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Integral Transforms

Fourier Integral

If $f(x)$ and $f'(x)$ are piecewise continuous in every finite interval, and $f(x)$ is absolutely integrable on R, i.e.

converges, then

$$\frac{1}{2}[f(x-)+f(x+)]=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{iwx}\left(\int_{-\infty}^{\infty}e^{-iwt}f(t)dt\right)dw$$

Remark: the above conditions are sufficient, but not necessary.

$$\mathcal{F}_c = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos kx \, dx = F_c(k)$$

$$\mathcal{F}_c^{-1} = \sqrt{\frac{2}{\pi}} \int_0^\infty F_c(k) \cos ktx \, dk = f(x)$$

} Fourier cosine transform

Fourier Sine Transform:

$$\mathcal{F}_s = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin kx \, dx = F_s(k)$$

$$\mathcal{F}_s^{-1} = \sqrt{\frac{2}{\pi}} \int_0^\infty F_s(k) \sin kx \, dk = f(x)$$

} Fourier sine transform.

► *Find the Fourier transform of the normalised Gaussian distribution*

$$f(t) = \frac{1}{\tau\sqrt{2\pi}} \exp\left(-\frac{t^2}{2\tau^2}\right), \qquad -\infty < t < \infty.$$

This Gaussian distribution is centred on $t = 0$ and has a root mean square deviation $\Delta t = \tau$. (Any reader who is unfamiliar with this interpretation of the distribution should refer to chapter 30.)

Using the definition (13.5), the Fourier transform of $f(t)$ is given by

$$\begin{aligned}\tilde{f}(\omega) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\tau\sqrt{2\pi}} \exp\left(-\frac{t^2}{2\tau^2}\right) \exp(-i\omega t)\, dt \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\tau\sqrt{2\pi}} \exp\left\{-\frac{1}{2\tau^2}\left[t^2 + 2\tau^2 i\omega t + (\tau^2 i\omega)^2 - (\tau^2 i\omega)^2\right]\right\} dt,\end{aligned}$$

where the quantity $-(\tau^2 i\omega)^2/(2\tau^2)$ has been both added and subtracted in the exponent in order to allow the factors involving the variable of integration t to be expressed as a complete square. Hence the expression can be written

$$\tilde{f}(\omega) = \frac{\exp(-\frac{1}{2}\tau^2\omega^2)}{\sqrt{2\pi}} \left\{\frac{1}{\tau\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left[-\frac{(t + i\tau^2\omega)^2}{2\tau^2}\right] dt\right\}.$$

The quantity inside the braces is the normalisation integral for the Gaussian and equals unity, although to show this strictly needs results from complex variable theory (chapter 24). That it is equal to unity can be made plausible by changing the variable to $s = t + i\tau^2\omega$ and assuming that the imaginary parts introduced into the integration path and limits (where the integrand goes rapidly to zero anyway) make no difference.

We are left with the result that

$$\tilde{f}(\omega) = \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-\tau^2\omega^2}{2}\right), \tag{13.7}$$

which is another Gaussian distribution, centred on zero and with a root mean square deviation $\Delta\omega = 1/\tau$. It is interesting to note, and an important property, that the Fourier transform of a Gaussian is another Gaussian. ◄

In the above example the root mean square deviation in t was τ, and so it is seen that the deviations or 'spreads' in t and in ω are inversely related:

$$\Delta\omega\, \Delta t = 1,$$

independently of the value of τ. In physical terms, the narrower in time is, say, an electrical impulse the greater the spread of frequency components it must contain. Similar physical statements are valid for other pairs of Fourier-related variables, such as spatial position and wave number. In an obvious notation, $\Delta k \Delta x = 1$ for a Gaussian wave packet.

The uncertainty relations as usually expressed in quantum mechanics can be related to this if the de Broglie and Einstein relationships for momentum and energy are introduced; they are

$$p = \hbar k \qquad \text{and} \qquad E = \hbar\omega.$$

13.1.3 The Dirac δ-function

Before going on to consider further properties of Fourier transforms we make a digression to discuss the Dirac δ-function and its relation to Fourier transforms. The δ-function is different from most functions encountered in the physical sciences but we will see that a rigorous mathematical definition exists; the utility of the δ-function will be demonstrated throughout the remainder of this chapter. It can be visualised as a very sharp narrow pulse (in space, time, density, etc.) which produces an integrated effect having a definite magnitude. The formal properties of the δ-function may be summarised as follows.

The Dirac δ-function has the property that

$$\delta(t) = 0 \quad \text{for } t \neq 0, \tag{13.11}$$

but its fundamental defining property is

$$\int f(t)\delta(t-a)\,dt = f(a), \tag{13.12}$$

provided the range of integration includes the point $t = a$; otherwise the integral

equals zero. This leads immediately to two further useful results:

$$\int_{-a}^{b} \delta(t)\,dt = 1 \quad \text{for all } a, b > 0 \tag{13.13}$$

and

$$\int \delta(t-a)\,dt = 1, \tag{13.14}$$

provided the range of integration includes $t = a$.

Equation (13.12) can be used to derive further useful properties of the Dirac δ-function:

$$\delta(t) = \delta(-t), \tag{13.15}$$

$$\delta(at) = \frac{1}{|a|}\delta(t), \tag{13.16}$$

$$t\delta(t) = 0. \tag{13.17}$$

▶ *Prove that* $\delta(bt) = \delta(t)/|b|$.

Let us first consider the case where $b > 0$. It follows that

$$\int_{-\infty}^{\infty} f(t)\delta(bt)\,dt = \int_{-\infty}^{\infty} f\left(\frac{t'}{b}\right)\delta(t')\frac{dt'}{b} = \frac{1}{b}f(0) = \frac{1}{b}\int_{-\infty}^{\infty} f(t)\delta(t)\,dt,$$

where we have made the substitution $t' = bt$. But $f(t)$ is arbitrary and so we immediately see that $\delta(bt) = \delta(t)/b = \delta(t)/|b|$ for $b > 0$.

Now consider the case where $b = -c < 0$. It follows that

$$\int_{-\infty}^{\infty} f(t)\delta(bt)\,dt = \int_{\infty}^{-\infty} f\left(\frac{t'}{-c}\right)\delta(t')\left(\frac{dt'}{-c}\right) = \int_{-\infty}^{\infty}\frac{1}{c}f\left(\frac{t'}{-c}\right)\delta(t')\,dt'$$

$$= \frac{1}{c}f(0) = \frac{1}{|b|}f(0) = \frac{1}{|b|}\int_{-\infty}^{\infty} f(t)\delta(t)\,dt,$$

where we have made the substitution $t' = bt = -ct$. But $f(t)$ is arbitrary and so

$$\delta(bt) = \frac{1}{|b|}\delta(t),$$

for all b, which establishes the result. ◀

(i) Differentiation:

$$\mathcal{F}[f'(t)] = i\omega\tilde{f}(\omega). \tag{13.28}$$

This may be extended to higher derivatives, so that

$$\mathcal{F}[f''(t)] = i\omega\mathcal{F}[f'(t)] = -\omega^2\tilde{f}(\omega),$$

and so on.

(ii) Integration:

$$\mathcal{F}\left[\int^t f(s)\,ds\right] = \frac{1}{i\omega}\tilde{f}(\omega) + 2\pi c\delta(\omega), \tag{13.29}$$

where the term $2\pi c\delta(\omega)$ represents the Fourier transform of the constant of integration associated with the indefinite integral.

(iii) Scaling:

$$\mathcal{F}[f(at)] = \frac{1}{a}\tilde{f}\left(\frac{\omega}{a}\right). \tag{13.30}$$

(iv) Translation:

$$\mathcal{F}[f(t+a)] = e^{ia\omega}\tilde{f}(\omega). \tag{13.31}$$

(v) Exponential multiplication:

$$\mathcal{F}\left[e^{\alpha t}f(t)\right] = \tilde{f}(\omega + i\alpha), \tag{13.32}$$

where α may be real, imaginary or complex.

▶ *Prove relation (13.28).*

Calculating the Fourier transform of $f'(t)$ directly, we obtain

$$\begin{aligned}\mathcal{F}[f'(t)] &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f'(t)\,e^{-i\omega t}\,dt \\ &= \frac{1}{\sqrt{2\pi}}\left[e^{-i\omega t}f(t)\right]_{-\infty}^{\infty} + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} i\omega\, e^{-i\omega t}f(t)\,dt \\ &= i\omega\tilde{f}(\omega),\end{aligned}$$

if $f(t) \to 0$ at $t = \pm\infty$, as it must since $\int_{-\infty}^{\infty} |f(t)|\,dt$ is finite. ◀

Example 15.4.2 HEAT FLOW PDE

To illustrate another transformation of a PDE into an ODE, let us Fourier transform the heat flow partial differential equation

$$\frac{\partial \psi}{\partial t} = a^2 \frac{\partial^2 \psi}{\partial x^2},$$

where the solution $\psi(x,t)$ is the temperature in space as a function of time. By taking the Fourier transform of both sides of this equation (note that here only ω is the transform variable conjugate to x because t is the time in the heat flow PDE), where

$$\Psi(\omega,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi(x,t) e^{i\omega x}\, dx,$$

this yields an ODE for the Fourier transform Ψ of ψ in the time variable t,

$$\frac{\partial \Psi(\omega,t)}{\partial t} = -a^2\omega^2 \Psi(\omega,t).$$

Integrating we obtain

$$\ln \Psi = -a^2\omega^2 t + \ln C, \qquad \text{or} \qquad \Psi = Ce^{-a^2\omega^2 t},$$

where the integration constant C may still depend on ω and, in general, is determined by initial conditions. In fact, $C = \Psi(\omega,0)$ is the initial spatial distribution of Ψ, so it is given by the transform (in x) of the initial distribution of ψ, namely, $\psi(x,0)$. Putting this solution back into our inverse Fourier transform, this yields

$$\psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} C(\omega) e^{-i\omega x} e^{-a^2\omega^2 t}\, d\omega.$$

For simplicity, we here take C ω-independent (assuming a delta-function initial temperature distribution) and integrate by completing the square in ω, as in Example 15.1.1,

making appropriate changes of variables and parameters ($a^2 \to a^2 t,\ \omega \to x, t \to -\omega$). This yields the particular solution of the heat flow PDE,

$$\psi(x,t) = \frac{C}{a\sqrt{2t}} \exp\left(-\frac{x^2}{4a^2 t}\right),$$

which appears as a clever guess in Chapter 8. In effect, we have shown that ψ is the inverse Fourier transform of $C \exp(-a^2\omega^2 t)$. ■

Example 15.4.3 INVERSION OF PDE

15.5 CONVOLUTION THEOREM

We shall employ convolutions to solve differential equations, to normalize momentum wave functions (Section 15.6), and to investigate transfer functions (Section 15.7).

Let us consider two functions $f(x)$ and $g(x)$ with Fourier transforms $F(t)$ and $G(t)$, respectively. We define the operation

$$f * g \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(y) f(x-y)\, dy \tag{15.52}$$

as the **convolution** of the two functions f and g over the interval $(-\infty, \infty)$. This form of an integral appears in probability theory in the determination of the probability density of two random, independent variables. Our solution of Poisson's equation, Eq. (9.148), may be interpreted as a convolution of a charge distribution, $\rho(\mathbf{r}_2)$, and a weighting function, $(4\pi\varepsilon_0|\mathbf{r}_1 - \mathbf{r}_2|)^{-1}$. In other works this is sometimes referred to as the **Faltung**, to use the

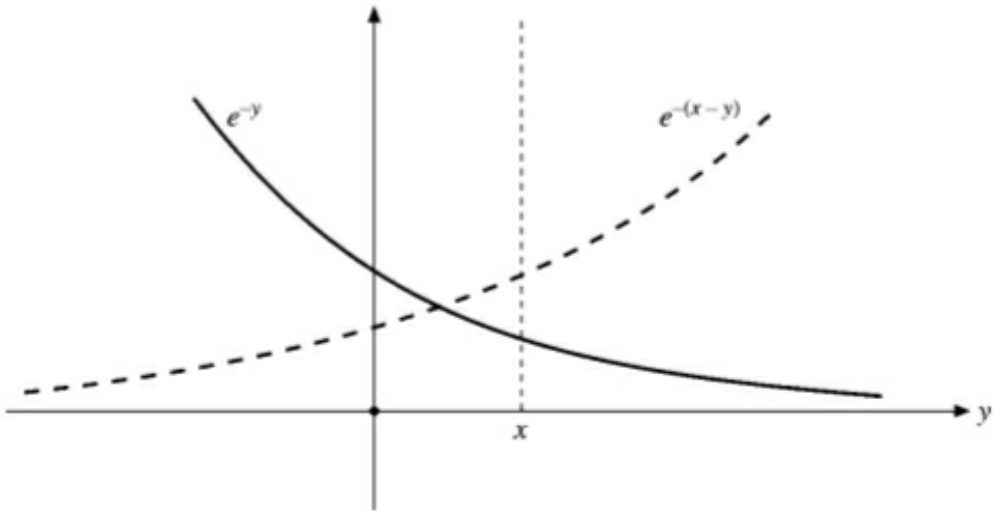

FIGURE 15.5

German term for "folding."[5] We now transform the integral in Eq. (15.52) by introducing the Fourier transforms:

$$\begin{aligned}\int_{-\infty}^{\infty} g(y)f(x-y)dy &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(y)\int_{-\infty}^{\infty} F(t)e^{-it(x-y)}\,dt\,dy \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(t)\left[\int_{-\infty}^{\infty} g(y)e^{ity}dy\right]e^{-itx}\,dt \\ &= \int_{-\infty}^{\infty} F(t)G(t)e^{-itx}\,dt, \qquad (15.53)\end{aligned}$$

interchanging the order of integration and transforming $g(y)$. This result may be interpreted as follows: The Fourier inverse transform of a **product** of Fourier transforms is the convolution of the original functions, $f * g$.

For the special case $x = 0$ we have

$$\int_{-\infty}^{\infty} F(t)G(t)\,dt = \int_{-\infty}^{\infty} f(-y)g(y)\,dy. \qquad (15.54)$$

The minus sign in $-y$ suggests that modifications be tried. We now do this with g^* instead of g using a different technique.

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Thank you!
All the best for Exams !