

Solutions of Homogeneous Linear Partial Differential Equations of second and higher order with constant coefficients

The general form of a Homogeneous Linear Partial Differential Equations of nth order with constant coefficients

$$a_0 \frac{\partial^n z}{\partial x^n} + a_1 \frac{\partial^n z}{\partial x^{n-1} \partial y} + a_2 \frac{\partial^n z}{\partial x^{n-2} \partial y^2} + \dots + a_{n-1} \frac{\partial^n z}{\partial x \partial y^{n-1}} + a_n \frac{\partial^n z}{\partial y^n} = R(x, y) \dots (1)$$

where $a_0, a_1, a_2, \dots, a_n$ are constants.

Here denote $D = \frac{\partial}{\partial x}$, D' (D dash) = $\frac{\partial}{\partial y}$, $D^r = \frac{\partial^r}{\partial x^r}$, $D'^r = \frac{\partial^r}{\partial y^r}$, \dots

$$a_0 D^n + a_1 D^{n-1} D' + a_2 D^{n-2} D'^2 + \dots + a_{n-1} D D'^{n-1} + a_n D'^n = R(x, y) \dots (2)$$

This is of the form $F(D, D')z = R(x, y) \dots (3)$

Then $F(D, D')z = 0 \dots (4)$ is the homogeneous Linear P D E.

Its Auxiliary equation can be obtained by replacing D by m and D' by 1

i.e., $F(m, 1) = 0 \dots (5)$, it has n roots $m_1, m_2, m_3, \dots, m_n$ which are real or complex.

Working procedure to find Complementary Function:

Case 1: If the roots are different, then the complementary function is

$$C. F. = \varphi_1(y + m_1 x) + \varphi_2(y + m_2 x) + \varphi_3(y + m_3 x) + \dots + \varphi_n(y + m_n x),$$

Where $\varphi_1, \varphi_2, \varphi_3, \dots, \varphi_n$ are n arbitrary functions.

Case 2: If the two roots are equal, $m_1 = m_2 = m$ (say) and the others are different then the complementary function is

$$C. F. = \varphi_1(y + mx) + x\varphi_2(y + mx) + \varphi_3(y + m_3 x) + \dots + \varphi_n(y + m_n x)$$

If the three roots are equal, $m_1 = m_2 = m_3 = m$ (say) and the others are different then the complementary function is

$$C. F. = \varphi_1(y + mx) + x\varphi_2(y + mx) + x^2\varphi_3(y + mx) + \varphi_4(y + m_4 x) + \dots + \varphi_n(y + m_n x)$$

Working procedure to find Particular Integral :

The Particular Integral of (1) is $P.I = \frac{1}{f(D, D')} R(x, y)$

Type 1: Let $R(x, y) = e^{ax+by}$

Then P. I = $\frac{1}{f(D, D')} e^{ax+by}$, Put $D = a$ and $D' = b$

If $f(a, b) \neq 0$, then P. I = $\frac{1}{f(a, b)} e^{ax+by}$

If $f(a, b) = 0$, then $D - \frac{a}{b} D'$ of its power will be a factor of $f(D, D')$

P.I = $\frac{x}{f'(a, b)} e^{ax+by} \dots\dots$

Type :II Let $R(x, y) = \sin(ax + by)$ or $\cos(ax + by)$

We have

$$D^2 \sin(ax+by) = -a^2 \sin(ax+by), \quad DD' \sin(ax + by) = -ab \sin(ax + by)$$

$$D'^2 \sin(ax+by) = -b^2 \sin(ax+by)$$

$$f(D^2, DD', D'^2) \sin(ax + by) = f(-a^2, -ab, -b^2) \sin(ax + by)$$

$$P.I = \frac{1}{f(D^2, DD', D'^2)} \sin(ax+by)$$

$$\text{Put } D^2 = -a^2, DD' = -ab, D'^2 = -b^2$$

$$= \frac{1}{f(-a^2, -ab, -b^2)} \sin(ax+by) \text{ (provided } f(-a^2, -ab, -b^2) \neq 0$$

$$\text{Similarly P.I} = \frac{1}{f(D^2, DD', D'^2)} \cos(ax+by)$$

$$= \frac{1}{f(-a^2, -ab, -b^2)} \cos(ax+by) \text{ (provided } f(-a^2, -ab, -b^2) \neq 0$$

Type :III Let $R(x, y) = x^m y^n$

$$P. I = \frac{1}{f(D, D')} x^m y^n = f(D, D')^{-1} x^m y^n$$

If $m \geq n$, re write $f(D, D')^{-1}$ by taking the highest power of D , as $[1 \pm \varphi(\frac{D'}{D})]^{-1}$

And expand using binomial series in powers of $\frac{D'}{D}$.

If $m < n$, re write $f(D, D')^{-1}$ by taking the highest power of D , as $[1 \pm \phi(\frac{D}{D'})]^{-1}$

And expand using binomial series in powers of $\frac{D}{D'}$.

Here $\frac{1}{D}f(x, y) = \int f(x, y)dx, y \text{ as constant}$ and $\frac{1}{D'}f(x, y) = \int f(x, y)dy, x \text{ as constant}$.

Ex: 1 Solve the p d e $\frac{\partial^3 z}{\partial x^3} - 3 \frac{\partial^3 z}{\partial^2 x \partial y} + 4 \frac{\partial^3 z}{\partial y^3} = e^{x+2y}$.

Sol: The given p d e is $(D^3 - 3 D^2 D' + 4 D'^3)z = e^{x+2y}$

Its Auxiliary equation can be obtained by replacing D by m and D' by 1

Auxiliary Equation is $m^3 - 3 m^2 + 4 = 0$

$= (m+1)(m-2)^2=0$, which implies $m=-1, 2, 2$.

C. F. $= \phi_1(y-x) + \phi_2(y+2x) + x\phi_3(y+2x)$.

P. I $= \frac{1}{f(D, D')} e^{ax+by}$, Put $D=a$ and $D' = b$

P. I $= \frac{1}{D^3 - 3 D^2 D' + 4 D'^3} e^{x+2y}$, Put $D=1$ and $D' = 2$

$= \frac{1}{1^3 - 3 \cdot 1^2 \cdot 2 + 4 (2)^3} e^{x+2y} = \frac{1}{27} e^{x+2y}$

The general solution is $z = C.F + P.I$

$z = \phi_1(y-x) + \phi_2(y+2x) + x\phi_3(y+2x) + \frac{1}{27} e^{x+2y}$.

EX : 2 Solve the p d e $\frac{\partial^2 z}{\partial x^2} - 2 \frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = e^{x+y}$.

Sol: The given p d e is $(D^2 - 2 D D' + D'^2)z = e^{x+y}$

Its Auxiliary equation can be obtained by replacing D by m and D' by 1

Auxiliary Equation is $m^2 - 2 m + 1 = 0$

$= (m-1)^2=0$, which implies $m=1, 1$.

C. F. $= \phi_1(y+x) + x\phi_2(y+x)$

$$P. I = \frac{1}{f(D,D')} e^{ax+by}, \text{ Put } D=a \text{ and } D' = b$$

$$P. I = \frac{1}{D^2-2D D'+D'^2} e^{x+y}, \text{ Put } D=1 \text{ and } D' =$$

$$= \frac{1}{1^2-2.1.1+1} e^{x+y} = \frac{1}{0} e^{x+y}$$

$$\text{So that } P. I = \frac{x}{f'(D,D')} e^{x+y} = \frac{1}{2D-2 D'} e^{x+y} \text{ (Derivative wrt } x \text{ only)}$$

Again , we get 0

$$\text{So that } P. I = \frac{x^2}{f''(D,D')} e^{x+y} = \frac{x^2}{2} e^{x+y}$$

$$P. I = \frac{x^2}{2} e^{x+y}$$

The general solution is $z = C.F + P.I$

$$z = \varphi_1(y+x) + x\varphi_2(y+x) + \frac{x^2}{2} e^{x+y}.$$

$$\text{EX: 3 Solve the p d e } \frac{\partial^2 z}{\partial x^2} + 4 \frac{\partial^2 z}{\partial x \partial y} - 5 \frac{\partial^2 z}{\partial y^2} = \sin(2x+3y)$$

$$\text{Sol: The given p d e is } (D^2 + 4D D' - 5 D'^2)z = \sin(2x + 3y)$$

Its Auxiliary equation can be obtained by replacing D by m and D' by 1

$$\text{Auxiliary Equation is } m^2 + 4m - 5 = 0$$

$$= (m-1)(m+5)=0, \text{ which implies } m=1, -5, \text{ real and distinct.}$$

$$C. F. = \varphi_1(y+x) + \varphi_2(y-5x)$$

$$P. I = \frac{1}{f(D,D')} \sin(ax+by), \text{ Put } D^2 = -a^2, DD' = -ab, D'^2 = -b^2$$

$$P. I = \frac{1}{D^2+4D D'-5D'^2} \sin(2x+3y), \text{ Put } D^2 = -2^2, DD' = -2 \times 3, D'^2 = -3^2$$

$$= \frac{1}{-4+4(-6)-5(-9)} \sin(2x+3y)$$

$$P.I = \frac{1}{17} \sin(2x+3y)$$

The general solution is $z = C.F + P.I$

$$z = \varphi_1(y+x) + \varphi_2(y-5x) + \frac{1}{17} \sin(x+y).$$

EX: 4. Solve the p d e $\frac{\partial^3 z}{\partial x^3} - 2 \frac{\partial^3 z}{\partial x^2 \partial y} = 4 \sin(x+y)$

Sol: The given p d e is $(D^3 - 2 D^2 D')z = 4 \sin(x + y)$

Its Auxiliary equation can be obtained by replacing D by m and D' by 1

Auxiliary Equation is $m^3 - 2m^2 = 0$

$= m^2(m - 2) = 0$, which implies $m=0, 0, 2$, real and distinct.

C. F. = $\varphi_1(y) + x\varphi_2(y) + \varphi_3(y + 2x)$

P. I = $\frac{1}{f(D, D')} \sin(ax + by)$, Put $D^2 = -a^2, DD' = -ab, D'^2 = -b^2$

P. I = $\frac{1}{D^3 - 2 D^2 D'} 4 \sin(x + y)$, Put $D^3 = D, D^2 = -1^2, D, D^2 D' = -1 D'$

$= 4 \frac{1}{-D - 2D(-1)} \sin(x + y)$

$= 4 \frac{1}{D} \sin(x + y)$

P.I = $4 \int \sin(x + y) dx = -4 \cos(x + y)$

P.I = $-4 \cos(x + y)$

The general solution is $z = C.F + P.I$

$z = \varphi_1(y) + x\varphi_2(y) + \varphi_3(y + 2x) - 4 \cos(x + y)$.

EX: 5. Solve the p d e $(D^2 + DD' - 6D'^2)z = x^2y$

Sol: The given p d e is $(D^2 + DD' - 6D'^2)z = x^2y$

Its Auxiliary equation can be obtained by replacing D by m and D' by 1

Auxiliary Equation is $m^2 + m - 6 = 0$

$= (m + 3)(m - 2) = 0$, which implies $m = -3, 2$, real and distinct.

C. F. = $\varphi_1(y - 3x) + \varphi_2(y + 2x)$.

P. I = $\frac{1}{f(D, D')} x^m y^n = f(D, D')^{-1} x^m y^n$

If $m \geq n$, re write $f(D, D')^{-1}$ by taking the highest power of D, as $[1 \pm \varphi(\frac{D'}{D})]^{-1}$

$$P. I = \frac{1}{(D^2 + DD' - 6D^2)} x^2 y$$

P.I=

$$\frac{1}{D^2 \left[1 + \frac{D'}{D} - \frac{6D'}{D^2} \right]} x^2 y$$

$$P.I = \frac{1}{D^2} \left[1 + \left(\frac{D'}{D} - \frac{6D'}{D^2} \right) \right]^{-1} (x^2 y)$$

$$= \frac{1}{D^2} \left[1 - \frac{D'}{D} \right] (x^2 y)$$

$$= \frac{1}{D^2} [x^2 y - \frac{D'}{D} (x^2 y)]$$

$$= \frac{1}{D} \int (x^2 y) dx - \frac{1}{D^2} \int x^2 dx = \frac{1}{D} \left(\frac{x^3}{3} y \right) - \frac{1}{D^2} \left(\frac{x^3}{3} \right)$$

On simplification we get P.I. = $\frac{yx^4}{12} - \frac{x^5}{60}$

Solution is $z = C.F + P.I = \varphi_1(y - 3x) + \varphi_2(y + 2x) + \frac{yx^4}{12} - \frac{x^5}{60}$

HW.

6. Solve P D E $\frac{\partial^3 z}{\partial x^3} - 3 \frac{\partial^3 z}{\partial^2 x \partial y} + 2 \frac{\partial^3 z}{\partial y^3} = e^{2x-y} + e^{x+y}$

7. Solve P D E $\frac{\partial^2 z}{\partial x^2} - 2 \frac{\partial^2 z}{\partial x \partial y} = 2 \sin x \cos 2y$

Hint: use $2 \sin A \cos B$ formula

8. Solve the p d e $(D^2 - 2DD')z = e^{2x} + x^3 y$

9. Solve the p d e $(D^2 - DD')z = e^{2x+y} + x^2 y + \cos x \cos 2y$.