

**Problems on Rule 4:**

1. Find the solution of  $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 4y = e^x \cos x$

**Solution:** Given equation is  $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 4y = e^x \cos x$

Given equation is non-homogeneous equation, whose homogeneous equation is,

$$(D^2 - 2D + 4)y = 0$$

A.E. is,

$$m^2 - 2m + 4 = 0$$

$$m = 1 \pm i\sqrt{3}$$

The complementary function is,

$$\text{C.F.} = e^x (c_1 \cos \sqrt{3}x + c_2 \sin \sqrt{3}x)$$

The particular integral is,

$$\begin{aligned} \text{P.I.} &= \frac{1}{f(D)} e^x \cos x \\ &= \frac{1}{D^2 - 2D + 4} e^x \cos x \end{aligned}$$

Put  $D = D + a = D + 1$

$$\begin{aligned} \text{P.I.} &= e^x \frac{1}{D^2 + 3} \cos x \\ &= e^x \frac{1}{-1 + 3} \cos x \\ &= \frac{e^x}{2} \cos x \end{aligned}$$

The general solution is,

$$y = \text{C.F.} + \text{I.F.}$$

$$y = e^x (c_1 \cos \sqrt{3}x + c_2 \sin \sqrt{3}x) + \frac{e^x}{2} \cos x$$

2. Find the solution of  $(D^2 - 5D + 6)y = xe^{4x}$

**Solution:** Given  $(D^2 - 5D + 6)y = xe^{4x}$

Given equation is non-homogeneous equation whose homogeneous equation is

$$(D^2 - 5D + 6)y = 0$$

A.E. is,

$$m^2 - 5m + 6 = 0$$

$$(m-3)(m-2) = 0$$

$$m = 3, 2$$

Then, C.F. =  $c_1 e^{3x} + c_2 e^{2x}$

$$\begin{aligned} P.I. &= \frac{1}{f(D)} F(x) \\ &= \frac{1}{f(D)} x e^{4x} \\ &= \frac{1}{D^2 - 5D + 6} x e^{4x} \\ &= \left( \frac{1}{D-3} - \frac{1}{D-2} \right) x e^{4x} \\ &= \frac{1}{D-3} x e^{4x} - \frac{1}{D-2} x e^{4x} \end{aligned}$$

$$\begin{aligned} P.I. &= e^{3x} \int x e^{4x} e^{-3x} dx - e^{2x} \int x e^{4x} e^{-2x} dx \\ &= e^{3x} \int x e^x dx - e^{2x} \int x e^{2x} dx \\ &= e^{3x} (x e^x - e^x) - e^{2x} \left( x \frac{e^{2x}}{2} - \frac{e^{2x}}{4} \right) \end{aligned}$$

$$P.I. = \frac{e^{4x}}{4} (2x - 3)$$

Therefore, the general solution is,

$$y = y_c + y_p$$

$$y = c_1 e^{3x} + c_2 e^{2x} + \frac{e^{4x}}{4} (2x - 3)$$

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### 4.2.3.1.

### Lecture-19

#### Problems on Rule 5:

1. Solve  $(D+1)^2 y = x \cos x$

**Solution:** Given  $(D+1)^2 y = x \cos x$

Given equation is non-homogeneous equation whose homogeneous equation is

$$(D+1)^2 = 0$$

A.E. is,

$$(m+1)^2 = 0$$

$$m = -1, -1$$

The complementary function is, C.F. =  $(c_1 + c_2 x)e^{-x}$

Consider,

$$\begin{aligned} P.I. &= \frac{1}{(D+1)^2} x \cos x \\ &= \left[ x - \frac{2(D+1)}{(D+1)^2} \right] \frac{1}{-1^2 + 2D + 1} \cos x \\ &= \left( x - \frac{2}{D+1} \right) \frac{\sin x}{2} \\ &= \frac{x}{2} \sin x - \frac{D-1}{D^2-1} \sin x \\ &= \frac{x}{2} \sin x + \frac{\cos x - \sin x}{2} \end{aligned}$$

Hence the general solution is,

$$y = C.F. + P.I$$

$$y = (c_1 + c_2 x)^2 e^{-x} + \frac{x}{2} \sin x + \frac{1}{2} (\cos x - \sin x)$$

2. Solve  $\frac{d^2 y}{dx^2} - 2 \frac{dy}{dx} + y = xe^x \cos x$

**Solution:** Given  $\frac{d^2 y}{dx^2} - 2 \frac{dy}{dx} + y = xe^x \cos x$

Given equation in the standard form can be written as  $(D^2 - 2D + 1)y = xe^x \cos x$

Which is a non-homogeneous equation.

Its homogeneous equation is  $(D^2 - 2D + 1)y = 0$

A.E. is,

$$m^2 - 2m + 1 = 0$$

$$m = 1, 1$$

Then, C.F. =  $(c_1 + c_2x)e^x$

Consider,

$$\begin{aligned} P.I. &= \frac{1}{D^2 - 2D + 1} xe^x \cos x \\ &= e^x \frac{1}{(D+1)^2 - 2(D+1) + 1} x \cos x \quad \text{put } D = D + 1 \\ &= e^x \cdot \frac{1}{D^2} (x \cos x) \\ &= e^x \frac{1}{D} \int x \cos x \end{aligned}$$

$$\begin{aligned} P.I. &= e^x \cdot \frac{1}{D} \left[ x \sin x - \int \sin x dx \right] \\ &= e^x \int (x \sin x + \cos x) dx \\ &= e^x \int x \sin x dx + e^x \int \cos x dx \end{aligned}$$

$$\begin{aligned} P.I. &= e^x \left[ x(-\cos x) - \int (-\cos x) dx \right] + e^x \sin x \\ &= e^x [-x \cos x + \sin x] + e^x \sin x \\ &= e^x [2 \sin x - x \cos x] \end{aligned}$$

The general solution is

$$y = \text{C.F.} + \text{P.I.}$$

$$y = (c_1 + c_2x)e^x + e^x (2 \sin x - x \cos x)$$

$$y = e^x [c_1 + c_2x + 2 \sin x - x \cos x]$$

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#### 4.2.4 Test Questions:

**a. Fill in the blanks type of questions:**

1. The solution of I.V.P.  $y'' + y = 0, y(0) = 2, y'(0) = -1$  is  $y =$  \_\_\_\_\_

**Answer:**  $2\cos x - \sin x$

2. The general solution of  $(9D^2 + 6D + 1)y = 0$  is \_\_\_\_\_

**Answer:**  $(c_1 + c_2x)e^{\frac{-x}{3}}$

3. The two linear solutions of  $(D^2 + a^2)y = 0$  are \_\_\_\_\_

**Answer:**  $\cos ax, \sin ax$

4. A particular value of  $\frac{1}{D^2 - 3D + 2}e^{2x}$  is \_\_\_\_\_

**Answer:**  $e^{2x}(x-1)$

5.  $\frac{1}{D^2 - 9}\cos 3x =$  \_\_\_\_\_

**Answer:**  $\frac{-\cos 3x}{18}$

6.  $\frac{1}{D^2 + a^2}\cos ax =$  \_\_\_\_\_

**Answer:**  $\frac{-x}{2a}\cos ax$

7.  $\frac{1}{(D-1)^2}e^x \sin x =$  \_\_\_\_\_

**Answer:**  $-e^x \sin x$

8. The P.I. of the equation  $(D^2 + 2D + 1)y = e^{-x} \cos x$  is \_\_\_\_\_

**Answer:**  $-e^{-x} \cos x$

9. A particular value of  $\frac{1}{D}\cos^3 x$  is \_\_\_\_\_

**Answer:**  $\frac{1}{12}(\sin 3x + 3\sin x)$

10. If  $\sin x$  and  $\cos x$  are two linear independent solutions of  $(aD^2 + bD + c)y = 0$  then

a=\_\_\_\_\_, b=\_\_\_\_\_ **Answer:**  $b = 0, \frac{c}{a} = 1$

**b. Multiple choice questions:**

1. Which of the following is the general solution to  $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - 10 = 0$ ?

In each case A and B are arbitrary constants.

(a)  $y = Ae^{2x} + Be^{-5x}$

(b)  $y = Ae^{-2x} + Be^{5x}$

(c)  $y = Ae^{2x} + Be^{5x}$

(d)  $y = Ae^{-2x} + Be^{-5x}$

**Answer: (a)**

2. Which of the following is the general solution to  $\frac{d^2y}{dx^2} + 10\frac{dy}{dx} + 25 = 0$ ?

In each case A and B are arbitrary constants.

(a)  $y = Ae^{-5x} + Be^{-5x}$

(b)  $y = Axe^{-5x} + Be^{-5x}$

(c)  $y = Ae^{5x} + Be^{5x}$

(d)  $y = Axe^{5x} + Be^{5x}$

**Answer: (a)**

3. Consider the differential equation  $\frac{d^2y}{dx^2} - 7y = 0$ . Which of the following is correct option?

- (a) The roots of the Auxiliary equation are 0 and 7.
- (b) There is no Auxiliary equation for a differential equation of this type.
- (c) The Auxiliary equation is a repeated root of  $\sqrt{7}$
- (d) The roots of the Auxiliary equation  $\sqrt{7}$  and  $-\sqrt{7}$

Answer: **(d)**

2. Find the particular solution to  $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} - 4y = 0$  satisfying  $y(0) = 0$  and  $y'(0) = 5$

- (a)  $y = e^t - e^{-4t}$
- (b)  $y = e^{-t} - e^{4t}$
- (c)  $y = e^{-4t} - e^t$
- (d)  $y = e^{4t} - e^{-t}$

Answer: **(d)**

3. The general solution of  $(D^3 - D)y = 0$  is

- (a)  $y = c_1 + c_2x + c_3x^2$
- (b)  $y = c_1 + c_2e^x + c_3e^{-x}$
- (c)  $y = c_1 + c_2 \cos x + c_3 \sin x$
- (d) None of these

Answer: **(b)**

4. The P.I. of  $\frac{1}{(D-1)^2} e^x =$

$$(a) xe^x$$

$$(b) \frac{x^2 e^{-x}}{2}$$

$$(c) \frac{x^2 e^x}{2}$$

$$(d) x^2 e^x$$

Answer: **(C)**

5. The P.I. of  $\frac{1}{D^2 + a^2} \sin ax =$

$$(a) \frac{1}{2a} \sin ax$$

$$(b) \frac{1}{2a^2} \cos ax$$

$$(c) \frac{1}{2a} \cos ax$$

$$(d) -\frac{1}{2a} \cos ax$$

Answer: **(d)**

8  $\frac{1}{D^2 + D} \sin x$

$$(a) \frac{1}{2}(\cos x + \sin x)$$

$$(b) -\frac{1}{2}(\cos x + \sin x)$$

$$(c) \frac{1}{2}(\cos x - \sin x)$$

$$(d) \frac{1}{2}(-\cos x + \sin x)$$

Answer: **(b)**

9 The P.I. of  $\frac{1}{D-3} 2^x$

$$(a) \frac{2^x}{x \log 2 + 3}$$

$$(b) \frac{2^x}{x \log 2 - 3}$$

$$(c) \frac{2^x}{\log 2 - 3}$$

$$(d) \frac{2^x}{\log 2 + 3}$$

Answer: **(C)**

10 The complete solution of  $(D-1)^2 y = 0$  is  $y =$

$$(a) (c_1 + c_2 x) e^{-x}$$

$$(b) (c_1 + c_2 x) \cos x$$

$$(c) (c_1 + c_2 x) e^x$$

$$(d) c_1 e^x + c_2 e^x$$

Answer: **(a)**

**c. True or False questions:**

1. The rule to find the particular integral  $y_p = \frac{1}{f(D)} e^{ax}$  is replace always  $D$  in

$f(D)$  by  $a$

Answer: **F**

$$2. \quad y_p = \frac{\cos ax}{D^2 + a^2} = \frac{x}{2a} \sin ax$$

Answer: **T**

$$3. \quad y_p = \frac{1}{D^2 - 2D + 2} e^x = x e^x$$

Answer: **F**

$$4. \quad y_p = \frac{1}{D^4 - 1} e^x \cos x = \frac{1}{5} e^x \cos x$$

Answer: **F**

$$5. \quad y_p = \frac{1}{(D^2 - 1)} (-1) = A e^x + B e^{-x} - 1$$

Answer: **T**

6.  $y_p = \frac{1}{(D^3 - 3D)} e^x = A + B e^{3x} - e^x$  Answer: **F**

7.  $y_p = \frac{1}{D} x = A + \frac{1}{2} x^2$  Answer: **T**

6.  $y_p = \frac{1}{D^2} \sin x = (A + Bx) - \cos x$  Answer: **F**

7.  $y_p = \frac{1}{(2-D)} x e^x = A e^{2x} + (x+1) e^x$  Answer: **T**

8. If the roots are real and different then A.E. is  $(c_1 + c_2 x) e^{mx}$  Answer: **F**

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#### 4.2.5 Review Questions:

##### d. Objective type of questions (Very short notes):

1. Solve  $\frac{d^2 y}{dx^2} - 8 \frac{dy}{dx} + 15y = 0$

**Solution:** the roots of the auxiliary equation are 3,5

The general solution is  $y = c_1 e^{3x} + c_2 e^{5x}$

2. Solve  $\frac{d^2 y}{dx^2} + y = 0, y(0) = 2, y\left(\frac{\pi}{2}\right) = -2$

**Solution:** the roots of the auxiliary equation are  $\pm i$

The general solution is  $y = c_1 \cos x + c_2 \sin x$

Applying  $y(0) = 2, y\left(\frac{\pi}{2}\right) = -2$ , we get,  $c_1 = 2, c_2 = -2$

3. Find the Particular Integral of  $(D^3 + 2D^2 + D)y = e^{2x}$

**Solution:** P.I. is given by,  $\frac{1}{D^3 + 2D^2 + D} e^{2x} = \frac{1}{18} e^{2x}$

4. Solve  $(D^2 + D + 1)y = x^4$

**Solution:** the roots of the auxiliary equation are  $\frac{-1 \pm \sqrt{3}i}{2}$

The complementary function is  $e^{-\frac{x}{2}} \left( c_1 \cos \left( \frac{\sqrt{3}}{2} x \right) + c_2 \sin \left( \frac{\sqrt{3}}{2} x \right) \right)$

And the particular integral is  $\frac{1}{D^2 + D + 1} x^4 = x^4 - 4x^3 + 24x - 72$

Hence the general solution is,

$$y = e^{-\frac{x}{2}} \left( c_1 \cos \left( \frac{\sqrt{3}}{2} x \right) + c_2 \sin \left( \frac{\sqrt{3}}{2} x \right) \right) + x^4 - 4x^3 + 24x - 72$$

5. Solve  $(D^2 + 2D + 2)y = e^{-x} + \sin 2x$

**Solution:** The roots of the auxiliary equation are  $-1 \pm i$  and the complementary function is

$$e^{-x} (c_1 \cos x + c_2 \sin x)$$

The particular integral for  $e^{-x}$  is  $e^{-x}$  and the particular integral for  $\sin 2x$  is

$$-\frac{1}{10} (2 \cos 2x + \sin 2x)$$

Hence the general solution is  $y = e^{-x} (c_1 \cos x + c_2 \sin x) - \frac{1}{10} (2 \cos 2x + \sin 2x)$

6. Solve  $(D^2 - 4)y = x \sinh x + 54x + 8$

**Solution:** The roots of the auxiliary equation are  $2, -2$  and the complementary function is

$$c_1 e^{2x} + c_2 e^{-2x}$$

The particular integral for  $x \sinh x$  is  $-\frac{e^x}{6} \left[ x + \frac{2}{3} \right] + \frac{e^{-x}}{6} \left[ x - \frac{2}{3} \right]$  and the particular integral of  $54x$  is  $-\frac{27}{2}x$  and the particular integral of  $8$  is  $-2$

Therefore, the general solution is,  $y = c_1 e^{2x} + c_2 e^{-2x} - \frac{e^x}{6} \left[ x + \frac{2}{3} \right] + \frac{e^{-x}}{6} \left[ x - \frac{2}{3} \right] - \frac{27}{2}x - 2$

7. Solve  $(D^4 - 1)y = e^x \cos x$

**Solution:** The roots of the auxiliary equation are  $1, -1, \pm i$  and the complementary function is

$$c_1 e^x + c_2 e^{-x} + c_3 \cos x + c_4 \sin x$$

And the particular integral of  $e^x \cos x$  is  $-\frac{e^x \cos x}{5}$

The general solution is,  $y = c_1 e^x + c_2 e^{-x} + c_3 \cos x + c_4 \sin x - \frac{e^x \cos x}{5}$

8. Solve  $(D^2 - 4D + 4)y = 8x^2 e^{2x} \sin 2x$

**Solution:** The roots of the auxiliary equation is  $2, 2$  and the complementary function is

$$(c_1 + c_2 x) e^{2x}$$

And the particular integral of  $8x^2 e^{2x} \sin 2x$  is  $-2e^{2x} \left[ 2x \cos 2x + \left( x^2 - \frac{3}{2} \right) \sin 2x \right]$

The general solution is,  $y = (c_1 + c_2 x) e^{2x} - 2e^{2x} \left[ 2x \cos 2x + \left( x^2 - \frac{3}{2} \right) \sin 2x \right]$

9. Solve  $(D^2 + 4)y = x \sin x$

**Solution:** The roots of auxiliary equation is  $\pm 2i$  and the complementary function is

$$c_1 \cos 2x + c_2 \sin 2x$$

The particular integral of  $x \sin x$  is  $\frac{1}{3} \left( -\frac{2}{3} \cos x + x \sin x \right)$

Therefore, the general solution is  $y = c_1 \cos 2x + c_2 \sin 2x + \frac{1}{3} \left( -\frac{2}{3} \cos x + x \sin x \right)$

10. Solve  $(D^2 - 5D + 6)y = e^x \sin x$

**Solution:** The roots of the auxiliary equation is 3, 2 and the complementary function is

$$c_1 e^{3x} + c_2 e^{2x}$$

The particular integral of  $e^x \sin x$  is  $\frac{e^x}{10} (\sin x + 3 \cos x)$

Therefore, the general solution is  $y = c_1 e^{3x} + c_2 e^{2x} + \frac{e^x}{10} (\sin x + 3 \cos x)$

**e. Analytical type questions:**

1. Solve  $(D^2 + 1)y = x^2 \cosh x$

2. Find the complete solution of  $(D^4 + 16)y = 0$

3. Find  $\frac{1}{D^2 - 4D + 4} x^2 \sin x$

4. Find the solution of  $\frac{d^2 y}{dx^2} + 4y = \sin 3x + \cos 2x$

5. Find the solution of  $\frac{d^2 y}{dx^2} + 5 \frac{dy}{dx} - 6y = \sin 4x \cos x$

6. Find the complete solution of  $y'' + 5y' - 6y = \sin 4x \sin x$

7. Find the complete solution of  $y'' + 2y = x e^{3x} + e^x \cos 2x$

8. Find the solution of  $\frac{d^2 y}{dx^2} - 4 \frac{dy}{dx} + 3y = 2x e^{3x} + 3e^x \cos 2x$

9. Find the complete solution of  $y'' + 4y = e^x \sin^2 x$

10. Find the complete solution of  $\frac{d^2 y}{dx^2} - 4 \frac{dy}{dx} - 5y = e^{2x} + 3 \cos(4x + 3)$

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a. **Essay type Questions:**

1. Solve  $(D^2 + 4D + 13)y = 2e^{-x}$

Solution: Given  $(D^2 + 4D + 13)y = 2e^{-x}$

A.E. is  $m^2 + 4m + 13 = 0$ , here  $a = 1, b = 4, c = 13$

$$\therefore m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-4 \pm \sqrt{16 - 52}}{2} = \frac{-2 \pm i6}{2} = -1 \pm i3$$

$$\therefore \text{C.F.} = e^{ax} (c_1 \cos \beta x + c_2 \sin \beta x) = e^{-x} ((c_1 \cos 3x + c_2 \sin 3x))$$

$$\text{P.I.} = \frac{1}{D^2 + 4D + 13} (2e^{-x}) = 2 \frac{1}{(-1)^2 + 4(-1) + 13} e^{-x} = 2 \frac{1}{10} e^{-x} = \frac{1}{5} e^{-x}$$

The general solution is  $y = \text{C.F.} + \text{P.I.} = e^{-x} ((c_1 \cos 3x + c_2 \sin 3x)) + \frac{1}{5} e^{-x}$

2. Solve

$$(D^3 + 3D - 4)y = \sinh 2x + 7$$

Solution: Given  $(D^3 + 3D - 4)y = \sinh 2x + 7$

A.E. is  $m^3 + 3m^2 - 4 = 0, m = -1, 2, 2.$

$$\text{C.F.} = c_1 e^{-x} + (c_2 + c_3 x) e^{2x}$$

$$\text{P.I. is } y_p = \frac{\left(\frac{e^{2x} - e^{-2x}}{2}\right) + 7}{D^3 + 3D - 4} = \frac{e^{2x}}{2(D^3 + 3D^2 - 4)} - \frac{e^{-2x}}{2(D^3 + 3D^2 - 4)} + \frac{7}{D^3 + 3D^2 - 4}$$

$$y_{p_1} = \frac{1}{2} \frac{e^{2x}}{(D+1)(D-2)^2} \quad [\text{put } D = 2]$$

$$= \frac{1}{2} \frac{x^2 e^{2x}}{3} = \frac{x^2 e^{2x}}{12}$$

$$y_{p_2} = \frac{\frac{-1}{2} e^{-2x}}{(D+1)(D-2)^2} \quad [\text{Put } D = -2]$$

$$= \frac{1}{2} \frac{e^{-2x}}{(-1)(16)} = \frac{e^{-2x}}{32}$$

$$y_{p_3} = \frac{7}{D^3 + 3D^2 - 4} = 7 \frac{1}{D^3 + 3D^2 - 4} e^{0.x}$$

$$= \frac{-7}{4}$$

General solution is  $y = y_c + y_{p_1} + y_{p_2} + y_{p_3}$

$$\therefore y = c_1 e^{-x} + (c_2 + c_3 x) e^{2x} + \frac{x^2 e^{2x}}{12} + \frac{e^{-2x}}{32} - \frac{7}{4}$$

3. Solve  $(D^3 - 3D^2 + 4)y = (1 + e^{-x})^3$

Solution: Given  $(D^3 - 3D^2 + 4)y = (1 + e^{-x})^3$

A.E. is  $m^3 - 3m^2 + 4 = 0$ ,

i.e.  $(m+1)(m-2)^2 = 0$ ,  $m = -1, 2, 2$ ,

$$(D^2 + 5D - 6)y = \sin 4x \sin x \quad \text{C.F.} = y = c_1 e^{-x} + (c_2 + c_3 x) e^{2x}$$

$$\begin{aligned} \text{Now P.I is } y_p &= \frac{1 + e^{-3x} + 3e^{-2x} + 3e^{-x}}{D^3 - 3D^2 + 4} \\ &= \frac{1}{D^3 - 3D^2 + 4} + \frac{e^{-3x}}{D^3 - 3D^2 + 4} + \frac{3e^{-2x}}{D^3 - 3D^2 + 4} + \frac{3e^{-x}}{D^3 - 3D^2 + 4} \\ &= y_{p_1} + y_{p_2} + y_{p_3} + y_{p_4} \end{aligned}$$

$$\text{Now } y_{p_1} = \frac{1}{D^3 - 3D^2 + 4} \quad \text{put } D = 0$$

$$= \frac{1}{4}$$

$$y_{p_2} = \frac{e^{-3x}}{D^3 - 3D^2 + 4} \quad (\text{put } D = -3)$$

$$= \frac{e^{-3x}}{-27 + -27 + 4} = \frac{e^{-3x}}{4}$$

$$y_{p_3} = \frac{3e^{-2x}}{D^3 - 3D^2 + 4} \quad (\text{put } D = -2)$$

$$= \frac{3e^{-2x}}{-8 - 12 + 4} = \frac{3e^{-2x}}{16}$$

$$y_{p_4} = \frac{3e^{-x}}{(D+1)(D^2 - 4D + 4)} \quad (\text{put } D = -1)$$

$$= \frac{xe^{-x}}{1 + 4 + 4} = \frac{xe^{-x}}{3}$$

General solution is  $y = c_1 e^{-x} + (c_2 + c_3 x) e^{2x} + \frac{1}{4} + \frac{e^{-3x}}{4} + \frac{3e^{-2x}}{16} + \frac{x e^{-x}}{3}$

4. Find the complete solution of

$$(D^2 + 5D - 6)y = \sin 4x \sin x$$

Solution: Given equation is

A.E. is  $m^2 + 5m - 6 = 0$ ,  $m = -6, 1$

The roots are real and distinct .

$\therefore$  C.F. is  $y_c = c_1 e^{-6x} + c_2 e^x$

P.I. is  $y_p = \frac{\sin 4x \sin x}{D^2 + 5D - 6} = \frac{1}{2} \cdot \frac{2 \sin 4x \sin x}{D^2 + 5D - 6}$

$$= \frac{1}{2} \left[ \frac{\cos 3x - \cos 5x}{D^2 + 5D - 6} \right] = \frac{1}{2} \left[ \frac{\cos 3x}{D^2 + 5D - 6} - \frac{\cos 5x}{D^2 + 5D - 6} \right]$$

$$y_{p_1} = \frac{1}{2} \left[ \frac{\cos 3x}{D^2 + 5D - 6} \right] \quad (\text{put } D^2 = -9)$$

$$= \frac{1}{2} \left[ \frac{\cos 3x}{5D - 15} \right] = \frac{1}{10} \left[ \frac{\cos 3x}{D - 3} \right] = \frac{1}{10} \left[ \frac{(D+3)\cos 3x}{D^2 - 9} \right] \quad (\text{put } D^2 = -9)$$

$$= \frac{1}{10} \left[ \frac{-3 \sin 3x + 3 \cos 3x}{-18} \right] = \frac{-\sin 3x + \cos 3x}{-60}$$

$$= \frac{\sin 3x - \cos 3x}{60}$$

$$y_{p_2} = -\frac{1}{2} \cdot \frac{\cos 5x}{D^2 + 5D - 6} \quad (\text{put } D^2 = -25)$$

$$= -\frac{1}{2} \cdot \frac{\cos 5x}{-25 + 5D - 6} = -\frac{1}{2} \cdot \frac{\cos 5x}{5D - 31}$$

$$= \frac{-25 \sin 5x + 31 \cos 5x}{2(-625 - 961)}$$

$$= \frac{25 \sin 5x - 31 \cos 5x}{3172}$$

General solution is  $y = c_1 e^{-6x} + c_2 e^x + \frac{\sin 3x - \cos 3x}{60} + \frac{25 \sin 5x - 31 \cos 5x}{3172}$

5. Solve the differential equation  $(D^3 + 1)y = \cos(2x - 1)$

Solution: Given differential equation is  $(D^3 + 1)y = \cos(2x - 1)$

$$\text{A.E. is } m^3 + 1 = 0$$

$$\Rightarrow (m+1)(m^2 - m + 1) = 0$$

$$\Rightarrow m = -1, \frac{1 \pm i\sqrt{3}}{2}$$

$$\therefore \text{C.F.} = c_1 e^{-x} + e^{x/2} \left[ c_2 \cos \frac{\sqrt{3}x}{2} + c_3 \sin \frac{\sqrt{3}x}{2} \right]$$

$$\text{P.I.} = \frac{1}{D^3 + 1} \cos(2x - 1)$$

Putting  $D^2 = -4$

$$\text{P.I.} = \frac{1}{1 - 4D} \cos(2x - 1) = \frac{1 + 4D}{1 - 16D^2} [\cos(2x - 1)]$$

Again putting  $D^2 = -4$

$$\text{P.I.} = \frac{1}{65} [\cos(2x - 1) - 8 \sin(2x - 1)]$$

General Solution is  $y = \text{C.F.} + \text{P.I.}$

$$= c_1 e^{-x} + e^{x/2} \left[ c_2 \cos \frac{\sqrt{3}x}{2} + c_3 \sin \frac{\sqrt{3}x}{2} \right] + \frac{1}{65} [\cos(2x - 1) - 8 \sin(2x - 1)]$$

6. Solve  $(D^3 - D^2 - 6D)y = 1 + x^2$

Solution: Given equation is  $(D^3 - D^2 - 6D)y = 1 + x^2$

$$\text{A.E. is } m^3 - m^2 - 6m = 0, \quad m(m^2 - m - 6) = 0$$

$$\Rightarrow m(m-3)(m+2) = 0,$$

$$\therefore m = 0, 3, -2$$

$$\therefore \text{C.F.} = y_c = c_1 e^{0x} + c_2 e^{3x} + c_3 e^{-2x}$$

$$\text{P.I.} = \frac{1+x^2}{D^3 - D^2 - 6D} = -\frac{1}{6D} \frac{1+x^2}{\left(1 + \frac{D-D^2}{6}\right)}$$

$$= -\frac{1}{6D} \left( 1 + \frac{D-D^2}{6} + \left(\frac{D-D^2}{6}\right)^2 - \dots \right) (1+x^2)$$

$$= \frac{-1}{6} \cdot \frac{1}{D} \left[ 1 - \left(\frac{D-D^2}{6}\right) + \left(\frac{D-D^2}{6}\right)^2 - \dots \right] (1+x^2)$$

$$= \frac{-1}{6} \cdot \frac{1}{D} \left[ 1 - \frac{D}{6} + \frac{D^2}{6} + \frac{D^2 + D^4 - 2D^3}{36} + \dots \right] (1+x^2)$$

$$= \frac{-1}{6} \cdot \left[ \frac{1}{D} - \frac{1}{6} + \frac{D}{6} + \frac{D}{36} - \frac{2D^2}{36} \right] (1+x^2) \text{ (ignoring the terms containing}$$

$D^3, D^4, \dots$ )

$$= \frac{-1}{6} \cdot \left[ x + \frac{x^3}{3} - \frac{1}{6} - \frac{1}{6}x^2 + \frac{2x}{6} + \frac{2x}{36} - \frac{2}{36} \right] (2)$$

$$= \frac{-1}{6} \left[ \frac{25x}{18} - \frac{x^2}{6} + \frac{x^3}{3} - \frac{1}{6} - \frac{1}{9} \right]$$

General solution is  $y = y_c + y_p$

$$y = c_1 + c_2 e^{3x} + c_3 e^{-2x} - \frac{1}{6} \left[ \frac{25x}{18} - \frac{x^2}{6} + \frac{x^3}{3} - \frac{1}{6} - \frac{1}{9} \right].$$

7. Solve  $(D^2 + 3D + 2)y = xe^x \sin x$

Solution: Given Equation is  $(D^2 + 3D + 2)y = xe^x \sin x$

$$\text{A.E. is } m^2 + 3m + 2 = 0 \Rightarrow (m+2)(m+1) = 0 \Rightarrow m = -2, -1$$

$$\therefore \text{C.F.} = c_1 e^{-2x} + c_2 e^{-x}$$

$$\text{Now P.I.} = \frac{xe^x \sin x}{D^2 + 3D + 2} = e^x \frac{x \sin x}{(D+1)^2 + 3(D+1) + 2}$$

$$= e^x \frac{x \sin x}{D^2 + 5D + 6}$$

$$= e^x \left[ x - \frac{2D+5}{D^2 + 5D + 6} \right] \frac{\sin x}{-1 + 5D + 6}$$

$$= e^x \left[ x - \frac{2D+5}{D^2 + 5D + 6} \right] \frac{\sin x}{5(D+1)}$$

$$= \frac{e^x}{5} \left[ x - \frac{2D+5}{D^2 + 5D + 6} \right] \frac{(D-1)\sin x}{D^2 - 1}$$

$$= \frac{e^x}{5} \left[ x - \frac{2D+5}{D^2 + 5D + 6} \right] \frac{\cos x - \sin x}{-2}$$

$$= \frac{e^x x (\sin x - \cos x)}{10} + \frac{e^x (2D+5)(\cos x - \sin x)}{10(D^2 + 5D + 6)}$$

$$= \frac{e^x x (\sin x - \cos x)}{10} + \frac{e^x}{10} \left[ \frac{-2 \sin x - 2 \cos x + 5 \cos x - 5 \sin x}{D^2 + 5D + 6} \right]$$

$$\begin{aligned}
&= \frac{e^x x (\sin x - \cos x)}{10} + \frac{e^x}{10} \left[ \frac{3 \cos x - 7 \sin x}{D^2 + 5D + 6} \right] \\
&= \frac{e^x x (\sin x - \cos x)}{10} + \frac{e^x}{10} \left[ \frac{3 \cos x - 7 \sin x}{-1 + 5D + 6} \right] \\
&= \frac{e^x x (\sin x - \cos x)}{10} + \frac{e^x}{10} \frac{D-1}{5(D+1)(D-1)} (3 \cos x - 7 \sin x) \\
&= \frac{e^x x (\sin x - \cos x)}{10} + \frac{e^x}{50} (D-1) \left( \frac{3 \cos x - 7 \sin x}{-2} \right) \\
&= \frac{e^x x (\sin x - \cos x)}{10} - \frac{e^x}{100} (-3 \sin x - 7 \cos x - 3 \cos x + 7 \sin x) \\
&= \frac{e^x x (\sin x - \cos x)}{10} - \frac{e^x}{100} (4 \sin x - 10 \cos x)
\end{aligned}$$

General solution is  $y = c_1 e^{-2x} + c_2 e^{-x} + \frac{e^x x (\sin x - \cos x)}{10} - \frac{e^x}{100} (4 \sin x - 10 \cos x)$

8. Solve  $\frac{d^2 y}{dx^2} + a^2 y = \sec ax$

Solution: Given equation in the standard form is  $(D^2 + a^2)y = \sec ax$

Which is a non-homogeneous and the homogeneous equation is

$$(D^2 + a^2)y = 0$$

A.E. is  $m^2 + a^2 = 0, m = \pm ia$

$\therefore$  C.F. =  $c_1 \cos ax + c_2 \sin ax$

$$\text{P.I.} = \frac{1}{D^2 + a^2} \sec ax = \frac{1}{2ia} \left[ \frac{1}{D - ia} \sec ax - \frac{1}{D + ia} \sec ax \right]$$

Now consider  $\frac{1}{D - ia} \sec ax = e^{iax} \int \sec ax e^{-iax} dx$

$$= e^{iax} \int \frac{\cos ax - i \sin ax}{\cos ax} dx$$

$$= e^{iax} \int (1 - i \tan ax) dx = e^{iax} \left( x + \frac{i}{a} \log \cos ax \right)$$

Changing  $i$  to  $-i$  we have  $\frac{1}{D+ia} \sec ax = e^{-iax} \left( x + \frac{i}{a} \log \cos ax \right)$

$$\begin{aligned} \text{P.I.} &= \frac{1}{2ia} \left[ e^{iax} \left( x + \frac{i}{a} \log \cos ax \right) - e^{-iax} \left( x + \frac{i}{a} \log \cos ax \right) \right] \\ &= \frac{x}{a} \left[ \frac{e^{iax} - e^{-iax}}{2i} \right] + \frac{1}{a^2} \log \cos ax \left[ \frac{e^{iax} + e^{-iax}}{2} \right] \\ &= \frac{x}{a} \sin ax + \frac{1}{a} \log (\cos ax) \cos ax \end{aligned}$$

Hence the general solution is  $y = c_1 \cos ax + c_2 \sin ax + \frac{x}{a} \sin ax + \frac{1}{a} \cos ax \log (\cos ax)$

9. Solve  $(D+1)^2 y = x \cos x$

Solution: Given  $(D+1)^2 y = x \cos x$

A.E. is  $(m+1)^2 = 0, m = -1, -1$

C.F. =  $(c_1 + c_2 x) e^{-x}$

$$\begin{aligned} \text{P.I.} &= \frac{1}{(D+1)^2} x \cos x \\ &= \left[ x - \frac{1}{f(D)} f'(D) \right] \frac{1}{f(D)} \cos x \\ &= \left[ x - \frac{2(D+1)}{(D+1)^2} \right] \frac{1}{-1^2 + 2D + 1} \cos x \\ &= \left[ x - \frac{2}{D+1} \right] \frac{\sin x}{2} \\ &= \frac{x}{2} \sin x - \frac{D-1}{D^2-1} \sin x \\ &= \frac{x}{2} \sin x + \frac{\cos x - \sin x}{2} \end{aligned}$$

Hence the general solution is  $y = (c_1 + c_2 x) e^{-x} + \frac{x}{2} \sin x + \frac{\cos x - \sin x}{2}$

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**b. Problems:**

1. Solve  $(D^2 - 1)y = x \sin x$
2. Solve the differential equation  $(D^4 - 1)y = e^x \cos x$
3. Solve  $y''' - y'' - y' + y = 1 + x^2$
4. Solve  $(D^2 + 4)y = \sin 3x + \cos 2x$
5. Solve  $(D^2 + 2D + 1)y = xe^x \sin x$
6. Solve  $(D^2 - 4D + 3)y = \sin 3x \cos 2x$
7. Solve  $(D^2 - 1)y = x \sin 3x + \cos x$
8. Solve  $(D^2 + 2)y = x^2 e^{3x} + e^x \cos 2x$
9. Solve  $D^2(D^2 + 4)y = 320(x^3 + 2x^2 + e^x)$
10. Solve  $y''' - 2y' + y = 1 + e^x$

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**4.2.6 Assignments:**

1. Solve  $(D^4 - 2D^3 - 3D^2 + 4D + 4)y = 0$
2. Solve  $(D^2 + 4)y = \tan 2x$
3. Solve  $(D^2 - 3D + 2)y = \cosh x$
4. Solve  $(D^3 + 4D)y = 5 + \sin 2x$
5. Solve  $\frac{d^2y}{dx^2} + y = e^{-x} + x^3$
6. Solve  $(D^2 + 1)y = \sin x \sin 2x + e^x x^2$

7. Solve  $\frac{d^2y}{dx^2} - y = x \sin x + (1 + x^2)e^x$

8. Solve  $(D^2 + 1)x = t \cos 2t$ , given  $x = 0, \frac{dx}{dt} = 0$  at  $t = 0$

9. Solve  $(D^4 + 2D^2 + 1)y = x^2 \cos^2 x$

10. Find the distance from the center at which the velocity in S.H.M. will be (i) half  
(ii) one-third of the maximum.

**GATE Questions (Where relevant):**

1. For the differential equations  $\frac{d^2x}{dt^2} + 6\frac{dx}{dt} + 8x = 0$  with initial conditions  $X(0) = 1$

$\left. \frac{dx}{dt} \right|_{t=0} = 0$  the solution is

(a)  $x(t) = 2e^{-6t} - e^{-2t}$

(b)  $x(t) = 2e^{-2t} - e^{-4t}$

(c)  $x(t) = -e^{-6t} + 2e^{-4t}$

(d)  $x(t) = e^{-2t} - 2e^{-4t}$

**Answer: (c)**

2. The solution for the differential equation  $\frac{d^2x}{dt^2} = -9x$  with initial condition  $X(0) = 1$  and

$\left. \frac{dx}{dt} \right|_{t=0} = 1$  is

(a)  $t^2 + t + 1$

(b)  $\sin 3t + \frac{1}{3} \cos 3t + \frac{2}{3}$

(c)  $\frac{1}{3} \sin 3t + \cos 3t$

(d)  $\cos 3t + t$

**Answer: (c)**

3. If the characteristic of the given differential equation  $\frac{d^2 y}{dx^2} + 2a \frac{dy}{dx} + y = 0$  has two equal

roots then the value of a are

- (a)  $\pm 1$
- (b)  $0,0$
- (c)  $\pm i$
- (d)  $\pm \frac{1}{2}$

**Answers: (a)**

4. If a and b are constants, the most general solution of the differential equation

$$\frac{d^2 x}{dt^2} + 2 \frac{dx}{dt} + x = 0 \text{ is}$$

- (a)  $ae^{-t}$
- (b)  $ae^{-t} + bte^{-t}$
- (c)  $ae^t + bte^{-t}$
- (d)  $ae^{-2t}$

**Answer: (b)**

5. With initial values  $y(0) = y'(0) = 1$ , the solution of the differential equation

$$\frac{d^2 y}{dx^2} + 4 \frac{dy}{dx} + 4 = 0 \text{ at } x = 1 \text{ is}$$

- (a) 0.53 to 0.55
- (b) 0.52 to 0.54
- (c) 0.50 to 0.55
- (d) 0.45 to 0.50

**Answer: (b)**

6. The solution of  $\frac{d^3 y}{dx^3} + \frac{dy}{dx} = 0$  is

(a)  $C_1 + C_2 \cos x + C_3 \sin x$

(b)  $C_1 \cos x + C_2 \sin x$

(c)  $C_1 + (C_2 + C_3 x) \cos x$

(d)  $C_1 + (C_2 + C_3 x) \sin x$

**Answer: A**

7. The solution of  $x \frac{d^3 y}{dx^3} + 4 \frac{dy}{dx} = 0$  is

(a)  $y(x) = C_1 - C_2 x^{-3}$

(b)  $y(x) = C_1 + C_2 e^{-3x}$

(c)  $y(x) = C_1 - C_2 e^{-3x}$

(d)  $y(x) = C_1 + C_2 x^{-3}$

**Answer: (a)**

8. The differential equation is  $\frac{d^2 y}{dx^2} + y = \sin^2 x$ . The particular integral

(a)  $\cos^2 x - \frac{\cos^4 x}{4} + \frac{\sin^4 x}{3}$

(b)  $\cos^2 x + \frac{\cos^4 x}{4} + \frac{\sin^4 x}{3}$

(c)  $\frac{\cos^2 x}{3} - \cos^4 x + \sin^4 x$

(d)  $\frac{\cos^2 x}{3} + \cos^4 x - \sin^4 x$

**Answer: (a)**

9. The differential equation is  $x^2 \frac{d^2 y}{dx^2} - 6y = 0$ . The particular integral is

(a)  $C_1 e^{-2x} + C_2 e^{3x}$

(b)  $C_1 x^2 + C_2 x^{-3}$

(c)  $C_1 x^{-2} + C_2 x^3$

(d)  $C_1 e^{2x} + C_2 e^{-3x}$

**Answer: (c)**

10. The differential equation is  $\frac{d^3 y}{dx^3} + 3 \frac{d^2 y}{dx^2} - 5 \frac{dy}{dx} - 39y = 30 \cos x$ . The complementary function is

(a)  $C_1 e^{-3x} + C_2 x e^{-3x} + C_3 e^{2x}$

(b)  $C_1 e^{3x} + C_2 x e^{3x} + C_3 e^{-2x}$

(c)  $C_1 e^{-3x} + C_2 e^{3x} \cos 2x + C_3 e^{3x} \sin 2x$

(d)  $C_1 e^{3x} + C_2 e^{-3x} \cos 2x + C_3 e^{-3x} \sin 2x$

**Answer: (d)**

#### 4.2.7 Review Questions:

**d. Objective type of questions (Very short notes):**

1. Solve  $\frac{d^2 y}{dx^2} - 8 \frac{dy}{dx} + 15y = 0$

**Solution:** the roots of the auxiliary equation are 3, 5

The general solution is  $y = c_1 e^{3x} + c_2 e^{5x}$

2. Solve  $\frac{d^2 y}{dx^2} + y = 0$ ,  $y(0) = 2$ ,  $y\left(\frac{\pi}{2}\right) = -2$

**Solution:** the roots of the auxiliary equation are  $\pm i$

The general solution is  $y = c_1 \cos x + c_2 \sin x$

Applying  $y(0) = 2$ ,  $y\left(\frac{\pi}{2}\right) = -2$ , we get,  $c_1 = 2, c_2 = -2$

3. Find the Particular Integral of  $(D^3 + 2D^2 + D)y = e^{2x}$

**Solution:** P.I. is given by,  $\frac{1}{D^3 + 2D^2 + D} e^{2x} = \frac{1}{18} e^{2x}$

4. Solve  $(D^2 + D + 1)y = x^4$

**Solution:** the roots of the auxiliary equation are  $\frac{-1 \pm \sqrt{3}i}{2}$

The complementary function is  $e^{-\frac{x}{2}} \left( c_1 \cos\left(\frac{\sqrt{3}}{2}x\right) + c_2 \sin\left(\frac{\sqrt{3}}{2}x\right) \right)$

And the particular integral is  $\frac{1}{D^2 + D + 1} x^4 = x^4 - 4x^3 + 24x - 72$

Hence the general solution is,

$$y = e^{-\frac{x}{2}} \left( c_1 \cos\left(\frac{\sqrt{3}}{2}x\right) + c_2 \sin\left(\frac{\sqrt{3}}{2}x\right) \right) + x^4 - 4x^3 + 24x - 72$$

5. Solve  $(D^2 + 2D + 2)y = e^{-x} + \sin 2x$

**Solution:** The roots of the auxiliary equation are  $-1 \pm i$  and the complementary function is

$$e^{-x} (c_1 \cos x + c_2 \sin x)$$

The particular integral for  $e^{-x}$  is  $e^{-x}$  and the particular integral for  $\sin 2x$  is

$$-\frac{1}{10} (2 \cos 2x + \sin 2x)$$

Hence the general solution is  $y = e^{-x} (c_1 \cos x + c_2 \sin x) - \frac{1}{10} (2 \cos 2x + \sin 2x)$

6. Solve  $(D^2 - 4)y = x \sinh x + 54x + 8$

**Solution:** The roots of the auxiliary equation are  $2, -2$  and the complementary function is

$$c_1 e^{2x} + c_2 e^{-2x}$$

The particular integral for  $x \sinh x$  is  $-\frac{e^x}{6} \left[ x + \frac{2}{3} \right] + \frac{e^{-x}}{6} \left[ x - \frac{2}{3} \right]$  and the particular

integral of  $54x$  is  $-\frac{27}{2}x$  and the particular integral of  $8$  is  $-2$

Therefore, the general solution is,  $y = c_1 e^{2x} + c_2 e^{-2x} - \frac{e^x}{6} \left[ x + \frac{2}{3} \right] + \frac{e^{-x}}{6} \left[ x - \frac{2}{3} \right] - \frac{27}{2}x - 2$

7. Solve  $(D^4 - 1)y = e^x \cos x$

**Solution:** The roots of the auxiliary equation are  $1, -1, \pm i$  and the complementary function is

$$c_1 e^x + c_2 e^{-x} + c_3 \cos x + c_4 \sin x$$

And the particular integral of  $e^x \cos x$  is  $-\frac{e^x \cos x}{5}$

The general solution is,  $y = c_1 e^x + c_2 e^{-x} + c_3 \cos x + c_4 \sin x - \frac{e^x \cos x}{5}$

8. Solve  $(D^2 - 4D + 4)y = 8x^2 e^{2x} \sin 2x$

**Solution:** The roots of the auxiliary equation is  $2, 2$  and the complementary function is

$$(c_1 + c_2 x) e^{2x}$$

And the particular integral of  $8x^2 e^{2x} \sin 2x$  is  $-2e^{2x} \left[ 2x \cos 2x + \left( x^2 - \frac{3}{2} \right) \sin 2x \right]$

The general solution is,  $y = (c_1 + c_2 x) e^{2x} - 2e^{2x} \left[ 2x \cos 2x + \left( x^2 - \frac{3}{2} \right) \sin 2x \right]$

9. Solve  $(D^2 + 4)y = x \sin x$

**Solution:** The roots of auxiliary equation is  $\pm 2i$  and the complementary function is

$$c_1 \cos 2x + c_2 \sin 2x$$

The particular integral of  $x \sin x$  is  $\frac{1}{3} \left( -\frac{2}{3} \cos x + x \sin x \right)$

Therefore, the general solution is  $y = c_1 \cos 2x + c_2 \sin 2x + \frac{1}{3} \left( -\frac{2}{3} \cos x + x \sin x \right)$

10. Solve  $(D^2 - 5D + 6)y = e^x \sin x$

**Solution:** The roots of the auxiliary equation is 3, 2 and the complementary function is

$$c_1 e^{3x} + c_2 e^{2x}$$

The particular integral of  $e^x \sin x$  is  $\frac{e^x}{10} (\sin x + 3 \cos x)$

Therefore, the general solution is  $y = c_1 e^{3x} + c_2 e^{2x} + \frac{e^x}{10} (\sin x + 3 \cos x)$

**e. Analytical type questions:**

11. Solve  $(D^2 + 1)y = x^2 \cosh x$

12. Find the complete solution of  $(D^4 + 16)y = 0$

13. Find  $\frac{1}{D^2 - 4D + 4} x^2 \sin x$

14. Find the solution of  $\frac{d^2 y}{dx^2} + 4y = \sin 3x + \cos 2x$

15. Find the solution of  $\frac{d^2 y}{dx^2} + 5 \frac{dy}{dx} - 6y = \sin 4x \cos x$

16. Find the complete solution of  $y'' + 5y' - 6y = \sin 4x \sin x$

17. Find the complete solution of  $y'' + 2y = xe^{3x} + e^x \cos 2x$

18. Find the solution of  $\frac{d^2 y}{dx^2} - 4 \frac{dy}{dx} + 3y = 2xe^{3x} + 3e^x \cos 2x$

19. Find the complete solution of  $y'' + 4y = e^x \sin^2 x$

20. Find the complete solution of  $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} - 5y = e^{2x} + 3\cos(4x+3)$

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c. **Essay type Questions:**

11. Solve  $(D^2 + 4D + 13)y = 2e^{-x}$

Solution: Given  $(D^2 + 4D + 13)y = 2e^{-x}$

A.E. is  $m^2 + 4m + 13 = 0$ , here  $a = 1, b = 4, c = 13$

$$\therefore m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-4 \pm \sqrt{16 - 52}}{2} = \frac{-2 \pm i6}{2} = -1 \pm i3$$

$\therefore$  C.F. =  $e^{ax} (c_1 \cos \beta x + c_2 \sin \beta x) = e^{-x} ((c_1 \cos 3x + c_2 \sin 3x))$

$$\text{P.I.} = \frac{1}{D^2 + 4D + 13} (2e^{-x}) = 2 \frac{1}{(-1)^2 + 4(-1) + 13} e^{-x} = 2 \frac{1}{10} e^{-x} = \frac{1}{5} e^{-x}$$

The general solution is  $y = \text{C.F.} + \text{P.I.} = e^{-x} ((c_1 \cos 3x + c_2 \sin 3x)) + \frac{1}{5} e^{-x}$

12. Solve

$$(D^3 + 3D - 4)y = \sinh 2x + 7$$

Solution: Given  $(D^3 + 3D - 4)y = \sinh 2x + 7$

A.E. is  $m^3 + 3m^2 - 4 = 0, m = -1, 2, 2.$

C.F. =  $c_1 e^{-x} + (c_2 + c_3 x) e^{2x}$

$$\text{P.I. is } y_p = \frac{\left(\frac{e^{2x} - e^{-2x}}{2}\right) + 7}{D^3 + 3D - 4} = \frac{e^{2x}}{2(D^3 + 3D^2 - 4)} - \frac{e^{-2x}}{2(D^3 + 3D^2 - 4)} + \frac{7}{D^3 + 3D^2 - 4}$$

$$y_{p_1} = \frac{1}{2} \frac{e^{2x}}{(D+1)(D-2)^2} \quad [\text{put } D = 2]$$

$$= \frac{1}{2} \frac{x^2 e^{2x}}{3} = \frac{x^2 e^{2x}}{12}$$

$$y_{p_2} = \frac{\frac{-1}{2} e^{-2x}}{(D+1)(D-2)^2} \quad [\text{Put } D = -2]$$

$$= \frac{1}{2} \frac{e^{-2x}}{(-1)(16)} = \frac{e^{-2x}}{32}$$

$$y_{p_3} = \frac{7}{D^3 + 3D^2 - 4} = 7 \frac{1}{D^3 + 3D^2 - 4} e^{0.x}$$

$$= \frac{-7}{4}$$

General solution is  $y = y_c + y_{p_1} + y_{p_2} + y_{p_3}$

$$\therefore y = c_1 e^{-x} + (c_2 + c_3 x) e^{2x} + \frac{x^2 e^{2x}}{12} + \frac{e^{-2x}}{32} - \frac{7}{4}$$

13. Solve  $(D^3 - 3D^2 + 4)y = (1 + e^{-x})^3$

Solution: Given  $(D^3 - 3D^2 + 4)y = (1 + e^{-x})^3$

A.E. is  $m^3 - 3m^2 + 4 = 0$ ,

i.e.  $(m+1)(m-2)^2 = 0$ ,  $m = -1, 2, 2$ ,

$$(D^2 + 5D - 6)y = \sin 4x \sin x \quad \text{C.F.} = y = c_1 e^{-x} + (c_2 + c_3 x)e^{2x}$$

$$\begin{aligned} \text{Now P.I is } y_p &= \frac{1 + e^{-3x} + 3e^{-2x} + 3e^{-x}}{D^3 - 3D^2 + 4} \\ &= \frac{1}{D^3 - 3D^2 + 4} + \frac{e^{-3x}}{D^3 - 3D^2 + 4} + \frac{3e^{-2x}}{D^3 - 3D^2 + 4} + \frac{3e^{-x}}{D^3 - 3D^2 + 4} \\ &= y_{p_1} + y_{p_2} + y_{p_3} + y_{p_4} \end{aligned}$$

$$\begin{aligned} \text{Now } y_{p_1} &= \frac{1}{D^3 - 3D^2 + 4} \quad \text{put } D = 0 \\ &= \frac{1}{4} \end{aligned}$$

$$\begin{aligned} y_{p_2} &= \frac{e^{-3x}}{D^3 - 3D^2 + 4} \quad (\text{put } D = -3) \\ &= \frac{e^{-3x}}{-27 + -27 + 4} = \frac{e^{-3x}}{4} \end{aligned}$$

$$\begin{aligned} y_{p_3} &= \frac{3e^{-2x}}{D^3 - 3D^2 + 4} \quad (\text{put } D = -2) \\ &= \frac{3e^{-2x}}{-8 - 12 + 4} = \frac{3e^{-2x}}{16} \end{aligned}$$

$$\begin{aligned} y_{p_4} &= \frac{3e^{-x}}{(D+1)(D^2 - 4D + 4)} \quad (\text{put } D = -1) \\ &= \frac{xe^{-x}}{1 + 4 + 4} = \frac{xe^{-x}}{3} \end{aligned}$$

General solution is  $y = c_1 e^{-x} + (c_2 + c_3 x) e^{2x} + \frac{1}{4} + \frac{e^{-3x}}{4} + \frac{3e^{-2x}}{16} + \frac{x e^{-x}}{3}$

14. Find the complete solution of

$$(D^2 + 5D - 6)y = \sin 4x \sin x$$

Solution: Given equation is

A.E. is  $m^2 + 5m - 6 = 0$ ,  $m = -6, 1$

The roots are real and distinct .

$\therefore$  C.F. is  $y_c = c_1 e^{-6x} + c_2 e^x$

P.I. is  $y_p = \frac{\sin 4x \sin x}{D^2 + 5D - 6} = \frac{1}{2} \cdot \frac{2 \sin 4x \sin x}{D^2 + 5D - 6}$

$$= \frac{1}{2} \left[ \frac{\cos 3x - \cos 5x}{D^2 + 5D - 6} \right] = \frac{1}{2} \left[ \frac{\cos 3x}{D^2 + 5D - 6} - \frac{\cos 5x}{D^2 + 5D - 6} \right]$$

$$y_{p_1} = \frac{1}{2} \left[ \frac{\cos 3x}{D^2 + 5D - 6} \right] \quad (\text{put } D^2 = -9)$$

$$= \frac{1}{2} \left[ \frac{\cos 3x}{5D - 15} \right] = \frac{1}{10} \left[ \frac{\cos 3x}{D - 3} \right] = \frac{1}{10} \left[ \frac{(D+3)\cos 3x}{D^2 - 9} \right] \quad (\text{put } D^2 = -9)$$

$$= \frac{1}{10} \left[ \frac{-3 \sin 3x + 3 \cos 3x}{-18} \right] = \frac{-\sin 3x + \cos 3x}{-60}$$

$$= \frac{\sin 3x - \cos 3x}{60}$$

$$y_{p_2} = -\frac{1}{2} \cdot \frac{\cos 5x}{D^2 + 5D - 6} \quad (\text{put } D^2 = -25)$$

$$= -\frac{1}{2} \cdot \frac{\cos 5x}{-25 + 5D - 6} = -\frac{1}{2} \cdot \frac{\cos 5x}{5D - 31}$$

$$= \frac{-25 \sin 5x + 31 \cos 5x}{2(-625 - 961)}$$

$$= \frac{25 \sin 5x - 31 \cos 5x}{3172}$$

General solution is  $y = c_1 e^{-6x} + c_2 e^x + \frac{\sin 3x - \cos 3x}{60} + \frac{25 \sin 5x - 31 \cos 5x}{3172}$

15. Solve the differential equation  $(D^3 + 1)y = \cos(2x - 1)$

Solution: Given differential equation is  $(D^3 + 1)y = \cos(2x - 1)$

$$\text{A.E. is } m^3 + 1 = 0$$

$$\Rightarrow (m+1)(m^2 - m + 1) = 0$$

$$\Rightarrow m = -1, \frac{1 \pm i\sqrt{3}}{2}$$

$$\therefore \text{C.F.} = c_1 e^{-x} + e^{x/2} \left[ c_2 \cos \frac{\sqrt{3}x}{2} + c_3 \sin \frac{\sqrt{3}x}{2} \right]$$

$$\text{P.I.} = \frac{1}{D^3 + 1} \cos(2x - 1)$$

Putting  $D^2 = -4$

$$\text{P.I.} = \frac{1}{1 - 4D} \cos(2x - 1) = \frac{1 + 4D}{1 - 16D^2} [\cos(2x - 1)]$$

Again putting  $D^2 = -4$

$$\text{P.I.} = \frac{1}{65} [\cos(2x - 1) - 8 \sin(2x - 1)]$$

General Solution is  $y = \text{C.F.} + \text{P.I.}$

$$= c_1 e^{-x} + e^{x/2} \left[ c_2 \cos \frac{\sqrt{3}x}{2} + c_3 \sin \frac{\sqrt{3}x}{2} \right] + \frac{1}{65} [\cos(2x - 1) - 8 \sin(2x - 1)]$$

16. Solve  $(D^3 - D^2 - 6D)y = 1 + x^2$

Solution: Given equation is  $(D^3 - D^2 - 6D)y = 1 + x^2$

$$\text{A.E. is } m^3 - m^2 - 6m = 0, \quad m(m^2 - m - 6) = 0$$

$$\Rightarrow m(m-3)(m+2) = 0,$$

$$\therefore m = 0, 3, -2$$

$$\therefore \text{C.F.} = y_c = c_1 e^{0x} + c_2 e^{3x} + c_3 e^{-2x}$$

$$\text{P.I.} = \frac{1+x^2}{D^3 - D^2 - 6D} = -\frac{1}{6D} \frac{1+x^2}{\left(1 + \frac{D-D^2}{6}\right)}$$

$$= -\frac{1}{6D} \left( 1 + \frac{D-D^2}{6} + \left(\frac{D-D^2}{6}\right)^2 - \dots \right) (1+x^2)$$

$$= \frac{-1}{6} \cdot \frac{1}{D} \left[ 1 - \left(\frac{D-D^2}{6}\right) + \left(\frac{D-D^2}{6}\right)^2 - \dots \right] (1+x^2)$$

$$= \frac{-1}{6} \cdot \frac{1}{D} \left[ 1 - \frac{D}{6} + \frac{D^2}{6} + \frac{D^2 + D^4 - 2D^3}{36} + \dots \right] (1+x^2)$$

$$= \frac{-1}{6} \cdot \left[ \frac{1}{D} - \frac{1}{6} + \frac{D}{6} + \frac{D}{36} - \frac{2D^2}{36} \right] (1+x^2) \text{ (ignoring the terms containing}$$

$D^3, D^4, \dots$ )

$$= \frac{-1}{6} \cdot \left[ x + \frac{x^3}{3} - \frac{1}{6} - \frac{1}{6}x^2 + \frac{2x}{6} + \frac{2x}{36} - \frac{2}{36} \right] (2)$$

$$= \frac{-1}{6} \left[ \frac{25x}{18} - \frac{x^2}{6} + \frac{x^3}{3} - \frac{1}{6} - \frac{1}{9} \right]$$

General solution is  $y = y_c + y_p$

$$y = c_1 + c_2 e^{3x} + c_3 e^{-2x} - \frac{1}{6} \left[ \frac{25x}{18} - \frac{x^2}{6} + \frac{x^3}{3} - \frac{1}{6} - \frac{1}{9} \right].$$

17. Solve  $(D^2 + 3D + 2)y = xe^x \sin x$

Solution: Given Equation is  $(D^2 + 3D + 2)y = xe^x \sin x$

$$\text{A.E. is } m^2 + 3m + 2 = 0 \Rightarrow (m+2)(m+1) = 0 \Rightarrow m = -2, -1$$

$$\therefore \text{C.F.} = c_1 e^{-2x} + c_2 e^{-x}$$

$$\text{Now P.I.} = \frac{xe^x \sin x}{D^2 + 3D + 2} = e^x \frac{x \sin x}{(D+1)^2 + 3(D+1) + 2}$$

$$= e^x \frac{x \sin x}{D^2 + 5D + 6}$$

$$= e^x \left[ x - \frac{2D+5}{D^2 + 5D + 6} \right] \frac{\sin x}{-1 + 5D + 6}$$

$$= e^x \left[ x - \frac{2D+5}{D^2 + 5D + 6} \right] \frac{\sin x}{5(D+1)}$$

$$= \frac{e^x}{5} \left[ x - \frac{2D+5}{D^2 + 5D + 6} \right] \frac{(D-1)\sin x}{D^2 - 1}$$

$$= \frac{e^x}{5} \left[ x - \frac{2D+5}{D^2 + 5D + 6} \right] \frac{\cos x - \sin x}{-2}$$

$$= \frac{e^x x (\sin x - \cos x)}{10} + \frac{e^x (2D+5)(\cos x - \sin x)}{10(D^2 + 5D + 6)}$$

$$= \frac{e^x x (\sin x - \cos x)}{10} + \frac{e^x}{10} \left[ \frac{-2 \sin x - 2 \cos x + 5 \cos x - 5 \sin x}{D^2 + 5D + 6} \right]$$

$$\begin{aligned}
&= \frac{e^x x (\sin x - \cos x)}{10} + \frac{e^x}{10} \left[ \frac{3 \cos x - 7 \sin x}{D^2 + 5D + 6} \right] \\
&= \frac{e^x x (\sin x - \cos x)}{10} + \frac{e^x}{10} \left[ \frac{3 \cos x - 7 \sin x}{-1 + 5D + 6} \right] \\
&= \frac{e^x x (\sin x - \cos x)}{10} + \frac{e^x}{10} \frac{D-1}{5(D+1)(D-1)} (3 \cos x - 7 \sin x) \\
&= \frac{e^x x (\sin x - \cos x)}{10} + \frac{e^x}{50} (D-1) \left( \frac{3 \cos x - 7 \sin x}{-2} \right) \\
&= \frac{e^x x (\sin x - \cos x)}{10} - \frac{e^x}{100} (-3 \sin x - 7 \cos x - 3 \cos x + 7 \sin x) \\
&= \frac{e^x x (\sin x - \cos x)}{10} - \frac{e^x}{100} (4 \sin x - 10 \cos x)
\end{aligned}$$

General solution is  $y = c_1 e^{-2x} + c_2 e^{-x} + \frac{e^x x (\sin x - \cos x)}{10} - \frac{e^x}{100} (4 \sin x - 10 \cos x)$

18. Solve  $\frac{d^2 y}{dx^2} + a^2 y = \sec ax$

Solution: Given equation in the standard form is  $(D^2 + a^2)y = \sec ax$

Which is a non-homogeneous and the homogeneous equation is

$$(D^2 + a^2)y = 0$$

A.E. is  $m^2 + a^2 = 0, m = \pm ia$

$\therefore$  C.F. =  $c_1 \cos ax + c_2 \sin ax$

$$\text{P.I.} = \frac{1}{D^2 + a^2} \sec ax = \frac{1}{2ia} \left[ \frac{1}{D - ia} \sec ax - \frac{1}{D + ia} \sec ax \right]$$

Now consider  $\frac{1}{D - ia} \sec ax = e^{iax} \int \sec ax e^{-iax} dx$

$$= e^{iax} \int \frac{\cos ax - i \sin ax}{\cos ax} dx$$

$$= e^{iax} \int (1 - i \tan ax) dx = e^{iax} \left( x + \frac{i}{a} \log \cos ax \right)$$

Changing  $i$  to  $-i$  we have  $\frac{1}{D+ia} \sec ax = e^{-iax} \left( x + \frac{i}{a} \log \cos ax \right)$

$$\begin{aligned} \text{P.I.} &= \frac{1}{2ia} \left[ e^{iax} \left( x + \frac{i}{a} \log \cos ax \right) - e^{-iax} \left( x + \frac{i}{a} \log \cos ax \right) \right] \\ &= \frac{x}{a} \left[ \frac{e^{iax} - e^{-iax}}{2i} \right] + \frac{1}{a^2} \log \cos ax \left[ \frac{e^{iax} + e^{-iax}}{2} \right] \\ &= \frac{x}{a} \sin ax + \frac{1}{a} \log (\cos ax) \cos ax \end{aligned}$$

Hence the general solution is  $y = c_1 \cos ax + c_2 \sin ax + \frac{x}{a} \sin ax + \frac{1}{a} \cos ax \log (\cos ax)$

19. Solve  $(D+1)^2 y = x \cos x$

Solution: Given  $(D+1)^2 y = x \cos x$

A.E. is  $(m+1)^2 = 0, m = -1, -1$

C.F.  $= (c_1 + c_2 x) e^{-x}$

$$\begin{aligned} \text{P.I.} &= \frac{1}{(D+1)^2} x \cos x \\ &= \left[ x - \frac{1}{f(D)} f'(D) \right] \frac{1}{f(D)} \cos x \\ &= \left[ x - \frac{2(D+1)}{(D+1)^2} \right] \frac{1}{-1^2 + 2D + 1} \cos x \\ &= \left[ x - \frac{2}{D+1} \right] \frac{\sin x}{2} \\ &= \frac{x}{2} \sin x - \frac{D-1}{D^2-1} \sin x \\ &= \frac{x}{2} \sin x + \frac{\cos x - \sin x}{2} \end{aligned}$$

Hence the general solution is  $y = (c_1 + c_2 x) e^{-x} + \frac{x}{2} \sin x + \frac{\cos x - \sin x}{2}$