

# Lecture-5

## 6.3.4 Directivity of a linear array

### (i) Directivity of a broadside array

The array factor of a broadside array can be written as

$$AF_n = \frac{1}{N} \left[ \frac{\sin\left(\frac{N}{2}kd \cos\theta\right)}{\sin\left(\frac{1}{2}kd \cos\theta\right)} \right] \quad (6.28)$$

For small spacing between the elements ( $d \ll \lambda$ ) the normalized array factor can be approximated as:

$$AF_n \approx \left[ \frac{\sin\left(\frac{N}{2}kd \cos\theta\right)}{\frac{N}{2}kd \cos\theta} \right] \quad (6.29)$$

Then the radiation intensity can be expressed as:

$$U(\theta) = |AF_n|^2 = \left[ \frac{\sin\left(\frac{N}{2}kd \cos\theta\right)}{\frac{N}{2}kd \cos\theta} \right]^2 = \left( \frac{\sin Z}{Z} \right)^2 \quad (6.30)$$

$$D_0 = 4\pi \frac{U_{\max}}{P_{\text{rad}}} = \frac{U_{\max}}{U_{\text{av}}} \quad (6.31)$$

where  $U_{\text{av}} = P_{\text{rad}} / (4\pi)$ .

The radiation intensity in the direction of maximum radiation,  $\theta = \pi/2$  in terms of  $AF_n$  is unity:

$$U_{\max} = U(\theta = \pi/2) = 1$$

$$\Rightarrow D_0 = U_{\text{av}}^{-1} \quad (6.32)$$

Radiation intensity averaged over all directions is calculated as

$$U_{\text{av}} = \frac{1}{4\pi} \int_0^{2\pi} \int_0^\pi \frac{\sin^2 Z}{Z^2} \sin\theta d\theta d\phi = \frac{1}{2} \int_0^\pi \left| \frac{\sin\left(\frac{N}{2}kd \cos\theta\right)}{\frac{N}{2}kd \cos\theta} \right|^2 \sin\theta d\theta$$

Changing the variables:

$$Z = \frac{N}{2}kd \cos\theta \Rightarrow dZ = -\frac{N}{2}kd \sin\theta d\theta \quad (6.33)$$

Then,

$$U_{av} = \frac{1}{2N} \frac{2}{kd} \int_{\frac{Nkd}{2}}^{\frac{Nkd}{2}} \left( \frac{\sin Z}{Z} \right)^2 dZ$$

$$U_{av} = \frac{1}{Nkd} \int_{\frac{Nkd}{2}}^{\frac{Nkd}{2}} \left( \frac{\sin Z}{Z} \right)^2 dZ \quad (6.34)$$

The function  $(Z^{-1} \sin Z)^2$  is a relatively fast decaying function as  $Z$  increases. For large array  $Nkd/2$  is too large and, (6.34) can be approximated by extending the limit to infinity.

$$U_{av} \approx \frac{1}{Nkd} \int_{-\infty}^{\infty} \left( \frac{\sin Z}{Z} \right)^2 dZ$$

Since  $\int_{-\infty}^{\infty} \left[ \frac{\sin(z)}{z} \right]^2 dz = \pi$

$$U_{av} \approx \frac{\pi}{Nkd} \quad (6.35)$$

$$D_0 = \frac{1}{U_{av}} \approx \frac{Nkd}{\pi} \approx 2N \left( \frac{d}{\lambda} \right) \quad (6.36)$$

Substituting the length of the array  $L = (N-1)d$  in (6.36)

$$D_0 = 2 \underbrace{\left( 1 + \frac{L}{d} \right)}_N \left( \frac{d}{\lambda} \right) \quad (6.37)$$

For large arrays ( $L \gg d$ ) the directivity can be reduced to

$$D_0 \approx 2L / \lambda \quad (6.38)$$

### (ii) Directivity of an ordinary endfire array

Consider an endfire array with maximum radiation at  $\theta = 0^\circ$ , i.e.  $\beta = -kd$ .

The radiation intensity of the endfire array can be expressed as

$$U(\theta) = |AF_n|^2 = \left\{ \frac{\sin \left[ \frac{N}{2} kd(\cos\theta - 1) \right]}{\left[ \frac{N}{2} kd(\cos\theta - 1) \right]} \right\}^2 = \left( \frac{\sin Z}{Z} \right)^2 \quad (6.39)$$

where  $Z = \frac{N}{2} kd(\cos\theta - 1)$ .

The maximum value of the radiation intensity is unity and it occurs at  $\theta = 0^\circ$ . The average value of the radiation intensity is given by:

$$U_{av} = \frac{P_{rad}}{4\pi} = \frac{1}{4\pi} \int_0^{2\pi} \int_0^\pi \left( \frac{\sin Z}{Z} \right)^2 \sin \theta d\theta d\phi = \frac{1}{2} \int_0^\pi \left( \frac{\sin Z}{Z} \right)^2 \sin \theta d\theta. \quad (6.40)$$

$$\text{As } Z = \frac{N}{2} kd (\cos \theta - 1) \quad \text{so} \quad dZ = -\frac{N}{2} kd \sin \theta d\theta$$

Then the average radiation intensity can be expressed as:

$$U_{av} = -\frac{1}{2} \frac{2}{Nkd} \int_0^{-Nkd/2} \left( \frac{\sin Z}{Z} \right)^2 dZ = \frac{1}{Nkd} \int_0^{Nkd/2} \left( \frac{\sin Z}{Z} \right)^2 dZ \quad (6.41)$$

For a large array  $Nkd$  is sufficiently large, then

$$U_{av} \approx \frac{1}{Nkd} \int_0^\infty \left( \frac{\sin Z}{Z} \right)^2 dZ = \frac{\pi}{2Nkd} \quad (6.42)$$

The directivity then have the value as:

$$D_0 \approx \frac{1}{U_{av}} = \frac{2Nkd}{\pi} = 4N \left( \frac{d}{\lambda} \right) \quad (6.43)$$

In terms of the array length  $L = (N-1)d$ , the directivity can be expressed as:

$$D_0 = 4 \left( 1 + \frac{L}{d} \right) \left( \frac{d}{\lambda} \right) \quad (6.44)$$

For a large array ( $L \gg \lambda$ ), the directivity reduces to:

$$D_0 \approx 4L / \lambda \quad (6.45)$$

It is observed that the directivity of an endfire array is approximately twice as large as the directivity of the broadside array.

### 6.3.5 Pattern characteristics of linear uniform arrays

#### A. Broadside array

#### NULLS

$$(AF_n = 0)$$

$$\theta_n = \cos^{-1}\left(\pm \frac{n \lambda}{N d}\right)$$

where  $n = 1, 2, 3, 4, \dots$  and  $n \neq N, 2N, 3N, \dots$

## MAXIMA

$$(AF_n = 1)$$

$$\theta_n = \cos^{-1}\left(\pm \frac{m \lambda}{d}\right)$$

where  $m = 0, 1, 2, 3, \dots$

## HALF-POWER POINTS:

$$\theta_h \approx \cos^{-1}\left(\pm \frac{1.391 \lambda}{\pi N d}\right)$$

where  $\frac{\pi d}{\lambda} \approx 1$

## HALF-POWER BEAMWIDTH:

$$\Delta \theta_h = 2 \left[ \frac{\pi}{2} - \cos^{-1}\left(\frac{1.391 \lambda}{\pi N d}\right) \right], \quad \frac{\pi d}{\lambda} \approx 1$$

## FIRST-NULL BEAMWIDTH

$$\Delta \theta_n = 2 \left[ \frac{\pi}{2} - \cos^{-1}\left(\frac{\lambda}{N d}\right) \right]$$

## B. Endfire array

### NULLS

$$(AF_n = 0)$$

$$\theta_n = \cos^{-1}\left(1 - \frac{n \lambda}{N D}\right)$$

where  $n = 1, 2, 3, \dots$  and  $n \neq N, 2N, 3N, \dots$

## MAXIMA

$$(AF_n = 1)$$

$$\theta_n = \cos^{-1}\left(1 - \frac{m\lambda}{d}\right),$$

where  $m = 0, 1, 2, 3, \dots$

## HALF-POWER POINTS:

$$\theta_h = \cos^{-1}\left(1 - \frac{1.391\lambda}{\pi Nd}\right)$$

where  $\frac{\pi d}{\lambda} \geq 1$

## HALF-POWER BEAMWIDTH:

$$\Delta\theta_h = 2\cos^{-1}\left(1 - \frac{1.391\lambda}{\pi Nd}\right)$$

where  $\frac{\pi d}{\lambda} \geq 1$

## FIRST-NULL BEAMWIDTH:

$$\Delta\theta_n = 2\cos^{-1}\left(1 - \frac{\lambda}{Nd}\right)$$