Lecture-6

6.3.6 N-Element Linear Array: Uniform Spacing, Nonuniform Amplitude

Uniform amplitude arrays produce small half-power beamwidth and possess the largest directivity. But in certain instances the sidelobe level of the radiation pattern has to be maintained at a desired level. The sidelobe level can be reduced by varying the amplitude excitations of the array elements. Nonuniform amplitude excitations of a linear antenna array produce a pattern with smaller sidelobe level and a slightly increased half power beamwidth in comparison to the uniform linear antenna array.

In this section, we will discuss arrays with uniform spacing but nonuniform amplitude distribution. Often, the broadside arrays are classified according to the type of their excitation amplitude. The categories are:

(a) Uniform amplitude array: Relatively high directivity, but the side-lobe levels are high;

(b) Dolph–Chebyscheff array: For a given number of elements, its maximum directivity is next to that of the uniform array. Side-lobe levels are the lowest in comparison with the other two types of arrays for a given directivity;

(c) Binomial array: Does not have good directivity but has very low side-lobe levels (when the element spacing is equal or less than $\lambda/2$, there are no side lobes at all).

Array Factor Analysis for Non-uniform Array

A simplification to the array factor can be made by symmetrically placing the array elements. Consider an array with an even number of isotropic elements ($2M$, where M is an integer) and suppose these are located symmetrically about the origin of the coordinate system (defined as the z-axis in this case, which is also the physical center of the array). The separation between the elements is d.

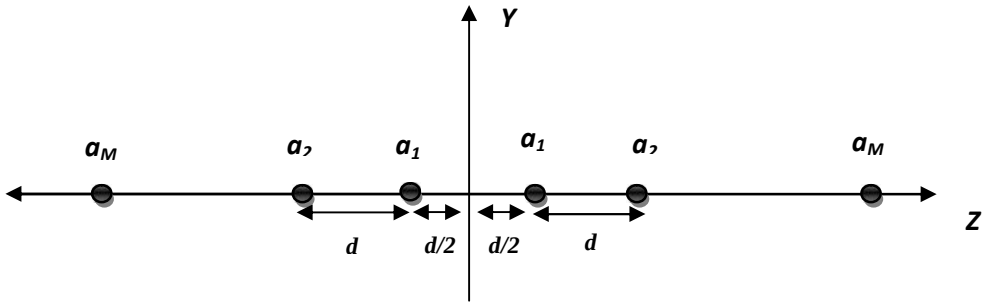


Fig.6.16 Nonuniform amplitude array of even number of elements.

Assuming that the amplitude excitation is symmetrical about the origin, the array factor for a nonuniform amplitude, broadside array can be written as:

$$(AF)_{2M} = a_1 e^{+j(\frac{kd\cos\bar{}}{2})} + a_2 e^{+j(\frac{3kd\cos\bar{}}{2})} + \cdots\cdots + a_M e^{+(\frac{(2M-1)kd\cos\bar{}}{2})} + a_1 e^{-j(\frac{kd\cos\bar{}}{2})} + a_2 e^{-j(\frac{3kd\cos\bar{}}{2})} + \cdots\cdots + a_M e^{-j(\frac{(2M-1)kd\cos\bar{}}{2})}$$

or $$(AF)_{2M} = 2\sum_{n=1}^{M} a_n \cos[\frac{(2n-1)}{2} kd\cos\bar{}]$$

In the normalized form, it reduces to

$$(AF)_{2M} = \sum_{n=1}^{M} a_n \cos[\frac{(2n-1)}{2} kd \cos^{-}] \tag{6.46}$$

where a_n' s are the excitation coefficients of the array element.

Similarly, for odd number of elements (*2M*+1), the array factor can be written as

$$(AF)_{2M+1} = \sum_{n=1}^{M+1} a_n \cos[(n-1)kd \cos^{-}] \tag{6.47}$$

The amplitude excitation of the center element is $2a_1$, in this case.

6.3.7 Binomial Array:

The binomial array was investigated and proposed by J. S. Stone to synthesize patterns without side lobes. Let us first consider a 2–element array with equal current amplitudes and spacing, the array factor is given by

$$AF = 1 + e^{j\cdot} \tag{6.48}$$

For a broadside array (= 0) with element spacing *d* less than one-half wavelength, the array factor has no sidelobes. This can be proved in the following way:

$$|AF|^2 = (1+\cos\cdot)^2 + \sin^2\cdot = 2(1+\cos\cdot) = 4\cos^2\left(\cdot/2\right)$$

where ψ = kd cosθ. The first null of this array factor can be obtained as:

$$\frac{1}{2}\cdot\frac{2\pi}{\lambda}\cdot d\cos^{-}_{n} = \pm\frac{\pi}{2} \quad \Rightarrow \quad {}^{-}_{n} = \pm\cos^{-1}\left(\frac{\lambda}{2d}\right)$$

As long as the $d < \lambda/2$, the first null does not exist. If d = λ/2, then null will be at $^{-} = 0^0$ and 180^0. Thus, in the "visible" range of $^{-}$, all secondary lobes are eliminated.

An array formed by taking the product of two arrays of this type gives:

$$AF = (1+e^{j\cdot})(1+e^{j\cdot}) = 1 + 2e^{j\cdot} + e^{j2\cdot} \tag{6.49}$$

This array factor, being the square of an array factor with no sidelobes, will also has no sidelobes. Mathematically, the array factor above represents a 3-element equally-spaced array driven by current amplitudes with ratios of 1:2:1. In a similar fashion, equivalent arrays with more elements may be formed.

2-element $AF = 1 + e^{j\cdot}$

3-element $AF = (1+e^{j\cdot})^2 = 1 + 2e^{j\cdot} + e^{j2\cdot}$

4-element $AF = \left(1 + e^{j\cdot}\right)^3 = 1 + 3e^{j\cdot} + 3e^{j2\cdot} + e^{j3\cdot}$

Similarly for N-element the array factor can be expressed as

N-element $AF = \left(1 + e^{j\cdot}\right)^{N-1}$ (6.50)

If $d \le \lambda/2$, the above AF does not have side lobes regardless of the number of elements N. The excitation amplitude distribution can be obtained easily by the expansion of the binome in (6.50). Making use of Pascal's triangle, this can be given by:

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

.....

The relative excitation amplitudes at each element of an (N+1) element array can be determined from this traiangle. An array with a binomial distribution of the excitation amplitudes is called a binomial array.

$$AF = \left(1 + e^{j\cdot}\right)^{N-1} = 1 + (N-1)e^{j\cdot} + \frac{(N-1)(N-2)}{2!}e^{j\cdot} + \frac{(N-1)(N-2)(N-3)}{3!}e^{j3\cdot} + \cdots \quad (6.51)$$

The excitation distribution as given by the binomial expansion gives the relative values of the amplitudes. It is immediately seen that there is too wide variation of the amplitude, which is a disadvantage of the binomial arrays. The overall efficiency of such an antenna would be low. Besides, the binomial array has a relatively wide beam. Its HPBW is the largest as compared to the uniform or the Dolph–Chebyshev array.

An approximate closed-form expression for the HPBW of a binomial array with $d = \lambda/2$ is

$$HPBW = \frac{1.06}{\sqrt{N-1}} = \frac{1.06}{\sqrt{2L/\lambda}} = \frac{1.75}{\sqrt{L/\lambda}} \quad (6.52)$$

where $L = (N\text{-}1)d$ is the array length.

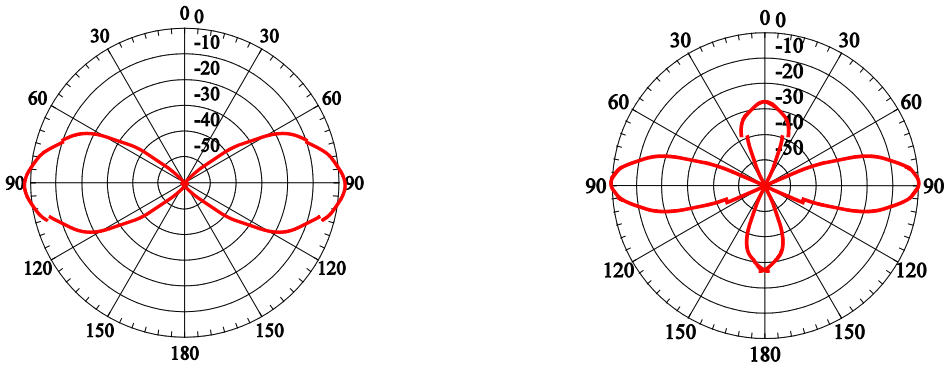
The directivity of a broadside binomial array with spacing $d = \lambda/2$ can be calculated as:

$$D_0 = \frac{2}{\int_0^{\pi} \left[\cos\left(\frac{\pi}{2}\cos\theta\right)\right]^{2(N-1)} d\theta}$$

$$D_0 = \frac{(2N-2)\cdot(2N-4)\cdot\cdots\cdot 2}{(2N-3)\cdot(2N-5)\cdot\cdots\cdot 1}$$

$$D_0 \approx 1.77\sqrt{N} = 1.77\sqrt{1+2L/a} \qquad (6.52)$$

The array factor of a 10 element broadside binomial array (N = 10) is shown below.



(a) $d = \lambda/2$ (b) $d = 3\lambda/4$

Fig.6.17 Radiation pattern for 10- element broadside binomial array

Lecture-7

6.3.8 Dolph–Chebyshev array:

Dolph proposed (in 1946) a method to design arrays with any desired sidelobe levels and any HPBWs. This method is based on the approximation of the pattern of the array by a Chebyshev polynomial of order *m*, high enough to meet the requirement for the side-lobe levels.

Chebyshev Polynomials: The Chebyshev polynomial of order *m* is defined by

$$T_m(z)=\begin{cases}(-1)^m \cosh\left(m\cdot\cos^{-1}(|z|)\right), & z\le -1\\ \cos\left(m\cdot\cos^{-1}(z)\right), & -1\le z\le 1,\\ \cosh\left(m\cdot\cos^{-1}(z)\right), & z\ge 1\end{cases} \tag{6.53}$$

A Chebyshev polynomial $T_m(z)$ of any order *m* can be derived from a recursion formula, provided $T_{m-1}(z)$ and $T_{m-2}(z)$ are known, in the following way:

$$T_m(z)=2zT_{m-1}(z)-T_{m-2}(z) \tag{6.54}$$

From the above equation we can express the Chebyshev polynomials as:

$$\begin{aligned}&m=0, T_0(z)=1\\ &m=1, T_1(z)=z\\ &m=2, T_2(z)=2z^2-1\\ &m=3, T_3(z)=4z^3-3z\\ &m=4, T_4(z)=8z^4-8z^2+1\\ &m=5, T_5(z)=16z^5-20z^3+5z\end{aligned} \tag{6.55}$$

The array factor expressed in (6.46) and (6.47) is the summation of cosine terms and the largest harmonics of the cosine terms is one less than the total number of elements in the array. Each cosine term having an argument which is the interger multiple of the fundamental frequency can be expressed as a series of cosine functions with the fundamental frequency as the argument. i.e.

$$\begin{aligned}&m=0 && \cos(mu)=1\\ &m=1 && \cos(mu)=\cos u\\ &m=2 && \cos(mu)=\cos 2u=2\cos 2u-1\\ &m=3 && \cos(mu)=\cos 3u=4\cos 3u-3\cos u\end{aligned} \tag{6.56}$$

If the $-1 \le z \le 1$, the Chebyshev polynomials are related to the cosine functions. So by comparing equation (6.55) and (6.56) we see that the Chebyshev argument *z* is related to cosine argument *u* by

$$z=\cos u \Rightarrow u=\cos^{-1} z$$

e.g. cos(*2u*) can be expressed as

$$\cos\left(2\cos^{-1} z\right)=2\left[\cos\left(\cos^{-1} z\right)\right]^2-1=2z^2-1=T_2(z)$$

Similarly:

$$\begin{aligned}
&m=0,\cos(mu)=T_0(z)=1\\
&m=1,\cos(mu)=\cos(u)=T_1(z)=z\\
&m=2,\cos(mu)=\cos(2u)=T_2(z)=2z^2-1\\
&m=3,\cos(mu)=\cos(3u)=T_3(z)=4z^3-3z\\
&m=4,\cos(mu)=\cos(4u)=T_4(z)=8z^4-8z^2+1\\
&m=5,\cos(mu)=\cos(5u)=T_5(z)=16z^5-20z^3+5z
\end{aligned} \tag{6.57}$$

Properties of the Chebyshev polynomials:

1) All polynomials of any order *m* pass through the point (1, 1).
2) Within the range -1 ≤ *z* ≤ 1, the polynomials have values within [–1, 1].
3) All nulls occur within -1 ≤ *z* ≤ 1.
4) The maxima and minima in the $z\in[-1,1]$ range have values +1 and –1, respectively.
5) The higher the order of the polynomial, the steeper the slope for $|z|>1$.

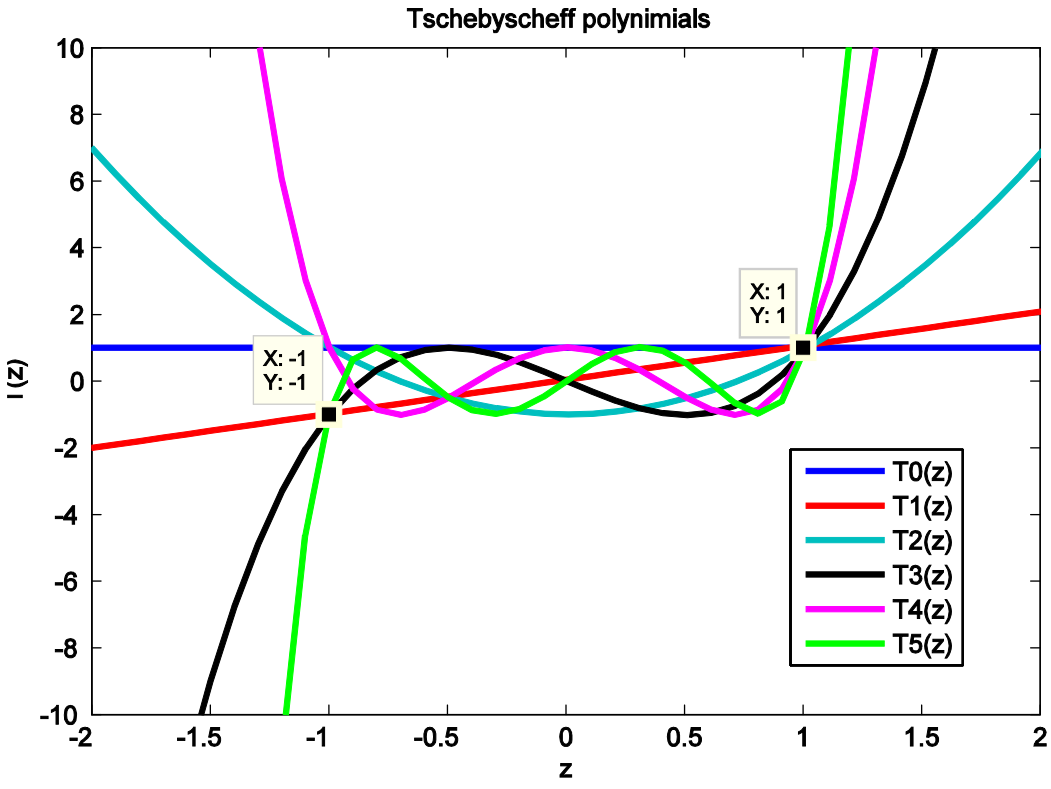


Fig. 6.18 Plot of Chebyshev polynomials

Lecture-8

6.3.9 Chebyshev Array Design

In this case, the main goal is to approximate the desired AF with a Chebyshev polynomial such that

- The side-lobe level meets the requirements, and
- The main beam width is as small as possible.

An array of N elements has an AF approximated with a Chebyshev polynomial of order *m*, which is always $m = N-1$.
where $N = 2M$, if N is even; and $N = 2M+1$, if N is odd.

Statement: Design a broadside Dolph-Tschebyscheff array of N elements with spacing d between the elements. The sidelobes are R_0 dB below the maximum of the major lobe. Find the excitation co-efficients and form the array factor.

Procedure:

1. Select the appropriate AF for the total number of elements N.

$$(AF)_{2M} = \sum_{n=1}^{M} a_n \cos[\frac{(2n-1)}{2} kd\cos\theta] \qquad N = 2M$$

$$(AF)_{2M+1} = \sum_{n=1}^{M+1} a_n \cos[(n-1)kd\cos\theta] \qquad N = 2M+1$$

2. Replace each cos(*mu*) term in the array factor by its appropriate expansion in terms of powers of cos(u).

3. Determine the point $z = z_0$ such that $T_m = R_0$ (Voltage ratio). Find z_0 such that

$$R_0 = T_{N-1}(z_0) = \cosh\left[(N-1)\cosh^{-1} z_0\right]$$

$$z_0 = \cosh\left[\frac{\cosh^{-1} R_0}{N-1}\right]$$

4. Substitute $\cos(u) = \frac{z}{z_0}$ in the array factor of step 2. This substitution normalizes the array factor sidelobes to a peak value of unity.

5. Equate the array factor of step 4 to $T_{N-1}(z)$ and determine the array coefficients.

Example-6: Design a Dolph–Chebyshev array ***(broadside) of N = 10 elements with a major-to-minor lobe ration of R_0=26dB. Find the excitation coefficients.***

Solution:

The order of the Chebyshev polynomial is *m* = N-1 = 9. The AF for an even-number array is:

$$AF_{2M} = \sum_{n=1}^{5} a_n \cos\left[(2n-1)u\right]$$

$$u = \frac{\pi d}{\lambda}\cos\theta, \quad M = 5.$$

Step 1: The array factor for 10 element array :

$$AF_{10} = a_1 \cos u + a_2 \cos 3u + a_3 \cos 5u + a_4 \cos 7u + a_5 \cos 9u$$

Expand the $\cos(mu)$ terms of powers of $\cos u$:

$$\cos 3u = 4\cos^3 u - 3\cos u,$$
$$\cos 5u = 16\cos^5 u - 20\cos^3 u + 5\cos u,$$
$$\cos 7u = 64\cos^7 u - 112\cos^5 u + 56\cos^3 u - 7\cos u,$$
$$\cos 9u = 256\cos^9 u - 576\cos^7 u + 432\cos^5 u - 120\cos^3 u + 9\cos u$$

Step 2: Determine z_0 :

$$R_0 = 26 \text{ dB}$$
$$\Rightarrow R_0 = 10^{26/20} \approx 20$$
$$\Rightarrow T_9(z_0) = 20$$
$$\cosh\left[9\cosh^{-1}(z_0)\right] = 20,$$
$$z_0 = \cosh 0.41 \quad \Rightarrow z_0 = 1.08515$$

Step 3: Express the AF from Step 1 in terms of $\cos u = z / z_0$

$$AF_{10} = \frac{z}{z_0}\left(a_1 - 3a_2 + 5a_3 - 7a_4 + 9a_5\right)$$
$$+ \frac{z^3}{z_0^3}\left(4a_2 - 20a_3 + 56a_4 - 120a_5\right)$$
$$+ \frac{z^5}{z_0^5}\left(16a_3 - 112a_4 + 432a_5\right)$$
$$+ \frac{z^7}{z_0^7}\left(64a_4 - 576a_5\right)$$
$$+ \frac{z^9}{z_0^9}\left(256a_5\right) =$$
$$= \underbrace{9z - 120z^3 + 432z^5 - 576z^7 + 256z^9}_{T_9(z)}$$

Step 4: Finding the coefficients by matching the power terms:

$256a_5 = 256z_0^9 \Rightarrow a_5 = 2.0860$

$64a_4 - 576a_5 = -576z_0^7 \Rightarrow a_4 = 2.8308$

$16a_3 - 112a_4 + 432a_5 = 432z_0^5 \Rightarrow a_3 = 4.1184$

$4a_2 - 20a_3 + 56a_4 - 120a_5 = -120z_0^7 \Rightarrow a_2 = 5.2073$

$a_1 - 3a_2 + 5a_3 - 7a_4 + 9a_5 = 9z_0^9 \Rightarrow a_1 = 5.8377$

Normalize coefficients with respect to edge element $(N = 5)$:

$a_5 = 1$; $a_4 = 1.357$; $a_3 = 1.974$; $a_2 = 2.496$; $a_1 = 2.789$

Lecture-9

6.4 Planar array

A planar array provides a large aperture and may be used for directional beam controlled by varying the relative phase of each element, symmetrical patterns with low side lobes, much higher directivity (narrow main beam) than that of their individual.

Array Factor

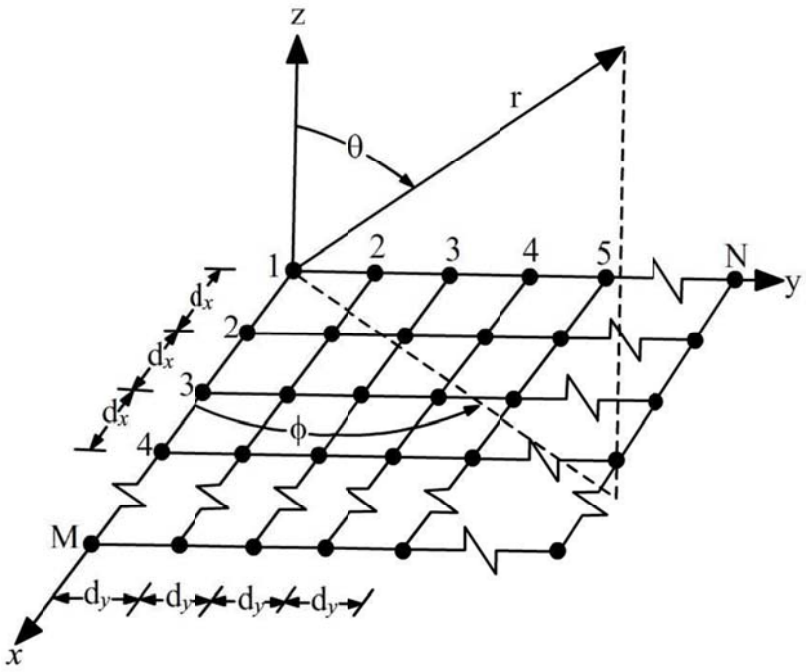


Fig.6.19 Planar array

If *M* elements are initially placed along x-axis, the array factor can be expressed as

$$AF_{x1} = \sum_{m=1}^{M} I_{m1} e^{j(m-1)(kd_x \sin\theta\cos\phi + \beta_x)} \tag{6.58}$$

It is assumed that all elements are equispaced with an interval of d_x and a progressive shift β_x. I_{m1} denotes the excitation coefficient of the element at the point with coordinates $x=(m-1)d_x, y=0.$, where $\sin\theta\cos\phi = \cos\gamma_x$ is the directional cosine with respect to the x-axis (γ_x is the angle between **r** and the *x*-axis). In the figure above, this is the element of the m^{th} row and the 1^{st} column of the array matrix. If *N* such arrays are placed at even intervals along the y-direction, a rectangular array is formed. We assume again that they are equispaced at a distance d_y and there is a progressive phase shift β_y along each row.

We also assume that the normalized current distribution along each of the x-directed arrays is the same but the absolute values correspond to a factor of $I_{1n}(n=1,...,N)$. Then, the AF of the entire $M\times N$ array is

$$AF = \sum_{n=1}^{N} I_{1n}\left[\sum_{m=1}^{M} I_{m1} e^{j(m-1)\left(kd_x \sin\theta\cos\phi+\beta_x\right)}\right]\cdot e^{j(n-1)\left(kd_y \sin\theta\sin\phi+\beta_y\right)} \tag{6.59}$$

or

$$AF = S_{x_M}\cdot S_{y_N}$$

$$\begin{aligned} S_{x_M} &= AF_{x1} = \sum_{m=1}^{M} I_{m1} e^{j(m-1)\left(kd_x \sin\theta\cos\phi+\beta_x\right)} \\ S_{y_N} &= AF_{1y} = \sum_{n=1}^{N} I_{1n} e^{j(n-1)\left(kd_y \sin\theta\cos\phi+\beta_y\right)} \end{aligned} \tag{6.60}$$

In the array factors above

$$\begin{aligned} \sin\theta\cos\phi &= \hat{\mathrm{x}}\cdot\hat{\mathrm{r}} = \cos\gamma_x \\ \sin\theta\cos\phi &= \hat{\mathrm{y}}\cdot\hat{\mathrm{r}} = \cos\gamma_y \end{aligned} \tag{6.61}$$

Thus, the pattern of a rectangular array is the product of the array factors of the linear arrays in the *x* and *y*-directions.

In the case of a uniform planar rectangular array, $I_{m1} = I_{1n} = I_0$ for all *m* and *n*, i.e., all elements have the same excitation amplitudes. Thus,

$$AF = I_0 \sum_{m=1}^{M} e^{j(m-1)\left(kd_x \sin\theta\cos\phi+\beta_x\right)} \times \sum_{n=1}^{N} e^{j(n-1)\left(kd_y \sin\theta\sin\phi+\beta_y\right)} \tag{6.62}$$

The normalized array factor is obtained as

$$\boxed{AF_n(\theta,\phi) = \left[\frac{1}{M}\frac{\sin\left(M\frac{\psi_x}{2}\right)}{\sin\left(\frac{\psi_x}{2}\right)}\right]\cdot\left[\frac{1}{N}\frac{\sin\left(N\frac{\psi_y}{2}\right)}{\sin\left(\frac{\psi_y}{2}\right)}\right]} \tag{6.63}$$

where

$$\begin{aligned} \psi_x &= kd_x \sin\theta\cos\phi + \beta_x \\ \psi_y &= kd_y \sin\theta\cos\phi + \beta_y \end{aligned}$$

The major lobe (principal maximum) and grating lobes of the terms

$$S_{x_M} = \frac{1}{M}\frac{\sin\left(M\frac{\psi_x}{2}\right)}{\sin\left(\frac{\psi_x}{2}\right)} \tag{6.64}$$

$$S_{y_N} = \frac{1}{N} \frac{\sin\left(N\frac{\psi_y}{2}\right)}{\sin\left(\frac{\psi_y}{2}\right)} \tag{6.65}$$

are located at angles such that

$$\begin{aligned} kd_x \sin\theta_m \cos\phi_m + \beta_x &= \pm 2m\pi, \qquad m = 0,1,..., \\ kd_y \sin\theta_n \sin\phi_n + \beta_y &= \pm 2n\pi, \qquad n = 0,1,..., \end{aligned} \tag{6.66}$$

The principal maximum corresponds to $m = 0$, $n = 0$.

In general, β_x and β_y are independent of each other. But, if it is required that the main beams of S_{x_M} and S_{y_N} intersect (which is usually the case), then the common main beam is in the direction: $\theta = \theta_0$ *and* $\phi = \phi_0$, $m = n = 0$.

If the principal maximum is specified by (θ_0, ϕ_0), then the progressive phases β_x and β_y must satisfy

$$\beta_x = -kd_x \sin\theta_0 \cos\phi_0 \tag{6.67}$$

$$\beta_y = -kd_y \sin\theta_0 \sin\phi_0. \tag{6.68}$$

When β_x and β_y are specified, the direction of the main beam can be found by simultaneously solving (6.67) and (6.68)

$$\tan\phi_0 = \frac{\beta_y d_x}{\beta_x d_y} \tag{6.69}$$

$$\sin\theta_0 = \pm\sqrt{\left(\frac{\beta_x}{kd_x}\right)^2 + \left(\frac{\beta_y}{kd_y}\right)^2} \tag{6.70}$$

The grating lobes can be located by substituting (6.67) and (6.68) in (6.66)

$$\tan\phi_{mn} = \frac{\sin\theta_0 \sin\phi_0 \pm n\lambda/d_y}{\sin\theta_0 \sin\phi_0 \pm m\lambda/d_x} \tag{6.71}$$

$$\sin\theta_{mn} = \frac{\sin\theta_0 \cos\phi_0 \pm m\lambda/d_x}{\cos\phi_{mn}} = \frac{\sin\theta_0 \sin\phi_0 \pm n\lambda/d_y}{\sin\phi_{mn}} \tag{6.72}$$

To avoid grating lobes, the spacing between the elements must be less than $\lambda\left(d_y < \lambda \text{ and } d_y < \lambda\right)$. In order a true grating lobe to occur, both equations (6.71) and (6.72) must have a real solution $(\theta_{mn}, \theta_{mn})$.