_k\}_{k=1}^{\infty} \text{ is in } l^2$$

From equation (1) we get,

$$\sum_{k=1}^{\infty} (s_k^{(n)} - s_k^{(m)})^2 < t^2$$

For  $l \in I$

$$\sum_{k=1}^l (s_k^n - s_k^m)^2 < \epsilon^2$$

Again, using

$$\lim_{n \rightarrow \infty} s_k^{(n)} = s_k \text{ we have}$$

$$\sum_{k=1}^{\infty} (s_k^{(n)} - s_k^{(m)})^2 < l^2$$

There for

$$\therefore \rho(s^n, s) = \|s^n - s\|_2 < \epsilon$$

Hence  $s', s^2$  converges to  $s$  in  $l^2$

Hence  $l^2$  is complete.

**Theorem -:** if  $\langle M, \rho \rangle$  is a complete metric space and  $H$  is closed subset of  $m$  then  $\langle A, \rho \rangle$  is also complete.

Proof-: let  $[x_n]_{n=1}^{\infty}$  is a Cauchy sequence of a point in  $\langle A, \rho \rangle$  since  $A$  is a subset of  $M$  is closed.

Hence,  $[x_n]_{n=1}^{\infty}$  is a Cauchy sequence of points converges to a point in  $A$ .

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I.e.  $\lim_{n \rightarrow \infty} \{x_n\}_{n=1}^{\infty} = x$  and  $x \in A$

Hence  $\langle A, \rho \rangle$  is a complete metric space.

**Example -:**  $[0,1]$  is closed subset of  $R'$  ( $R'$  is complete) hence  $[0,1]$  is also a complete metric space.

**Theorem – :** let  $\langle M, \rho \rangle$  be a complete metric space for each  $n \in I$  , let  $F_n$  be closed bounded subset of  $M$  such that

- (i)  $F_1 \supset F_2 \supset F_3 \dots \dots \dots \supset F_n \supset F_{n+1}$  and
- (ii) Diameter of  $F_n \rightarrow 0$  as  $n \rightarrow \infty$  then  $\bigcap_{n=1}^{\infty} F_n$  contains precisely one point

Proof -: For each  $n \in I$

Let can be any point in  $F_n$

By condition (i)

$$a_n, a_{n+1}, a_{n+2} \dots \dots \dots \in F_n$$

Given  $\epsilon > 0 \exists N \in I$  such that  $\text{diam} F_n < \epsilon$

$\therefore \{a_n\}_{n=1}^{\infty}$  is a Cauchy sequence

Given  $M$  is a complete, i.e. there exist  $a \in M$  S.T.

$$\lim_{n \rightarrow \infty} a_n = a$$

As  $f_n$  is closed and bounded, then the limit point  $a \in f_n$

$$a \in \bigcap_{n=1}^{\infty} F_n$$

If  $b \in M$  and  $b \neq a$

Then  $\rho(a, b) > \text{diam} f_n$

If  $n$  is sufficiently large

Therefore  $b \notin \bigcap_{n=1}^{\infty} F_n$  .

i.e.  $\bigcap_{n=1}^{\infty} F_n$  contain precisely one.

Contraction –:

If  $\langle m, \rho \rangle$  be a metric space if  $T$  is a mapping from  $N \rightarrow M$  ,  $T$  is said to be contraction on  $M$  if  $\alpha \in R$  st

$$\rho(T_x, T_y) \leq \alpha \rho(x, y)$$

Notation -:

$$T_x = T(x)$$

$$T^2 = T \circ T$$

$$T^3 = T \circ T^2$$

$$T^m = T \circ T^{m-1}$$

Example -: if  $u = \{u_n\}_{n=1}^\infty \in l^2$

$$u = \left\{ \frac{u_n}{2} \right\}_{n=1}^\infty$$

If  $v = \{v_n\}_{n=1}^\infty \in l^2$  is any other point in  $l^2$

$$\begin{aligned} \rho(T_u, T_v) &= \|T_u + T_v\|_2 \\ &= \left[ \sum_{n=1}^\infty \left( \frac{u_n}{2} + \frac{v_n}{2} \right)^2 \right]^{\frac{1}{2}} \\ &= \left( \left( \frac{1}{2} \right)^n \right)^{\frac{1}{2}} \left[ \sum_{n=1}^\infty (u_n + v_n)^2 \right]^{\frac{1}{2}} \\ &= \frac{1}{2} \|u_n + v_n\|_2 \\ &= \frac{1}{2} \rho(u_n + v_n) \end{aligned}$$

Hence it is a contraction.

**Theorem -:**

Let  $\langle M, \rho \rangle$  be complete metric space if  $p$  is a contraction on  $m$ , then there is one and only one point  $x$  in  $M$ , such that  $T_x = x$ .

Proof -: let  $x, y \in M$

Given  $T$  is a contraction on  $M$

$$\text{Here } \rho(T_x^2, T_y^2) \leq \alpha \rho(T_x, T_y) \leq \alpha^2 \rho(x, y)$$

For  $n \in I$

$$\rho(T_x^n, T_y^n) \leq \alpha \rho(T_x^{n-1}, T_y^{n-1}) \leq \alpha^2 \rho(T_x^{n-2}, T_y^{n-2}) \leq \alpha^n \rho(x, y)$$

Now choose any point  $x_0 \in M$ , such that  $T_{x_1}, T_{x_0}$

$$\therefore x_1 = T_{x_0} \quad x_2 = T_{x_1}^2 \dots \dots \dots x_n = T_{x_0}^{n+1}$$

We shall 1<sup>st</sup> show that  $\{x_n\}_{n=1}^\infty$  is a Cauchy sequence

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For  $m, n \in I$  (and  $m > n$ ) let  $m = n + p$

We have  $\rho(x_n, x_m) = \rho(x_n, x_{n+p}) \leq \rho(x_n, x_{n+1}) + \dots + \rho(x_{n+1}, x_{n+p})$

$$\rho(x, y) \leq \rho(x, z) + \rho(y, z)$$

$$\leq \alpha^n \rho(x_0, x_1) + \alpha^{n+1} \rho(x_0, x_1) + \dots + \alpha^{n+p-1} \rho(x_0, x_1)$$

$$= \alpha^n \rho(x_0, x_1) \sum_{h=0}^{\infty} \alpha^h$$

$$= \alpha^n \rho(x_0, x_1) (1 - \alpha)^{-1}$$

We know  $\lim_{n \rightarrow \infty} \alpha^n = 0$

$\therefore \rho(x_n, x_m) \leq \epsilon$  (small +ve value taking  $n, m$  sufficient large)

Hence  $\{x_n\}_{n=1}^{\infty}$  is a Cauchy sequence

Given  $M$  is complete and every Cauchy sequence converges,  $\exists x \in M$

Such that  $\lim_{n \rightarrow \infty} x_n$

$$\lim_{n \rightarrow \infty} T x_n = T x \dots \dots \dots (2)$$

$\{T x_n\}_{n=1}^{\infty}$  is sub-sequence of  $\{x_n\}_{n=0}^{\infty}$

Hence it must converge to the same limit point to which  $x_n$  converges

*i. e.*  $\lim_{n \rightarrow \infty} T x_n = x$

Combining (2) and (3) we get

$$T x = x$$

Let  $y \in M$  such that  $y \neq x$

That we have to show that  $y$  is not a fixed point

Let us assume that  $y$  is a fixed point

$$\Rightarrow T x = y$$

$$\Rightarrow \rho(x, y) = \rho(T x, T y) \leq \alpha \rho(x, y)$$

$$\therefore \rho(x, y) \neq 0 \text{ finally we get } \rho(x, y) \leq \alpha \rho(x, y)$$

$$1 \leq \alpha$$

Hence it is a contradiction

$\therefore$  our assumption is wrong

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$\therefore y$  is not a fixed point

Hence only previous fixed in  $M$

Compact metric space:-

The metric space  $\langle M, \rho \rangle$  is said to be compact, if  $\langle M, \rho \rangle$  is totally bounded and complete and complete metric space then we say it is compact metric space.

1:  $R'$  is not a compact metric space because it is not totally bounded, where it is a complete metric space.

2:  $[1,2]$  is any closed sub set in  $R'$ . we known  $R'$  is complete

Hence  $[1,2]$  is also a complete metric space

Again:  $[1,2]$  set is totally bounded hence it is compact.

Any open interval is not compact as it doesn't contain all its limit points i.e. open interval is not a complete metric space, hence it is a totally bounded set.

Theorem

The metric space  $\langle M, \rho \rangle$  is compact iff every sequence of points in  $M$ . as sub sequence converging a point in  $M$ .

Necessary part

Suppose that  $M$  is compact

We have to show that every sequence of point in  $M$

As sub sequence converges a point in  $M$ .

Let  $\{x_n\}_{n=1}^{\infty}$  be sequence of point in  $M$

Since  $M$  is totally bounded

$\{x_n\}_{n=1}^{\infty}$  has a cauchy sub sequence  $\{x_k\}_{k=1}^{\infty}$

Again as  $M$  is complete it is convergent to a point in  $M$

Hence every sequence of  $M$  is subsequence convergence to point in  $M$

Sufficient part:- Conversely every sequence in  $M$  is sub sequence converges a point in  $M$

We have to show that  $M$  is compact

Since every sequence of point in has subsequence  $M$  is totally bounded

Again it is given that every sub sequence is convergent in  $M$  i.e.

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$$\lim_{k \rightarrow \infty} x_{n_k} = x$$

$\Rightarrow \{x_n\}$  is convergent to same limit point  $x$

i.e.  $\lim_{x \rightarrow \infty} x_n = x$

$\therefore M$  is complete

So  $M$  is both totally bounded and complete

Hence  $M$  is bounded

**Heine-Bonell property-:** The metric space  $M$ , is said to have heine-bonell property, if whenever  $F$  is open covering of  $M$ , then there exist, finite number of sub sets  $G_1, G_2, \dots, G_n$  is still a covering of  $M$ .

**Theorem -:** If  $M$  is compact metrics then  $M$  as the Heine-Borel property

Proof -: Let us suppose the contradict, i.e. for some open covering no finite number sets in  $F$

From a covering of  $M$

Given  $M$  is totally bounded

$\Rightarrow M$  can be written as  $U$  finite number of bounded sub set each of which diameter less than 1.

One of these subset let it be  $A_1$ , it is covered by finite number set  $F$

Otherwise  $M$  could be covered by finite number of sets in  $F$

We know diameter of  $\overline{A_1} = \text{diameter } A_1$

$\overline{A_1}$  Closer is closed subset of  $M$

Whose diameter less than 1

Which can't be covered by finite number of sets in  $F$

Since it is totally bounded

By same assumption, there exist  $A_2$  of  $\overline{A_1}$  with diameter less than  $\frac{1}{2}$

$A_2$  Can't be covered by finite number sub sets in  $F$

Continuing in this way for  $n \in I$

The existence  $\overline{A_n} \subset M$

$$\Rightarrow \overline{A_1} \supset \overline{A_2} \supset \overline{A_3} \dots \dots \dots \overline{A_{n-1}} \supset \overline{A_{n+1}}$$

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Where  $\text{diam} \bar{A}_n < \frac{1}{2}$

$x \in \bigcap_{n=1}^{\infty} \bar{A}_n$  Contains precisely a one point

Let it be  $x$  i.e.  $x \in \bigcap_{n=1}^{\infty} \bar{A}_n$

Since  $f$  is covering of  $M$  there is a set  $c_1$  in  $f$

There is a set  $G$  in  $f$  such that  $G \in f$   
 $G$  is open  $C$

$\therefore F$  is open covers

$\exists r > 0$  such that  $B(x, r)$  is contained in  $G$

But  $L \in I$  satisfies  $\frac{1}{n} < r$  then diameter of  $\bar{A}_N < \frac{1}{N} < r$

Since  $x \in \bar{A}_N = \bigcap_{n=1}^{\infty} \bar{A}_N$

We have  $\bar{A}_N \subset B[x, r] \subset G$

Hence  $G$  alone covers  $n$  closure which is contradiction.

Hence our assumption is wrong

If  $m$  is compact, therefore it has finite no of subset which covers  $m$

i.e.  $m$  satisfies Heine-Bonell property

$\therefore M$  is covered  $M$  has Heine-Bonell property

**Theorem -:** if the metric space  $M$  has the Heine-Bonell property then  $M$  is compact.

Proof -: let  $M$  be the metric space with Heine-Bonell property

Let  $\{x_n\}_{n=1}^{\infty}$  be a sequence of point of  $M$ .

Let us suppose each point  $x$  in .

There were an open ball  $B_x$

For only finitely many values of  $n$

The family of open ball  $B_x$  which contained  $\{x_n\}$

For finitely many values of  $M$ .

The family of such open ball  $B_x$  be an open covering  $M$  is open covering of  $M$ .

By hypothesis  $M$  could be cover by finite number of these open ball  $B_x$

Since which is impossible.

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∴ Each  $B_x$  containing  $x_n$  for only finite number of values of  $M$

Hence there must be some points  $x$  in ,

Such that for every open ball about  $x$  contained  $\{x_n\}$  for infinitely many values of  $M$ .

$\exists n_1 \in I$  such that  $\{x_{n_1} \in B[x, 1]\}$

Similarly  $n_2 > n_1$  such that  $\{x_{n_2} \in B[x, \frac{1}{2}]\}$

In the singular  $n_k, n_{k-1} \in I$  such that  $x_{n_k} \in B[x, \frac{1}{k}]$

∴ The sub-sequence  $\{x_{n_k}\}_{n=1}^{\infty}$  is Cauchy sub-sequence  $\{x_n\}_{n=1}^{\infty}$  which converges to  $x$  in  $M$

Hence  $M$  is complete.

Definition -: let  $A$  be family of  $f$  a sub sets of these  $M$  is said to have infinite intersection property, if the intersection of any finite number set in  $F$  is never empty, I.e.  $F_1, F_2 \dots \dots F_n \in F$   $\cap_{n=1}^F F_n \neq \emptyset$  we can say  $n$  has finite intersection property

Theorem-: let  $F$  be continues function from the compact metric space  $M_1$  , into a metric space  $M_2$  , then the range of  $F(M_1)$  in  $M_2$  is also compact.

Proof- let  $G$  be any open covering of  $F(M_1)$  for each  $G \in G$  , the set  $F^{-1}G$  is also open subset of  $M$

The family of all  $(G) F^{-1}G$  is an open covering of  $M_1$

Hence every open covering  $M_1$  admits the finite sub-covering i.e. finite number of sub sets  $F^{-1}(G_1), F^{-1}(G_2) \dots \dots \dots F^{-1}(G_n)$  for a covering for  $M$ .

But  $G_1, G_2 \dots \dots \dots G_N$  from covering for  $(M_1)$  .

The family of open set  $G$  covering  $F(M_1)$  is finite,

Hence  $F(M_1)$  is compact.

### Continuous function on compact metric space

**Theorem** -: let  $f$  be a continuous function from the compact metric space  $M$  into the metric space  $M_2$ , then range of  $f(M_1)$  is also compact.

Proof-: Let  $G$  be any open covering of  $f(n)$  for each  $G$  the set  $f^{-1}(G)$  is an open subset of  $N$

The family of all such  $f(G)$  such for  $G \in G$  is an open covering of  $N$

Given  $N_1$  is compact

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Every open covering of  $N$  admits finite sub covers

$\therefore$  finite number of sets  $f^{-1}G, \dots, f^{-1}(G_n)$

From a covering of  $M$

But  $G_1 G_2 \dots G_n$  from a covering of  $f(m) \dots$  i.e.

The set  $G_1 G_2 \dots G_n$  of finite set covering  $f(m)$ .

Here,  $f(m_1)$  is compact.

**Continuity of the inverse function -:**

**Theorem -:** If  $F$  is a one-one continues function from the compact metric space  $M$ ,

Onto the metric space  $M_2$  then  $F'$  is continues on  $M_2$  and hence  $F$  is a homomorphism of  $m$  onto  $M_2$

Proof -: To show that  $F^{-1}$  is continues we must show that

If  $f$  is any closed subset of  $m$

Then the inverse image of  $f$  under  $f^{-1}$  is  $f(f)$

We have to show that if  $f$  is closed in  $H$

Then  $F(f)$  is closed in  $M_2$

$F$  itself is a compact metric space

The range  $F(f)$  is also a compact metric space.

Example  $f(x) = \sqrt{x}$

We will prove that  $f$  is continues on  $[0, \infty]$

For any  $N \in I$  the function  $f$  defined by  $I(x) = x^2$  is a one-one continues function from the compact space  $[0, N]$  into  $[0, N^2]$ .

Hence  $f^{-1}$  is continues

Definition -: let  $F$  be a function from set  $A$  into a metric space  $M$ .  $F$  is said to be bounded if its range  $F(A)$  is bounded subset of  $M$ .

Note -:  $f$  be a continuous function form the compact metric space  $M_2$  then the range of  $F(M_1)$  of is a bounded subset of  $M_2$

Note -: if the real valued function  $f$  is continuous in a closed bounded interval in  $R'$ , the  $f$  must be bounded.

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Definition -:  $f$  is real valued function on set  $A$ ,  $f$  attains a maximum value at  $a \in A$  if  $f(a) \geq f(x)$  ( $x \in A$ )

Example -:  $f(x) = x^2$  ( $-1 < x < 1$ ) then attains maximum at  $x = 1$ , and  $x = -1$

$g(x) = x^2$  ( $-\infty < x < \infty$ )  $g$  Clearly doesn't attains maximum value.

Note -: Real valued function on set  $A$  is not bounded above then  $f$  cannot attain a maximum at any point of  $A$

Theorem; - prove that, if  $A$  is a set of  $M$  in metric space  $\langle M, \rho \rangle$  then  $diam A = diam \bar{A}$

Proof - let  $r > 0$

Let  $x, y \in A$

$$\Rightarrow B\left(x, \frac{r}{2}\right) \cap A \neq \emptyset, B\left(y, \frac{r}{2}\right) \cap A \neq \emptyset$$

Choose  $x', y'$  such that  $x' \in B\left(x, \frac{r}{2}\right) \cap A, y' \in B\left(y, \frac{r}{2}\right) \cap A$

$$\begin{aligned} \text{Now } d(x, y) &\leq d(x, x') + d(x', y) \\ &\leq d(x, x') + d(y, y') + d(x', y) \\ &\leq \frac{r}{2} + \frac{r}{2} + diam(A) = r + diam(A) \end{aligned}$$

$$\Rightarrow diam(\bar{A}) < diam(A) + r$$

$$\text{i.e. } diam(\bar{A}) \leq diam(A) \dots \dots \dots (1)$$

but we know  $A \subset \bar{A}$

$$diam(A) < diam(\bar{A}) \dots \dots \dots (2)$$

From (1) and (2)

$$diam(A) = diam(\bar{A})$$

## Uniform continuity

**Definition** -: let  $\langle M_1, \rho_1 \rangle$  and  $\langle M_2, \rho_2 \rangle$  be metric spaces if  $F: M_1 \rightarrow M_2$  we say that  $f$  is uniformly continuous on  $M$ , if given  $\epsilon > 0 \exists \delta > 0$  such that  $\rho_2[f(x), f(a)] < \epsilon$ .  
theorem

Let  $\langle m, \rho_1 \rangle$  be compact metric space if  $f$  is a continuous function from  $m$ , into metric space  $\langle m_2, \rho_2 \rangle$  then  $f$  is uniformly continuous on  $M$ .

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Proof

$f$  is continuous at each  $a \in N$  given  $\epsilon > 0, \exists r > 0$  s.t

$$\rho_2[f(x), f(a)] < \frac{\epsilon}{2}$$

Let the family of open ball,  $R\left[a, \frac{r}{2}\right] \forall a \in M, G$  an open covering of  $M_1$ .

Here are finite no of such open ball i.e.  $B\left[a_1, \frac{r_1}{2}\right] \dots \dots B$  which form a covering of  $M$ .

Now let  $\delta = \min\left(\frac{r_1}{2}, \dots, \frac{r_n}{2}\right)$

For any  $a \in N$

We have for some  $j=1 \dots n$   $a \in B$

$$\text{So } \rho(a, a_j) < \frac{r}{2}$$

Now, if  $\rho_1(x, a) < \delta$ ,

$$\text{Then } \rho(x, a) < \frac{r_j}{2}$$

$$\Rightarrow \rho(x, a_j) < r_j$$

We have

$$\rho_2(f(x), f(a_j)) < \frac{\epsilon}{2}$$

And  $\rho_2(f(x), f(a)) < \epsilon$

$$\Rightarrow \rho(f(x), f(a)) < \epsilon \text{ for } \delta = \min\left(\frac{r_1}{2}, \dots, \frac{r_n}{2}\right)$$

We have  $\forall a \in M$

$$\rho_2(f(x), f(a)) < \epsilon \quad (\rho(x, a) < \delta)$$

$\therefore f$  is uniformly continuous







