

Content: Physics of Fluid

Fluid = Liquid + Gas

→ Density  $\rho$   $\rho = \begin{cases} \text{constant} \rightarrow \text{incompressible} \\ \text{variable} \rightarrow \text{Compressible} \\ \text{variable density flows} \end{cases}$

→ Viscosity  $\mu$  → resistance to flow of fluid

$$\mu = \left( \frac{Ns}{m^2} \right) = \text{poise}$$

→ Transport property like density, viscosity, thermal conductivity

Conservation law



$$\dot{m}_{in} = \dot{m}_{out} \quad \frac{dM}{dt} = \dot{m}_{in} - \dot{m}_{out}$$
$$\frac{dM}{dt} = 0$$

Let us

Navier - Stokes Equation 1

Mass conservation → ~~Cont~~ continuity equation

Material derivative

$$\left[ \frac{D\rho}{Dt} + \rho \frac{\partial U_i}{\partial x_i} \right] \rightarrow \text{compressible}$$

$$\downarrow \rho = \text{const}, \frac{D\rho}{Dt} = 0$$
$$\rho \frac{\partial U_i}{\partial x_i} = 0 \rightarrow \text{incompressible}$$

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# Navier-Stokes Equation - II

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Momentum conservation  $\rightarrow$  Momentum equation

$$\rho \frac{\partial u_j}{\partial t} + \rho u_i \frac{\partial u_j}{\partial x_i} = -\frac{\partial p}{\partial x_j} - \frac{\partial \tau_{ij}}{\partial x_i}$$

I
II
III
IV

$$\tau_{ij} = -\mu \left( \frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j} \right) + \frac{2}{3} \delta_{ij} \mu \frac{\partial u_k}{\partial x_k}$$

- 1) Local change with time or with the velocity with time
- 2) Momentum convection  $\rightarrow$  is the convection of the velocity
- 3) surface force or pressure gradient.
- 4) Molecular - dependent momentum exchange (diffusion)
- 5) Mass force or force term due to gravity

## Navier-Stokes equation - III

Momentum equation for incompressible fluid

$$\frac{\partial \tau_{ij}}{\partial x_i} = -\mu \frac{\partial}{\partial x_i} \left( \frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j} \right) + \frac{2}{3} \delta_{ij} \mu \frac{\partial u_k}{\partial x_k}$$

for mass conservation

$$\text{law } \frac{\partial u_i}{\partial x_i} = 0 \quad \frac{\partial \tau_{ij}}{\partial x_i} = -\mu \frac{\partial^2 u_j}{\partial x_i^2}$$

$$-\mu \frac{\partial}{\partial x_i} \frac{\partial u_i}{\partial x_i} = -\mu \frac{\partial^2 u_j}{\partial x_i^2}$$

Energy conservation <sup>(3)</sup> → Energy equation  
(temp)

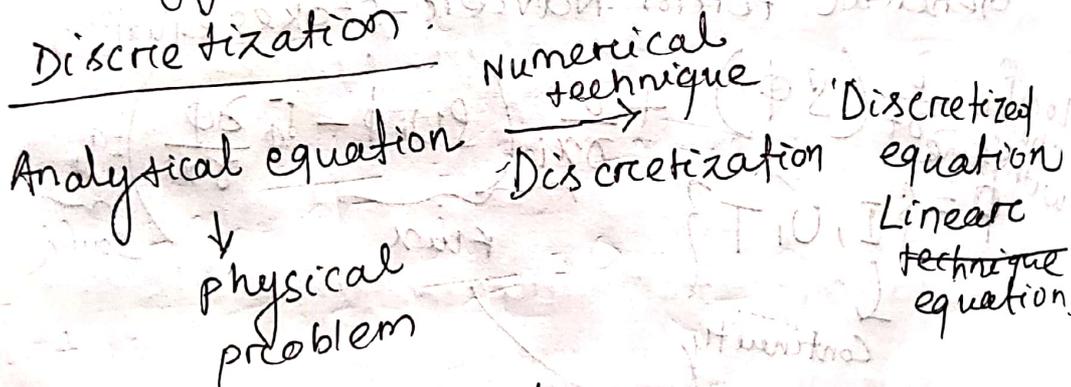
$$\rho c_m \frac{\partial T}{\partial t} + \rho c_m u_i \frac{\partial T}{\partial x_i} = -p \frac{\partial u_i}{\partial x_i} + \frac{\partial^2 T}{\partial x_i^2} - \tau_{ij} \frac{\partial u_j}{\partial x_i}$$

I
II
III

IV
V

- i) Local energy change with time
- ii) convective term
- iii) pressure work
- iv) Heat flux (diffusion)
- v) Irreversible transfer of mechanical energy into heat

Discretization



Discretization Method

- Finite Difference → Applied to simple, structured grids
- Finite Element → Any geometries
- Finite volume → Conservation, any geometries

Finite volume Method (FVM) → Domain is divided into a number of non-overlapping control volumes.

→ The differential equation is integrated over each control volume

→ piecewise profiles expressing the variation of the unknown between the grid points are used to evaluate the required integral.

Advantage → It is physical soundness.

Disadvantage → It is applicable to any geometry & it is acceptable to solve fluid flow problem.

General form Navier-Stokes equation

$$\underbrace{\frac{\partial}{\partial t} (\rho \phi)}_{\substack{\text{Local change} \\ \text{with time}}} + \underbrace{\frac{\partial}{\partial x_i} (\rho u_i \phi - \Gamma \frac{\partial \phi}{\partial x_i})}_{\text{flux}} = \underbrace{q_\phi}_{\text{source}}$$

$\phi = \{I, U, T\}$   
↓  
Continuity

Integral form of Navier-Stokes equation

Governing equation.

$$\int_V \frac{\partial}{\partial t} (\rho \phi) dV + \int_S (\rho u_i \phi - \Gamma \frac{\partial \phi}{\partial x_i}) n_i ds$$

$$= \int_V q_\phi dV$$

Source in CV.

flux over the CV surface

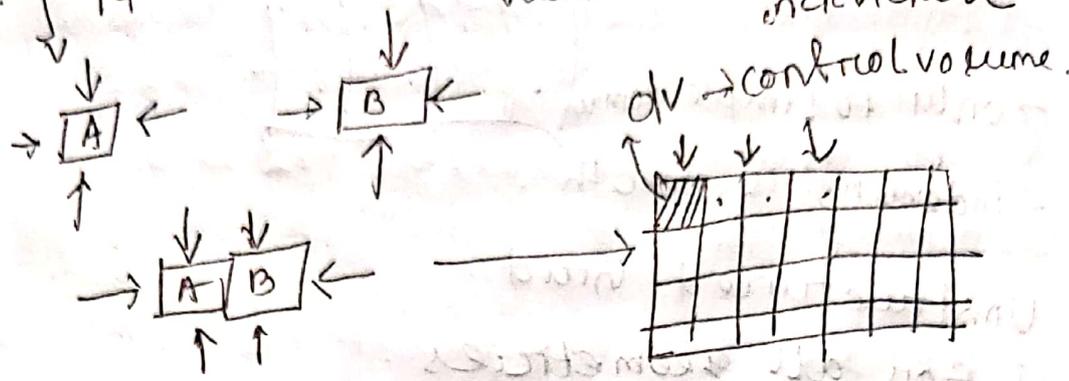
Local change with time CV

where CV = control volume

Conservation of Finite volume method (S)

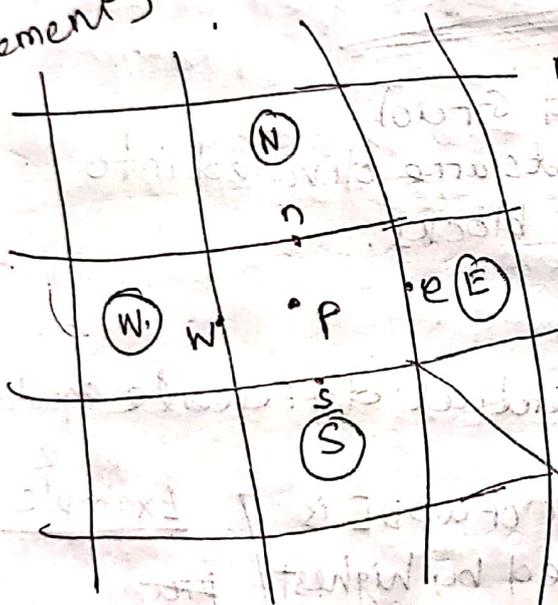
$$\int_V \frac{\partial}{\partial t} (\rho \phi) dV + \int_S \left( \rho U_i \phi - I \frac{\partial \phi}{\partial x_i} \right) \cdot n_i ds$$

=  $\int_V \rho \phi dV$  this eq<sup>n</sup> satisfies with individual volume. individual



Approximation of volume integral.

element S.



$$m = \int_{V_i} \rho dV = \rho p V$$

$$m u_e = \int_{V_i} \rho u_i dV = \rho p u_p V$$

Approximation of surface integral.

$$\int_{V_i} \Delta p dV = \oint_{S_i} p \phi ds = \sum_k p u_s \Delta x_k, k = n, s, e, w$$

Interpolation

Upwind  $U_e = \begin{cases} U_p & \text{if } (\bar{U}, \bar{n})_e > 0 \\ U_E & \text{if } (\bar{U}, \bar{n})_e < 0 \end{cases}$

Central  $U_e = U_E \lambda_e + U_p (1 - \lambda_e)$

$$\lambda_e = \frac{\kappa_e - \kappa_p}{\kappa_E - \kappa_p}$$

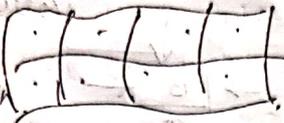
Grids

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Structure of Grid.

all nodes have same number of elements around it.

→ only for simple domain

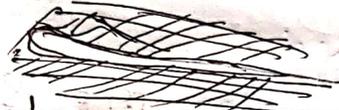


→ indexing is smooth.

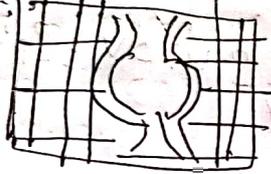
Unstructured Grid.

For all geometries.

\* Irregular data structure.  
\* Aerofoil.



Blocked structured Grid.



elements are divided into no. of blocks.

Classification partial differential equation

- Order

The order of a PDE is determined by highest order partial derivative present in that equation

Example

First order:

$$\frac{\partial \phi}{\partial x} - c \frac{\partial \phi}{\partial y} = 0$$

Second order:

$$\frac{\partial^2 \phi}{\partial x^2} - \phi \frac{\partial \phi}{\partial y} = 0$$

Third order:

$$\left[ \frac{\partial^3 \phi}{\partial x^3} \right]^2 + \frac{\partial^2 \phi}{\partial x^2} = 0$$

Linearity of the  
or functions of  
only then it is  
are functions of  
or any of it  
or same order  
 $a, b, c = \text{const.}$   
linear

Linear sec  
Parabolic,

$$a \frac{\partial^2 \phi}{\partial x^2} + b \frac{\partial \phi}{\partial x} + c \phi + g = 0$$

$$+ f \phi + g = 0$$

where  $C$   
or function  
only. The  
determin

$$\rightarrow b^2 - 4ac$$

$$\rightarrow b^2 - 4ac$$

$$b^2 - 4ac$$

Elliptic

the  
 $\beta$   
bo

(7)

linearity of the coefficients are const or functions of the independent variables only then it is linear. If the coefficients are functions of the dependent variables or any of its derivatives of either lower or same order, then eqn is nonlinear.

$a, b, c = \text{const}$ .  $a \frac{\partial^2 \phi}{\partial x^2} + b \frac{\partial^2 \phi}{\partial x \partial y} + c \frac{\partial^2 \phi}{\partial y^2} + d = 0$   
linear

linear second-order PDEs: elliptic, parabolic,

$a \frac{\partial^2 \phi}{\partial x^2} + b \frac{\partial^2 \phi}{\partial x \partial y} + c \frac{\partial^2 \phi}{\partial y^2} + d \frac{\partial \phi}{\partial x} + e \frac{\partial \phi}{\partial y} + f \phi + g = 0$

where coefficient are either constant or functions of the independent variables only. The three canonical forms are determined by the following criteria

- $\rightarrow b^2 - 4ac < 0$ , elliptic
- $\rightarrow b^2 - 4ac = 0$ , parabolic
- $b^2 - 4ac > 0$ , hyperbolic

Elliptic

Example on elliptical Laplace eqn problems, the

$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$  Poisson's equation  $\phi$  function  $f(x, y)$  must

$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = g(x, y)$  satisfy

the differential eqn over a closed domain & boundary conditions on the closed boundary of that domain.

Parabolic

Example  
Unsteady  
Heat conduction

$$\frac{\partial^2 \phi}{\partial t} = \alpha \frac{\partial^2 \phi}{\partial x^2}$$

initial boundary condition is known

In parabolic problems, the solution advances outward indefinitely from known initial values, always satisfies the known boundary condition.

Hyperbolic : wave equation

$$\frac{\partial^2 \phi}{\partial t^2} = \gamma^2 \frac{\partial^2 \phi}{\partial x^2}$$

The solution domain of hyperbolic PDE has the same open-ended nature as in parabolic PDE. However two initial conditions are required to start the solution of problem.

Classification of N-S equation

The complete Navier-stokes equations in three space coordinates (x, y, z) & time (t) are a system of three non-linear second-order equations in four independent variables. so the normal classification rules don't apply directly to them. But they possess properties such as hyperbolic, parabolic & elliptic

Hyperbolic : Unsteady, inviscid compressible flow, A compressible flow can sustain & N-equation is hyperbolic in nature

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For steady inviscid compressible flows, the equations are hyperbolic if the speed is supersonic & elliptic for subsonic speed.

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