

CHAPTER 4

INTRODUCTION TO GREEN'S FUNCTION

Green's functions are named after the mathematician, **George Green**, who first developed the concept in the 1830s. In the modern study of linear partial differential equations, Green's functions are studied largely from the point of view of fundamental solutions instead. It is an important mathematical tool that has application in many areas of theoretical physics including mechanics, electromagnetism, solid-state physics, thermal physics, and the theory of elementary particles. For the solution of Boundary value problems associated with either ordinary or partial differential equations, one requires a brief knowledge about **Green's function**. Unfortunately it took many years to emerge from the realms of more formal and abstract mathematical analysis as a potential everyday tool for the practical study of Boundary value problem.

4.1 FUNDAMENTAL CONCEPT

Initially we solve, by fairly elementary methods, a typical one dimensional boundary value problem for the understanding of Green's function.

Consider the differential equation,

$$Lu(x) = f(x) \tag{4.1.1}$$

where L is an ordinary linear differential operator, $f(x)$ is a known function while $u(x)$ is an unknown function. To solve above equation, one

method is to find the inverse operator L^{-1} in the form of an integral operator with a kernel $G(x, \xi)$ such that,

$$u(x) = L^{-1}f(x) = \int G(x, \xi)f(\xi)d\xi \quad (4.1.2)$$

The kernel of this integral operator is called **Green's function** for the differential operator. Thus the solution to the non-homogeneous differential equation (4.1.1) can be written down, once the Green's function for the problem is known. For this reason, the Green's function is also sometimes called the fundamental solution associated to the operator L .

4.2 GREEN FUNCTION FOR ORDINARY DIFFERENTIAL EQUATIONS

Consider the problem of forced, transverse vibration of a taut string of length ' l '. If we remove the time dependent parts of the solution by the usual separation of variation method, we obtain the following differential equation containing the transverse displacement of the string, u , as unknown,

$$\frac{d^2u(x)}{dx^2} + k^2u(x) = -f(x) \quad ; 0 \leq x \leq l \quad (4.2.1)$$

If the ends of the strings are kept fixed, then above equation must be solved subject to the boundary conditions:

$$u(0) = u(l) = 0 \quad (4.2.2)$$

By employing method of variation of parameters, we will assume that a solution to the problem actually exists and that furthermore it has a precise form

$$u(x) = A(x) \cos kx + B(x) \sin kx \quad (4.2.3)$$

If we differentiate (4.2.3) twice with respect to x and in passing assume that

$$A' \cos kx + B' \sin kx = 0$$

then (4.2.3) constitute a solution provided that,

$$-kA' \sin kx + kB' \cos kx = -f(x) \quad (4.2.4)$$

Solving assumption and equation (4.2.3) we find that

$$A'(x) = \frac{f(x) \sin kx}{k}; B'(x) = -\frac{f(x) \cos kx}{k} \quad (4.2.5)$$

Hence solution of (4.2.1) can be written in the form

$$u(x) = \frac{\cos kx}{k} \int_{c_1}^x f(y) \sin ky dy - \frac{\sin kx}{k} \int_{c_2}^x f(y) \cos ky dy \quad (4.2.6)$$

where c_1 and c_2 are constants which must be chosen as to ensure that the boundary conditions are satisfied.

Inserting BC; $u(0) = 0$

We find that we must choose c_1 such that

$$\int_{c_1}^0 f(y) \sin ky \, dy = 0 \Rightarrow c_1 = 0 \quad (4.2.7)$$

Hence (4.2.6) reduces to

$$u(x) = \frac{\cos kx}{k} \int_0^x f(y) \sin ky \, dy - \frac{\sin kx}{k} \int_{c_2}^x f(y) \cos ky \, dy \quad (4.2.8)$$

which imply we must choose $c_1 = 0$

Using second BC in (4.2.8) we have

$$\int_{c_2}^l f(y) \cos ky \, dy = \frac{\cos kl}{\sin kl} \int_0^l f(y) \sin ky \, dy \quad (4.2.9)$$

After slight manipulation we can re write the above equation as

$$\begin{aligned} \int_{c_2}^0 f(y) \cos ky \, dy &= \frac{1}{\sin kl} \int_0^l f(y) [\sin ky \cos kl - \cos ky \sin kl] \, dy \\ \Rightarrow \int_{c_2}^0 f(y) \cos ky \, dy &= \frac{1}{\sin kl} \int_0^l f(y) [\sin k(y-l)] \, dy \end{aligned} \quad (4.2.10)$$

\therefore Solution can now be written in the form

$$\begin{aligned} u(x) &= \frac{\cos kx}{k} \int_0^x f(y) \sin ky \, dy - \frac{\sin kx}{k} \int_0^x f(y) \cos ky \, dy \\ &\quad - \frac{\sin kx}{k \sin kl} \int_0^l f(y) \sin k(y-l) \, dy \end{aligned}$$

$$\begin{aligned}
&= \int_0^x f(y) \frac{\sin kl \sin k(y-x)}{k \sin kl} dy + \int_x^l f(y) \frac{\sin kx \sin k(l-y)}{k \sin kl} dy \\
&= \int_0^l f(y) G(x, y) dy \tag{4.2.11}
\end{aligned}$$

where $G(x, y)$ can be introduced as,

$$\begin{aligned}
G(x, y) &= \frac{\sin ky \sin k(l-x)}{k \sin kl} ; 0 \leq y \leq x \\
&= \frac{\sin kx \sin k(l-y)}{k \sin kl} ; x \leq y \leq l
\end{aligned}$$

This function $G(x, y)$ is a two point function of position, known as the Green's function for the equation (4.2.1) and the boundary conditions. Its existence is assured, provided $\sin kl \neq 0$. Thus we see that when $G(x, y)$ exists and when it is known explicitly then we can immediately write down the solution to our boundary value problem along with given boundary conditions. One of the main advantages of above expressed Green function is that it is independent of the Forcing term $f(x)$ and depends only upon the particular differential equation along with boundary conditions imposed. Once $G(x, y)$ has been determined; always provided that the resulting integral in (4.2.11) exists.

Before extending the concept of Green's Function to the parabolic equations some basic concepts that are utilized in the solution have to be explored.

4.3 CONCEPT OF EIGEN VALUES AND EIGEN FUNCTIONS

Let $\phi(x)$ satisfy a second order ordinary differential equations with two homogeneous boundary conditions:

$$\begin{aligned}\frac{d^2 \phi}{d x^2} &= -\lambda \phi \\ \phi(0) &= 0 \\ \phi(L) &= 0\end{aligned}\tag{4.3.1}$$

It is a boundary value problem, since the two conditions are not given at the same place, but at the two different places, $x = 0$ and $x = L$. There is no simple theory which guarantees that the solution exists or is unique to this type of problem. In particular, we note that $\phi(x) \equiv 0$ satisfies the ODE and both homogeneous boundary conditions, no matter what the separation constant λ is, even if $\lambda < 0$, it is referred to as the trivial solution of the boundary value problem. It corresponds to $u(x,t) \equiv 0$, where $u(x,t) \equiv \phi(x) G(t)$. If the solution of given problem had been unique, then $\phi(x) \equiv 0$ would be the only solution; we would not be able to obtain a nontrivial solutions of linear homogeneous PDE by separation of variables method.

Fortunately there are other solutions. However, there do not exist a non trivial solution for all values of λ . Instead we will show that there

are certain special values of λ , called **eigen values** of the given boundary value problem for which there are non-trivial solutions, $\phi(x)$. A non-trivial $\phi(x)$, which exists for certain values of λ , is known as **eigen functions** corresponds to the eigen value λ .

For the determination of eigen value of given problem, we observe that the given equation is linear and homogeneous with constant coefficients; two independent solutions are usually obtained in the form of exponentials; $\phi(x) = \exp(rx)$. Substituting this into the differential equation yields the characteristic polynomial $r^2 = -\lambda$. The solutions corresponding to two roots have significantly different properties depending on the values of λ .

Case:1 $\lambda > 0$

In this case exponential solutions have imaginary exponents $\exp(\pm i\sqrt{\lambda}x)$ and solution oscillates. For real solutions we can choose $\cos\sqrt{\lambda}x$ and $\sin\sqrt{\lambda}x$ in general.

Thus general solution in this case is :

$$\phi(x) = C_1 \cos \sqrt{\lambda}x + C_2 \sin \sqrt{\lambda}x \quad (4.3.2)$$

Boundary condition at $x = 0 \Rightarrow C_1 = 0$

so $\phi(x) = C_2 \sin \sqrt{\lambda}x$

Then boundary condition at

$$x = L \Rightarrow C_2 \sin \sqrt{\lambda} L = 0$$

$$\therefore \text{either } C_2 = 0 \text{ or } \sin \sqrt{\lambda} L = 0.$$

If $C_2 = 0$ then $\phi(x) \equiv 0$, this is a trivial solution.

$$\text{Now } \sin \sqrt{\lambda} L = 0 \Rightarrow \sqrt{\lambda} L = n\pi$$

We are searching for those values of λ that have non-trivial solutions, therefore eigen value λ must satisfy

$$\sin \sqrt{\lambda} x = 0.$$

$$\Rightarrow \lambda = \left(\frac{n\pi}{L} \right)^2 \quad n = 1, 2, 3, \dots$$

Hence the eigen vector corresponding to eigen value λ is

$$\phi(x) = C_2 \sin \left(\frac{n\pi x}{L} \right)$$

where C_2 is an arbitrary constant.

Case:2 $\lambda = 0$

$$\text{In this case } \phi(x) = C_1 + C_2 x$$

corresponding to double zero roots; $r=0$ of the characteristic polynomial.

$$\text{Boundary condition at } x = 0 \Rightarrow C_1 = 0$$

then boundary condition at $x = L \Rightarrow C_2 L = 0$.

Since length L is positive this gives a trivial solution $\phi(x) \equiv 0$.

Thus $\lambda = 0$ is not an eigen value for the problem.

Case:3 $\lambda < 0$

In this case the roots of the characteristic polynomials are $r = \pm\sqrt{-\lambda}$, so solutions are $\exp(\sqrt{-\lambda}x)$ and $\exp(-\sqrt{-\lambda}x)$. We may prefer equivalent notation $\sqrt{|\lambda|}$.

Considering $\lambda < 0$ and suppose $\lambda = -S$, gives $S > 0$

Therefore we have

$$\phi(x) = C_1 \exp(\sqrt{S}x) + C_2 \exp(-\sqrt{S}x)$$

In terms of Hyperbolic function this can be rewritten as

$$\phi(x) = C_3 \cosh(\sqrt{S}x) + C_4 \sinh(-\sqrt{S}x)$$

Boundary condition at $x = 0 \Rightarrow C_3 = 0$

Then, boundary condition at $x = L \Rightarrow C_4 \sinh(\sqrt{S}L) = 0$.

Since $\sqrt{S}L = 0$ and since \sinh is never zero for a positive argument, it follows that $C_4 = 0 \Rightarrow \phi(x) = 0$.

\therefore The only solution to (4.3.2) for $\lambda < 0$ that solves the homogeneous boundary conditions is the trivial solution.

4.4 METHOD OF EIGEN FUNCTION

EXPANSION

Consider a problem of non homogeneous linear partial differential equation with homogeneous boundary conditions.

$$\begin{aligned}\frac{\partial v}{\partial t} &= k \frac{\partial^2 v}{\partial x^2} + \bar{Q}(x, t) \\ v(0, t) &= 0; v(L, t) = 0 \\ v(x, 0) &= g(x)\end{aligned}\tag{4.4.1}$$

Now related Homogeneous problem is given by

$$\begin{aligned}\frac{\partial u}{\partial t} &= k \frac{\partial^2 u}{\partial x^2} \\ u(0, t) &= 0; \\ u(L, t) &= 0\end{aligned}\tag{4.4.2}$$

the eigen functions of related homogeneous problem satisfy

$$\begin{aligned}\frac{d^2 \phi}{dx^2} &= -\lambda \phi \\ \phi(0) &= 0 \\ \phi(L) &= 0\end{aligned}$$

Now from above discussion we know that the eigen values are

$$\lambda = \left(\frac{n\pi}{L}\right)^2 \quad n = 1, 2, 3, \dots \text{ and the corresponding eigen functions are}$$

$$\phi_n(x) = \sin\left(\frac{n\pi x}{L}\right) \text{ which are known.}$$

The method of eigen function expansion, employed to solve the non homogeneous problem with homogeneous boundary conditions consists in expanding the unknown solution $v(x,t)$ in a series of the related homogeneous eigen functions:

$$v(x,t) = \sum_{n=1}^{\infty} a_n(t)\phi_n(x) \quad (4.4.3)$$

For each fixed t , $v(x,t)$ is a function of x , and hence $v(x,t)$ will have a generalized Fourier series.

In this case we have an ordinary Fourier sine series, and generalized Fourier coefficients are a_n , which varies as t changes. Here $a_n(t)$ are not the time dependent separated solutions $e^{-k(n\pi/L)^2 t}$ but they are just generalized Fourier coefficients for $v(x,t)$ which can be determined as follows

Orthogonality of Sines :

Let,

$$v(x,0) = g(x) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{L} \quad (4.4.4)$$

We will assume that standard mathematical operations are also valid for infinite series. Equation represents one equation in an infinite number of unknowns but it should be valid at every value of x . If we substitute a thousand different values of x into above equation, each of the thousand

equations would hold, but there would still be an infinite number of unknowns. This is not an efficient way to determine a_n . Instead, we frequently will employ an extremely important technique based on noticing that the eigen functions $\sin \frac{n\pi x}{L}$ satisfying the following integral property.

$$\int_0^L \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L} dx = \begin{cases} 0 & ; \quad m \neq n & (4.4.5(a)) \\ L/2 & ; \quad m = n & (4.4.5(b)) \end{cases}$$

where m and n are positive integers.

To use this conditions (4.4.5) to determine a_n , we multiply both sides of (4.4.4) by $\sin \frac{m\pi x}{L}$ (for any fixed integer m , independent of the ‘dummy’ index n).

$$g(x) \sin \frac{m\pi x}{L} = \sum_{n=1}^{\infty} a_n \left(\sin \frac{n\pi x}{L} \right) \left(\sin \frac{m\pi x}{L} \right)$$

Now we integrate with respect to x , from $x = 0$ to $x = L$:

$$\int_0^L g(x) \sin \frac{m\pi x}{L} dx = \sum_{n=1}^{\infty} a_n \int_0^L \left(\sin \frac{n\pi x}{L} \right) \left(\sin \frac{m\pi x}{L} \right) dx$$

For finite series the integral of a sum of terms equals the sum of integrals. We assume that this is valid for this infinite series. Now we evaluate the infinite sum. From the integral property (4.4.5) we see that each term of the sum is zero whenever $n \neq m$. In summing over n ,

eventually n equals m . It is only for that one value of n ($n = m$) that there is a contribution to the infinite sum. The only term that appears on the right hand side of occurs when n is replaced by m :

$$\int_0^L g(x) \sin \frac{m \pi x}{L} dx = a_m \int_0^L \sin^2 \frac{m \pi x}{L} dx$$

Since the integral on the right equals $L/2$, we can solve for a_n :

$$a_m = \frac{\int_0^L g(x) \sin \frac{m \pi x}{L} dx}{\int_0^L \sin^2 \frac{m \pi x}{L} dx} = \frac{2}{L} \int_0^L g(x) \sin \frac{n \pi x}{L} dx \quad (4.4.6)$$

The integral in (4.4.6) is considered to be known since $g(x)$ is the given initial condition.

4.5 GREEN FUNCTIONS FOR PARABOLIC DIFFERENTIAL EQUATION

A concept of Green's Function method now can be extended to a parabolic equation and its associated boundary value problem for brief understanding.

Consider a problem associated with the Diffusion equation:

$$\nabla^2 u = \frac{\partial u}{\partial t} \quad (4.5.1)(a)$$

which holds through out a finite, bounded region D . The boundary conditions imposed on the solution function u , is taken to be,

$$\begin{aligned} u(p,t) &= 0 & p \in \partial D, & \text{ for all } t \\ u(P,0) &= f(P) & P \in D \end{aligned} \quad (4.5.1)(b)$$

Where second equation represent the initial distribution of u throughout D consequently a “boundary value problem” for a parabolic equation can be typified as:

$$\nabla^2 u(P) = \frac{\partial u}{\partial t} \quad P \in D$$

$$\begin{aligned} u(p,t) &= 0 & p \in \partial D, & \text{ for all } t \\ u(P,0) &= f(P) & P \in D \end{aligned} \quad (4.5.2)$$

For solution of such problem we assume an expansion for the solution function, u , in the form,

$$u(P,t) = \sum_{n=1}^{\infty} c_n(t)u_n(P) \quad (4.5.3)$$

where $c_n(t)$ is an undetermined function of t and the functions u_n are orthonormal eigen functions, with corresponding eigen values λ_n defined by the problem:

$$\nabla^2 \psi(P) + \lambda \psi(P) = 0 \quad P \in D, \quad \psi(P) = 0 \quad P \in \partial D \quad (4.5.4)$$

Employing orthonormality property of the eigen function u_n , we obtain from (4.5.3),

$$c_n(t) = \int_D u(P, t) u_n(P) d\tau_P \quad (4.5.5)$$

Differentiating with respect to time we have,

$$\begin{aligned} c_n'(t) &= \int_D u'(P, t) u_n(P) d\tau_P \\ &= \int_D \nabla^2 u(P, t) u_n(P) d\tau_P \end{aligned}$$

Applying Green's identity to $u(P, t)$ and $u_n(P)$ throughout the region D , and implementing the boundary conditions imposed on the functions we can write:

$$\begin{aligned} c_n'(t) &= \int_D \nabla^2 u(P, t) u_n(P) d\tau_P \\ &= \int_D u(P, t) \nabla^2 u_n(P) d\tau_P \end{aligned}$$

$$\therefore c_n'(t) = -\lambda_n c_n(t)$$

The solution for this first order Ordinary differential equation for $c_n(t)$ is:

$$c_n(t) = c_n(0) \exp(-\lambda_n t) \quad (4.5.6)$$

Imposing Initial conditions on $u(P, t)$ we find ,

$$c_n(0) = f_n = \int_D f(P) u_n(P) d\tau_P \quad (4.5.7)$$

substituting (4.5.6) and (4.5.7) in (4.5.3) we obtain

$$\begin{aligned} u(P, t) &= \sum_{n=1}^{\infty} \exp(-\lambda_n t) u_n(P) \int_D f(Q) u_n(Q) d\tau_Q \\ &= \int_D f(Q) \left\{ \sum_{n=1}^{\infty} \exp(-\lambda_n t) u_n(P) u_n(Q) \right\} d\tau_Q \end{aligned}$$

which gives

$$K(P, Q; t) = \sum_{n=1}^{\infty} \exp(-\lambda_n t) u_n(P) u_n(Q) H(t)$$

as our required Green's Function for the parabolic Equation. Here Heviside function $H(t)$ is included to emphasize the fact that the solution is identically zero for $t < 0$.

Since $H(t) = 1$ for $t = 0$

$$\lim_{t \rightarrow 0^+} K(P, Q; t) = \sum_{n=1}^{\infty} u_n(Q) u_n(P)$$

and hence solution satisfies initial conditions of the problem.

Finally examining more general problem posed by

$$\frac{\partial u(P)}{\partial t} - \nabla^2 u(P) = f(P, t) \quad P \in D$$

$$u(p, t) = g(p, t) \quad p \in \partial D, \quad \text{for all } t$$

$$u(P, 0) = h(P) \quad P \in D$$

If we consider this problem with one as connected with the distribution of temperature u , throughout the region D with prescribed initial and surface temperature distribution we can interpret $K(P, Q; t)$ as being the temperature, at time t , at the point P due to a heat source at the point Q . To construct a solution of above problem, we find it convenient to integrate $K(P, Q; t-T) (u_t - \nabla^2 u)$ over the product domain $D \times T$ of the region D and time interval T where $0 \leq T \leq t - \epsilon$.

Now integrating by parts we have

$$\begin{aligned}
& \int_{0^+}^{t-\epsilon} \int_D K(P, Q; t-T) (u_T - \nabla^2 u) d\tau_Q dT \\
&= \int_D \{ [K(P, Q; t-T) u(Q, T)]_0^{t-\epsilon} + \int_{0^+}^{t-\epsilon} K_T(P, Q; t-T) u(Q, T) dT \} d\tau_Q - \\
& \int_{0^+}^{t-\epsilon} \left\{ \int_{\partial D} \left(K \frac{\partial u}{\partial n_q} - u \frac{\partial K}{\partial n_q} \right) dS_q + \int_D u \nabla^2 K d\tau_Q \right\} dT
\end{aligned}$$

Substituting given data with initial and boundary conditions as $\epsilon \rightarrow 0$:

$$\begin{aligned}
& \int \int_D K(P, Q; t-T) f(Q, T) d\tau_Q dT \\
&= \int_D K(P, Q; 0) u(Q, T) d\tau_Q - \int_D K(P, Q; t) h(Q) d\tau_Q + \int_0^t dt \left\{ \int_{\partial D} g(q, T) \frac{\partial}{\partial n_q} \{ K(P, q; t-T) \} dS_q \right.
\end{aligned}$$

(4.5.8)