

## Linear Differential Equations with constant coefficients

### Introduction:

Applications of linear homogeneous and non-homogeneous differential equations, in particular, to simple harmonic motion, electrical circuits and deflection beams are presented.

### Defintion of Linear Differential Equation:

A linear differential equation is one in which the dependent variable and its derivatives occur in the first degree and not multiplied together. Thus, the general linear differential equation of nth order is

$$\frac{d^n y}{dx^n} + P_1(x) \frac{d^{n-1} y}{dx^{n-1}} + P_2(x) \frac{d^{n-2} y}{dx^{n-2}} + \dots + P_n(x) y = F(x) \quad \dots(1)$$

If  $P_1, P_2, \dots, P_n$  are constants the equation (1) is called a linear non-homogeneous differential equation with constant coefficients.

### Operator Notation:

Using the operator  $D$ ,  $D = \frac{d}{dx}$ ,  $D^2 = \frac{d^2}{dx^2}$ ,  $\dots$ ,  $D^n = \frac{d^n}{dx^n}$

So that  $Dy = \frac{dy}{dx}$ ,  $D^2 y = \frac{d^2 y}{dx^2}$ ,  $\dots$ ,  $D^n y = \frac{d^n y}{dx^n}$

Now equation (1) can be written as

$$\left[ D^n + P_1(x)D^{n-1} + P_2(x)D^{n-2} + \dots + P_n(x) \right] y = F(x)$$

Or  $f(D)y = F(x) \quad \dots(2)$

where  $f(D) = D^n + P_1 D^{n-1} + \dots + P_n$ , which is a polynomial in  $D$ .

### Auxiliary Equation:

In equation (2) replacing  $D$  by  $m$ , when  $f(x) = 0$ , we get  $f(m) = 0$  which is called Auxiliary equation.

**Roots of Auxiliary Equation  $f(m) = 0$**

**Case1:**  $m_1, m_2, m_3, \dots$  all roots are real and different, then

$$\text{Complementary Function} = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \dots$$

**Case2:**  $m_1, m_2, m_3, \dots$  (two real and equal roots, the rest are all different). Then

$$\text{Complementary Function} = (c_1 + c_2 x) e^{m_1 x} + c_3 e^{m_3 x} + \dots$$

**Case3:**  $m_1, m_1, m_1, m_4, \dots$  (three real and equal the rest are all different). Then

$$\text{Complementary Function} = (c_1 + c_2 x + c_3 x^2) e^{m_1 x} + c_4 e^{m_4 x} + \dots$$

**Case4:**  $\alpha + i\beta, \alpha - i\beta, m_3, m_4, \dots$  (a pair of imaginary roots and the rest is all different)

$$\text{Complementary Function} = e^{\alpha x} (c_1 \cos \beta x + c_2 \sin \beta x) + c_3 e^{m_3 x} + \dots$$

**Case5:**  $\alpha \pm i\beta, \alpha \pm i\beta, m_5, m_6, \dots$  (two pairs of equal imaginary roots, the rest are all different.)

$$\text{Complementary Function} = e^{\alpha x} ((c_1 + c_2 x) \cos \beta x + (c_3 + c_4 x) \sin \beta x) + c_5 e^{m_5 x} + \dots$$

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**Solution of Homogeneous equations with second and higher order Differential equations:**

Consider a homogeneous differential equation with constant coefficients given by  $f(D)y = 0$

**Working Procedure:**

**Step1:** write the given equation in the operator form  $(D^n + P_1 D^{n-1} + \dots + P_n)y = 0$

**Step2:** write the auxiliary equation corresponding to it,

$$m^n + P_1 m^{n-1} + \dots + P_n = f(m) = 0$$

**Step3:** solve the A.E.  $f(m) = 0$  and recognize nature of roots.

**Step4:** write the C.F.

**Problem1:** Solve  $\frac{d^4 y}{dx^4} - a^4 y = 0$

**Solution:** Given  $\frac{d^4 y}{dx^4} - a^4 y = 0$

The given homogeneous differential equation can be written as by using operator notation,

$$(D^4 - a^4)y = 0$$

Corresponding Auxiliary equation is,

$$m^4 - a^4 = 0$$
$$(m^2 - a^2)(m^2 + a^2) = 0$$

whose roots are  $m = \pm a, \pm ai$

Here two roots are real and distinct and the rest are complex conjugates.

Hence the general solution is  $y = c_1 e^{ax} + c_2 e^{-ax} + c_3 \cos ax + c_4 \sin ax$

**Problem2:** Solve  $y'' + y' - 2y = 0$ , given  $y(0) = 4$ ,  $y'(0) = 1$

**Solution:** Writing the given equation in the operator form, we have  $(D^2 + D - 2)y = 0$

The auxiliary equation is,

$$m^2 + m - 2 = 0$$

$$(m-1)(m-2) = 0$$

Whose roots are  $m = -1, -2$  which are real and distinct.

Hence the general solution is  $y = c_1 e^x + c_2 e^{-2x}$  .....(1)

where  $c_1$  and  $c_2$  are constants of integration which can be evaluated using the conditions

$$y(0) = 4, y'(0) = 1$$

From equation (1) we have, using the given conditions,

$$4 = c_1 e^0 + c_2 e^{-2(0)} = c_1 + c_2$$

$$c_1 + c_2 = 4 \quad \text{.....(2)}$$

Differentiating equation (1) with respect to  $x$ , we get

$$y' = c_1 e^x - 2c_2 e^{-2x} \quad \text{.....(3)}$$

Now, using the given conditions, we get

$$1 = c_1 e^0 - 2c_2 e^{-2x}$$

$$c_1 - 2c_2 = 1 \quad \text{.....(4)}$$

Solving the equations (2) and (4), we obtain  $c_1 = 3, c_2 = 1$

Substituting these values in (1), we have the solution as  $y = 3e^x + e^{-2x}$

**General method for solving Non-homogeneous linear differential equation:**

An equation of the form  $f(D)y = F(x)$

Where  $F(x)$  is any function of  $x$ , then,  $P.I. = \frac{1}{f(D)} F(x)$

Here, resolve  $f(D)$  into partial fractions and operate each partial fractions, while

$$\frac{1}{D-a} F(x) = e^{ax} \int F(x) e^{-ax} dx$$

**Problem:** Solve  $\frac{d^2 y}{dx^2} + a^2 y = \sec ax$

**Solution:** Given equation in the standard form is  $(D^2 + a^2) y = \sec ax$

Which is a non-homogeneous equation and the homogeneous equation is  $(D^2 + a^2) y = 0$

A.E. is,

$$m^2 + a^2 = 0$$
$$m = \pm ia$$

Hence, C.F. =  $c_1 \cos ax + c_2 \sin ax$  .....(1)

The particular Integral is,

$$P.I = \frac{1}{D^2 + a^2} \sec ax$$
$$= \frac{1}{(D + ai)(D - ai)} \sec ax$$
$$= \frac{1}{2ia} \left[ \frac{1}{D - ia} - \frac{1}{D + ia} \right] \sec ax$$
$$= \frac{1}{2ia} \left[ \frac{1}{D - ia} \sec ax - \frac{1}{D + ia} \sec ax \right] \quad \text{.....(2)}$$

Consider,

$$\begin{aligned}
\frac{1}{D-ia} \sec ax &= e^{iax} \int \sec ax e^{-iax} dx \\
&= e^{iax} \int \frac{\cos ax - i \sin ax}{\cos ax} dx \\
&= e^{iax} \int (1 - i \tan ax) dx \\
&= e^{iax} \left( x + \frac{i}{a} \log \cos ax \right) \quad \dots\dots(3)
\end{aligned}$$

Changing  $i$  to  $-i$ ,

We have

$$\frac{1}{D+ia} \sec ax = e^{-iax} \left\{ x - \frac{i}{a} \log \cos ax \right\} \quad \dots\dots(4)$$

Using (3) and (4) in (2), we get,

$$\begin{aligned}
P.I. &= \frac{1}{2ia} \left[ e^{iax} \left\{ x + \frac{i}{a} \log \cos ax \right\} - e^{-iax} \left\{ x - \frac{i}{a} \log \cos ax \right\} \right] \\
&= \frac{x}{a} \left[ \frac{e^{iax} - e^{-iax}}{2i} \right] + \frac{1}{a^2} \log \cos ax \left[ \frac{e^{iax} + e^{-iax}}{2} \right] \\
&= \frac{x}{a} \sin ax + \frac{1}{a} \log(\cos ax) \cos ax
\end{aligned}$$

Hence the general solution is,

$$y = C.F. + I.F.$$

$$y = c_1 \cos ax + c_2 \sin ax + \frac{x}{a} \sin ax + \left( \frac{1}{a^2} \right) \cos ax \log \cos ax$$

### Rules for finding particular integrals:

#### Particular Integral:

An equation of the form  $f(D)y = F(x)$ , then,  $P.I. = \frac{1}{f(D)} F(x)$ .

**Rule 1:** When  $F(x) = e^{ax}$

$$P.I. = \frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax}, \text{ if } f(a) \neq 0$$

if  $f(a) = 0$ ,

the above rule fails and hence  $P.I. = \frac{1}{f(D)} e^{ax} = x \frac{1}{f'(a)} e^{ax}$ , if  $f'(a) \neq 0$

if  $f'(a) = 0$ , then  $\frac{1}{f(D)} e^{ax} = x^2 \frac{1}{f''(a)} e^{ax}$ , if  $f''(a) \neq 0$  and so on.

**Rule 2:** When  $F(x) = \sin(ax+b)$  or  $\cos(ax+b)$

$$P.I. = \frac{1}{\phi(D^2)} \sin(ax+b) = \frac{1}{\phi(-a^2)} \sin(ax+b), \text{ if } \phi(-a^2) \neq 0$$

$$P.I. = \frac{1}{\phi(D^2)} \cos(ax+b) = \frac{1}{\phi(-a^2)} \cos(ax+b), \text{ if } \phi(-a^2) \neq 0$$

if  $\phi(-a^2) = 0$

$$P.I. = \frac{1}{\phi(D^2)} \sin(ax+b) = x \frac{1}{\phi'(-a^2)} \sin(ax+b), \text{ if } \phi'(-a^2) \neq 0$$

$$P.I. = \frac{1}{\phi(D^2)} \cos(ax+b) = x \frac{1}{\phi'(-a^2)} \cos(ax+b), \text{ if } \phi'(-a^2) \neq 0$$

if  $\phi'(-a^2) = 0$

$$\frac{1}{\phi'(D^2)} \sin(ax+b) = x^2 \frac{1}{\phi''(-a^2)} \sin(ax+b), \text{ if } \phi''(-a^2) \neq 0 \text{ And, so on.}$$

Also we have,

$$\frac{1}{D^2 + a^2} \sin(ax + b) = -\frac{x}{2a} \cos(ax + b)$$

$$\frac{1}{D^2 + a^2} \cos(ax + b) = \frac{x}{2a} \sin(ax + b)$$

**Rule 3:** When  $F(x) = x^m$ ,  $m$  being a positive integer,  $P.I. = \frac{1}{f(D)} x^m = [f(D)]^{-1} x^m$

To evaluate, expand  $[f(D)]^{-1}$  in ascending powers of  $D$  as far as  $D^m$  and operate on  $x^m$  term by term.

**Rule 4:** When  $F(x) = e^{ax}V$ , where  $V$  is a function of  $x$

$$P.I. = \frac{1}{f(D)} e^{ax}V = e^{ax} \frac{1}{f(D+a)} V$$

Then evaluate  $\frac{1}{f(D+a)} V$  as rule 1, 2, and 3.

**Rule 5:** When  $F(x) = xV$ , where  $V$  is a function of  $x$

$$P.I. = \frac{1}{f(D)} (xV) = \left[ x - \frac{1}{f(D)} f'(D) \right] \frac{1}{f(D)} V$$

### Problems on Rule 1

1. Find the P.I. of  $2 \frac{d^3 y}{dx^3} - 3 \frac{d^2 y}{dx^2} + y = e^x + 1$

**Solution:** Given  $2 \frac{d^3 y}{dx^3} - 3 \frac{d^2 y}{dx^2} + y = e^x + 1$

Writing the given equation in the operator form,  $(2D^3 - 3D^2 + 1)y = e^x + e^0$

$$\begin{aligned}
 P.I. &= \frac{1}{2D^3 - 3D^2 + 1} (e^x + e^0) \\
 &= \frac{1}{2D^3 - 3D^2 + 1} e^x + \frac{1}{2D^3 - 3D^2 + 1} e^0
 \end{aligned}$$

By using rule 1, put  $D = a = 1$ . Then,

$$\begin{aligned}
 P.I. &= \frac{1}{2(1)^3 - 3(1)^2 + 1} e^x + \frac{1}{2(0) - 3(0) + 1} e^0 \\
 &= \frac{xe^x}{6D^2 - 6D} + 1 \\
 &= \frac{xe^x}{6(1)^2 - 6(1)} + 1 \\
 &= \frac{x^2}{12D - 6} e^x + 1 \\
 &= \frac{x^2}{12(1) - 6} e^x + 1 \\
 P.I. &= \frac{x^2}{6} e^x + 1
 \end{aligned}$$

2. Solve  $(4D^2 - 4D + 1)y = 0$

**Solution:** Given  $(4D^2 - 4D + 1)y = 0$

This is a non-homogeneous equation. The corresponding homogeneous equation is,

$$4m^2 - 4m + 1 = 0$$

$$(2m - 1)^2 = 0$$

$$m = \frac{1}{2}, \frac{1}{2}$$

then, C.F. =  $(c_1 + c_2 x)e^{\frac{x}{2}}$

Consider,

$$\begin{aligned}
\text{P.I.} &= \frac{1}{(2D-1)^2} 100 \\
&= \frac{1}{(2D-1)^2} \cdot 100 \cdot e^{0x} \\
&= \frac{100}{1^2} \\
&= 100
\end{aligned}$$

The general solution is,

$$y = \text{C.F.} + \text{P.I.}$$

$$y = (c_1 + c_2 x) e^{\frac{x}{2}} + 100$$

1. Solve  $(D^2 + 4)y = \cos 2x$

**Solution:** Given  $(D^2 + 4)y = \cos 2x$

Given equation is non-homogeneous equation, whose homogeneous equation is,

$$(D^2 + 4)y = 0$$

The auxiliary equation is,

$$\begin{aligned}
m^2 + 4 &= 0, \\
m &= \pm 2i
\end{aligned}$$

So, the complementary function is,

$$\text{C.F.} = c_1 \cos 2x + c_2 \sin 2x$$

The particular Integral is,

$$\begin{aligned}
\text{P.I.} &= \frac{1}{D^2 + 4} \cos 2x \\
&= \frac{1}{-4 + 4} \cos 2x \quad \text{put } D^2 = -2^2 \\
&= x \cdot \frac{1}{2D} \cos 2x \\
&= \frac{x}{2} \cdot \frac{1}{2} \sin 2x
\end{aligned}$$

The general solution is,  $y = c_1 \cos 2x + c_2 \sin 2x + \frac{x}{4} \sin 2x$

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**Problems on Rule 3:**

1. Find the solution of

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} = x^2 + 2x + 4$$

Sol: Given  $\frac{d^2y}{dx^2} + \frac{dy}{dx} = x^2 + 2x + 4$

given equation is non-homogeneous equation whose homogeneous equation is

$$m^2 + m = 0,$$

$$m(m+1) = 0,$$

$$m = 0, -1$$

Then, C.F. =  $c_1e^{x(0)} + c_2e^{x(-1)} = c_1 + c_2e^{-x}$

Consider,

$$\begin{aligned} P.I. &= \frac{1}{f(D)} F(x) \\ &= \frac{1}{f(D)} (x^2 + 2x + 4) \\ &= \frac{1}{D^2 + D} (x^2 + 2x + 4) \\ &= \frac{1}{D(1+D)} (x^2 + 2x + 4) \end{aligned}$$

$$\begin{aligned}
\text{P.I.} &= \frac{1}{D}[1+D]^{-1}(x^2+2x+4) \\
&= \frac{1}{D}[1-D+D^2-\dots](x^2+2x+4) \\
&= \frac{1}{D}[x^2+2x+4-2x-2+2] \\
&= \frac{1}{D}[x^2+4] = D^{-1}(x^2+4) \\
&= \int(x^2+4)dx \\
&= \frac{x^3}{3}+4x
\end{aligned}$$

Therefore, the general solution is,

$$\begin{aligned}
y &= y_c + y_p \\
&= c_1 + c_2 e^{-x} + \frac{x^3}{3} + 4x
\end{aligned}$$

2. Solve  $(D-2)^2 y = 8(e^{2x} + \sin 2x + x^2)$

**Solution:** Given  $(D-2)^2 y = 8(e^{2x} + \sin 2x + x^2)$  .....(1)

Given equation is non-homogeneous equation whose homogeneous equation is

A.E. is,

$$\begin{aligned}
(m-2)^2 &= 0, \\
m &= 2, 2
\end{aligned}$$

Then, C.F. =  $(c_1 + c_2 x)e^{2x}$

And the particular integral is,

$$P.I. = 8 \left[ \frac{1}{(D-2)^2} e^{2x} + \frac{1}{(D-2)^2} \sin 2x + \frac{1}{(D-2)^2} x^2 \right] \quad \dots\dots(2)$$

Consider,

$$\frac{1}{(D-2)^2} e^{2x} = x^2 \frac{1}{2(1)} e^{2x} = \frac{x^2}{2} e^{2x} \quad \dots\dots(3)$$

Consider,

$$\begin{aligned} \frac{1}{(D-2)^2} \sin 2x &= \frac{1}{D^2 - 4D + 4} \sin 2x \\ &= \frac{1}{(-2)^2 - 4D + 4} \sin 2x \\ &= -\frac{1}{4} \cdot \frac{1}{D} \sin 2x \\ &= -\frac{1}{4} \int \sin 2x dx \\ &= \frac{\cos 2x}{8} \quad \dots\dots(4) \end{aligned}$$

Consider,

$$\begin{aligned} \frac{1}{(D-2)^2} x^2 &= \frac{1}{4} \left(1 - \frac{D}{2}\right)^{-2} x^2 \\ &= \frac{1}{4} \left[ 1 + (-2) \left(\frac{D}{2}\right) + \frac{(-2)(-3)}{2} \left(\frac{-D}{2}\right)^2 + \dots \right] x^2 \\ &= \frac{1}{4} \left( 1 + D + \frac{3D^2}{4} + \dots \right) x^2 \\ &= -\frac{1}{4} \cdot \frac{1}{D} \sin 2x \\ &= -\frac{1}{4} \int \sin 2x dx \\ &= \frac{1}{4} \left( x^2 + 2x + \frac{3}{2} \right) \quad \dots\dots(5) \end{aligned}$$

Now using (3), (4), (5), in (2), we get

$$\begin{aligned} \text{P.I.} &= 8 \left[ \frac{x^2}{2} e^{2x} + \frac{\cos 2x}{8} + \frac{1}{4} \left( x^2 + 2x + \frac{3}{2} \right) \right] \\ &= 4x^2 e^{2x} + \cos 2x + 2x^2 + 4x + 3 \end{aligned}$$

Hence the general solution is

$$y = \text{C.F.} + \text{P.I.}$$

$$y = (c_1 + c_2 x)e^{2x} + 4x^2 e^{2x} + \cos 2x + 2x^2 + 4x + 3$$

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