

## 44 Further Studies of Laplace Transform

Properties of the Laplace transform enable us to find Laplace transforms without having to compute them directly from the definition. In this section, we establish properties of Laplace transform that will be useful for solving ODEs.

### Laplace Transform of the Heaviside Step Function

The Heaviside step function is a piecewise continuous function defined by

$$h(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

Figure 44.1 displays the graph of  $h(t)$ .

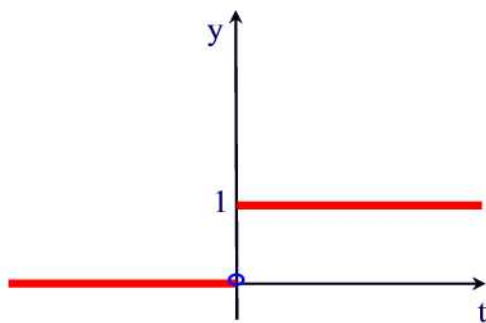


Figure 44.1

Taking the Laplace transform of  $h(t)$  we find

$$\mathcal{L}[h(t)] = \int_0^{\infty} h(t)e^{-st} dt = \int_0^{\infty} e^{-st} dt = \left[ -\frac{e^{-st}}{s} \right]_0^{\infty} = \frac{1}{s}, \quad s > 0.$$

A Heaviside function at  $\alpha \geq 0$  is the shifted function  $h(t - \alpha)$  ( $\alpha$  units to the right). For this function, the Laplace transform is

$$\mathcal{L}[h(t - \alpha)] = \int_0^{\infty} h(t - \alpha)e^{-st} dt = \int_{\alpha}^{\infty} e^{-st} dt = \left[ -\frac{e^{-st}}{s} \right]_{\alpha}^{\infty} = \frac{e^{-s\alpha}}{s}, \quad s > 0.$$

### Laplace Transform of $e^{at}$

The Laplace transform for the function  $f(t) = e^{at}$  is

$$\mathcal{L}[e^{at}] = \int_0^{\infty} e^{-(s-a)t} dt = \left[ -\frac{e^{-(s-a)t}}{s-a} \right]_0^{\infty} = \frac{1}{s-a}, \quad s > a.$$

### Laplace Transforms of $\sin at$ and $\cos at$

Using integration by parts twice we find

$$\begin{aligned}\mathcal{L}[\sin at] &= \int_0^\infty e^{-st} \sin at dt \\ &= \left[ -\frac{e^{-st} \sin at}{s} - \frac{ae^{-st} \cos at}{s^2} \right]_0^\infty - \frac{a^2}{s^2} \int_0^\infty e^{-st} \sin at dt \\ &= -\frac{a}{s^2} - \frac{a^2}{s^2} \mathcal{L}[\sin at] \\ \left( \frac{s^2+a^2}{s^2} \right) \mathcal{L}[\sin at] &= \frac{a}{s^2} \\ \mathcal{L}[\sin at] &= \frac{a}{s^2+a^2}, \quad s > 0\end{aligned}$$

A similar argument shows that

$$\mathcal{L}[\cos at] = \frac{s}{s^2+a^2}, \quad s > 0.$$

### Laplace Transforms of $\cosh at$ and $\sinh at$

Using the linear property of  $\mathcal{L}$  we can write

$$\begin{aligned}\mathcal{L}[\cosh at] &= \frac{1}{2} (\mathcal{L}[e^{at}] + \mathcal{L}[e^{-at}]) \\ &= \frac{1}{2} \left( \frac{1}{s-a} + \frac{1}{s+a} \right), \quad s > |a| \\ &= \frac{s}{s^2-a^2}, \quad s > |a|\end{aligned}$$

A similar argument shows that

$$\mathcal{L}[\sinh at] = \frac{a}{s^2-a^2}, \quad s > |a|.$$

### Laplace Transform of a Polynomial

Let  $n$  be a positive integer. Using integration by parts we can write

$$\int_0^\infty t^n e^{-st} dt = - \left[ \frac{t^n e^{-st}}{s} \right]_0^\infty + \frac{n}{s} \int_0^\infty t^{n-1} e^{-st} dt.$$

By repeated use of L'Hôpital's rule we find  $\lim_{t \rightarrow \infty} t^n e^{-st} = \lim_{t \rightarrow \infty} \frac{n!}{s^n e^{st}} = 0$  for  $s > 0$ . Thus,

$$\mathcal{L}[t^n] = \frac{n}{s} \mathcal{L}[t^{n-1}], \quad s > 0.$$

Using induction on  $n = 0, 1, 2, \dots$  one can easily establish that

$$\mathcal{L}[t^n] = \frac{n!}{s^{n+1}}, \quad s > 0.$$

Using the above result together with the linearity property of  $\mathcal{L}$  one can find the Laplace transform of any polynomial.

The next two results are referred to as the first and second shift theorems. As with the linearity property, the shift theorems increase the number of functions for which we can easily find Laplace transforms.

**Theorem 44.1** (*First Shifting Theorem*)

If  $f(t)$  is a piecewise continuous function for  $t \geq 0$  and has exponential order at infinity with  $|f(t)| \leq Me^{at}$ ,  $t \geq C$ , then for any real number  $\alpha$  we have

$$\mathcal{L}[e^{\alpha t} f(t)] = F(s - \alpha), \quad s > a + \alpha$$

where  $\mathcal{L}[f(t)] = F(s)$ .

**Proof.**

From the definition of the Laplace transform we have

$$\mathcal{L}[e^{\alpha t} f(t)] = \int_0^{\infty} e^{-st} e^{\alpha t} f(t) dt = \int_0^{\infty} e^{-(s-\alpha)t} f(t) dt.$$

Using the change of variable  $\beta = s - \alpha$  the previous equation reduces to

$$\mathcal{L}[e^{\alpha t} f(t)] = \int_0^{\infty} e^{-\beta t} f(t) dt = F(\beta) = F(s - \alpha), \quad s > a + \alpha \blacksquare$$

**Theorem 44.2** (*Second Shifting Theorem*)

If  $f(t)$  is a piecewise continuous function for  $t \geq 0$  and has exponential order at infinity with  $|f(t)| \leq Me^{at}$ ,  $t \geq C$ , then for any real number  $\alpha \geq 0$  we have

$$\mathcal{L}[f(t - \alpha)h(t - \alpha)] = e^{-\alpha s} F(s), \quad s > a$$

where  $\mathcal{L}[f(t)] = F(s)$  and  $h(t)$  is the Heaviside step function.

**Proof.**

From the definition of the Laplace transform we have

$$\mathcal{L}[f(t - \alpha)h(t - \alpha)] = \int_0^{\infty} f(t - \alpha)h(t - \alpha)e^{-st} dt = \int_{\alpha}^{\infty} f(t - \alpha)e^{-st} dt.$$

Using the change of variable  $\beta = t - \alpha$  the previous equation reduces to

$$\begin{aligned}\mathcal{L}[f(t - \alpha)h(t - \alpha)] &= \int_0^\infty f(\beta)e^{-s(\beta+\alpha)}d\beta \\ &= e^{-s\alpha} \int_0^\infty f(\beta)e^{-s\beta}d\beta = e^{-s\alpha}F(s), \quad s > a \blacksquare\end{aligned}$$

### Example 44.1

Find

(a)  $\mathcal{L}[e^{2t}t^2]$  (b)  $\mathcal{L}[e^{3t} \cos 2t]$  (c)  $\mathcal{L}^{-1}[e^{-2t}s^2]$

**Solution.**

(a) By Theorem 44.1, we have  $\mathcal{L}[e^{2t}t^2] = F(s - 2)$  where  $\mathcal{L}[t^2] = \frac{2!}{s^3} = F(s)$ ,  $s > 0$ . Thus,  $\mathcal{L}[e^{2t}t^2] = \frac{2}{(s-2)^3}$ ,  $s > 2$ .

(b) As in part (a), we have  $\mathcal{L}[e^{3t} \cos 2t] = F(s - 3)$  where  $\mathcal{L}[\cos 2t] = F(s - 3)$ . But  $\mathcal{L}[\cos 2t] = \frac{s}{s^2+4}$ ,  $s > 0$ . Thus,

$$\mathcal{L}[e^{3t} \cos 2t] = \frac{s - 3}{(s - 3)^2 + 4}, \quad s > 3$$

(c) Since  $\mathcal{L}[t] = \frac{1}{s^2}$ , by Theorem 44.2, we have

$$\frac{e^{-2t}}{s^2} = \mathcal{L}[(t - 2)h(t - 2)].$$

Therefore,

$$\mathcal{L}^{-1} \left[ \frac{e^{-2t}}{s^2} \right] = (t - 2)h(t - 2) = \begin{cases} 0, & 0 \leq t < 2 \\ t - 2, & t \geq 2 \blacksquare \end{cases}$$

The following result relates the Laplace transform of derivatives and integrals to the Laplace transform of the function itself.

### Theorem 44.3

Suppose that  $f(t)$  is continuous for  $t \geq 0$  and  $f'(t)$  is piecewise continuous of exponential order at infinity with  $|f'(t)| \leq Me^{at}$ ,  $t \geq C$  Then

(a)  $f(t)$  is of exponential order at infinity.

(b)  $\mathcal{L}[f'(t)] = s\mathcal{L}[f(t)] - f(0) = sF(s) - f(0)$ ,  $s > \max\{a, 0\} + 1$ .

(c)  $\mathcal{L}[f''(t)] = s^2\mathcal{L}[f(t)] - sf(0) - f'(0) = s^2F(s) - sf(0) - f'(0)$ ,  $s > \max\{a, 0\} + 1$ .

(d)  $\mathcal{L} \left[ \int_0^t f(u)du \right] = \frac{\mathcal{L}[f(t)]}{s} = \frac{F(s)}{s}$ ,  $s > \max\{a, 0\} + 1$ .

**Proof.**

(a) By the Fundamental Theorem of Calculus we have  $f(t) = f(0) - \int_0^t f'(u)du$ . Also, since  $f'$  is piecewise continuous,  $|f'(t)| \leq T$  for some  $T > 0$  and all  $0 \leq t \leq C$ . Thus,

$$\begin{aligned} |f(t)| &= \left| f(0) - \int_0^t f'(u)du \right| = \left| f(0) - \int_0^C f'(u)du - \int_C^t f'(u)du \right| \\ &\leq |f(0)| + TC + M \int_C^t e^{au} du \end{aligned}$$

Note that if  $a > 0$  then

$$\int_C^t e^{au} du = \frac{1}{a}(e^{at} - e^{aC}) \leq \frac{e^{at}}{a}$$

and so

$$|f(t)| \leq [|f(0)| + TC + \frac{M}{a}]e^{at}.$$

If  $a = 0$  then

$$\int_C^t e^{au} du = t - C$$

and therefore

$$|f(t)| \leq |f(0)| + TC + M(t - C) \leq (|f(0)| + TC + M)e^t.$$

Now, if  $a < 0$  then

$$\int_C^t e^{au} du = \frac{1}{a}(e^{at} - e^{aC}) \leq \frac{1}{|a|}$$

so that

$$|f(t)| \leq (|f(0)| + TC + \frac{M}{|a|})e^t$$

It follows that

$$|f(t)| \leq Ne^{bt}, \quad t \geq 0$$

where  $b = \max\{a, 0\} + 1$ .

(b) From the definition of Laplace transform we can write

$$\mathcal{L}[f'(t)] = \lim_{A \rightarrow \infty} \int_0^A f'(t)e^{-st} dt.$$

Since  $f'(t)$  may have jump discontinuities at  $t_1, t_2, \dots, t_N$  in the interval  $0 \leq t \leq A$ , we can write

$$\int_0^A f'(t)e^{-st} dt = \int_0^{t_1} f'(t)e^{-st} dt + \int_{t_1}^{t_2} f'(t)e^{-st} dt + \dots + \int_{t_N}^A f'(t)e^{-st} dt.$$

Integrating each term on the RHS by parts and using the continuity of  $f(t)$  to obtain

$$\begin{aligned} \int_0^{t_1} f'(t)e^{-st} dt &= f(t_1)e^{-st_1} - f(0) + s \int_0^{t_1} f(t)e^{-st} dt \\ \int_{t_1}^{t_2} f'(t)e^{-st} dt &= f(t_2)e^{-st_2} - f(t_1)e^{-st_1} + s \int_{t_1}^{t_2} f(t)e^{-st} dt \\ &\vdots \\ \int_{t_{N-1}}^{t_N} f'(t)e^{-st} dt &= f(t_N)e^{-st_N} - f(t_{N-1})e^{-st_{N-1}} + s \int_{t_{N-1}}^{t_N} f(t)e^{-st} dt \\ \int_{t_N}^A f'(t)e^{-st} dt &= f(A)e^{-sA} - f(t_N)e^{-st_N} + s \int_{t_N}^A f(t)e^{-st} dt \end{aligned}$$

Also, by the continuity of  $f(t)$  we can write

$$\int_0^A f(t)e^{-st} dt = \int_0^{t_1} f(t)e^{-st} dt + \int_{t_1}^{t_2} f(t)e^{-st} dt + \dots + \int_{t_N}^A f(t)e^{-st} dt.$$

Hence,

$$\int_0^A f'(t)e^{-st} dt = f(A)e^{-sA} - f(0) + s \int_0^A f(t)e^{-st} dt.$$

Since  $f(t)$  has exponential order at infinity,  $\lim_{A \rightarrow \infty} f(A)e^{-sA} = 0$ . Hence,

$$\mathcal{L}[f'(t)] = s\mathcal{L}[f(t)] - f(0).$$

(c) Using part (b) we find

$$\begin{aligned} \mathcal{L}[f''(t)] &= s\mathcal{L}[f'(t)] - f'(0) \\ &= s(sF(s) - f(0)) - f'(0) \\ &= s^2F(s) - sf(0) - f'(0), \quad s > \max\{a, 0\} + 1 \end{aligned}$$

(d) Since  $\frac{d}{dt} \left( \int_0^t f(u) du \right) = f(t)$ , by part (b) we have

$$F(s) = \mathcal{L}[f(t)] = s\mathcal{L} \left\{ \int_0^t f(u) du \right\}$$

and therefore

$$\mathcal{L}\left[\int_0^t f(u)du\right] = \frac{\mathcal{L}[f(t)]}{s} = \frac{F(s)}{s}, \quad s > \max\{a, 0\} + 1 \quad \blacksquare$$

The argument establishing part (b) of the previous theorem can be extended to higher order derivatives.

**Theorem 44.4**

Let  $f(t), f'(t), \dots, f^{(n-1)}(t)$  be continuous and  $f^{(n)}(t)$  be piecewise continuous of exponential order at infinity with  $|f^{(n)}(t)| \leq Me^{at}$ ,  $t \geq C$ . Then

$$\mathcal{L}[f^{(n)}(t)] = s^n \mathcal{L}[f(t)] - s^{n-1}f(0) - s^{n-2}f'(0) - \dots - f^{(n-1)}(0), \quad s > \max\{a, 0\} + 1.$$

We next illustrate the use of the previous theorem in solving initial value problems.

**Example 44.2**

Solve the initial value problem

$$y'' - 4y' + 9y = t, \quad y(0) = 0, \quad y'(0) = 1.$$

**Solution.**

We apply Theorem 44.4 that gives the Laplace transform of a derivative. By the linearity property of the Laplace transform we can write

$$\mathcal{L}[y''] - 4\mathcal{L}[y'] + 9\mathcal{L}[y] = \mathcal{L}[t].$$

Now since

$$\begin{aligned} \mathcal{L}[y''] &= s^2\mathcal{L}[y] - sy(0) - y'(0) = s^2Y(s) - 1 \\ \mathcal{L}[y'] &= sY(s) - y(0) = sY(s) \\ \mathcal{L}[t] &= \frac{1}{s^2} \end{aligned}$$

where  $\mathcal{L}[y] = Y(s)$ , we obtain

$$s^2Y(s) - 1 - 4sY(s) + 9Y(s) = \frac{1}{s^2}.$$

Rearranging gives

$$(s^2 - 4s + 9)Y(s) = \frac{s^2 + 1}{s^2}.$$

Thus,

$$Y(s) = \frac{s^2 + 1}{s^2(s^2 - 4s + 9)}$$

and

$$y(t) = \mathcal{L}^{-1} \left[ \frac{s^2 + 1}{s^2(s^2 - 4s + 9)} \right] \blacksquare$$

In the next section we will discuss a method for finding the inverse Laplace transform of the above expression.

### Example 44.3

Consider the mass-spring oscillator without friction:  $y'' + y = 0$ . Suppose we add a force which corresponds to a push (to the left) of the mass as it oscillates. We will suppose the push is described by the function

$$f(t) = -h(t - 2\pi) + u(t - (2\pi + a))$$

for some  $a > 2\pi$  which we are allowed to vary. (A small  $a$  will correspond to a short duration push and a large  $a$  to a long duration push.) We are interested in solving the initial value problem

$$y'' + y = f(t), \quad y(0) = 1, \quad y'(0) = 0.$$

#### Solution.

To begin, determine the Laplace transform of both sides of the DE:

$$\mathcal{L}[y'' + y] = \mathcal{L}[f(t)]$$

or

$$s^2 Y - sy(0) - y'(0) + Y(s) = -\frac{1}{s}e^{-2\pi s} + \frac{1}{s}e^{-(2\pi+a)s}.$$

Thus,

$$Y(s) = \frac{e^{-(2\pi+a)s}}{s(s^2 + 1)} - \frac{e^{-2\pi s}}{s(s^2 + 1)} + \frac{s}{s^2 + 1}.$$

Now since  $\frac{1}{s(s^2+1)} = \frac{1}{s} - \frac{s}{s^2+1}$  we see that

$$Y(s) = e^{-(2\pi+a)s} \left[ \frac{1}{s} - \frac{s}{s^2 + 1} \right] - e^{-2\pi s} \left[ \frac{1}{s} - \frac{s}{s^2 + 1} \right] + \frac{s}{s^2 + 1}$$



and therefore

$$\begin{aligned}
 y(t) &= h(t - (2\pi + a)) \left[ \mathcal{L}^{-1} \left( \frac{1}{s} - \frac{s}{s^2+1} \right) \right] (t - (2\pi + a)) \\
 &- h(t - 2\pi) \left[ \mathcal{L}^{-1} \left( \frac{1}{s} - \frac{s}{s^2+1} \right) \right] (t - 2\pi) + \cos t \\
 &= h(t - (2\pi + a)) [1 - \cos(t - (2\pi + a))] - u(t - 2\pi) [1 - \cos(t - 2\pi)] \\
 &+ \cos t \blacksquare
 \end{aligned}$$

We conclude this section with the following table of Laplace transform pairs.

f(t)	F(s)
$h(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$	$\frac{1}{s}, s > 0$
$t^n, n = 1, 2, \dots$	$\frac{n!}{s^{n+1}}, s > 0$
$e^{\alpha t}$	$\frac{s}{s-\alpha}, s > \alpha$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}, s > 0$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}, s > 0$
$\sinh(\omega t)$	$\frac{\omega}{s^2-\omega^2}, s >  \omega $
$\cosh(\omega t)$	$\frac{s}{s^2-\omega^2}, s >  \omega $
$e^{\alpha t} f(t),$ with $ f(t)  \leq M e^{\alpha t}$	$F(s - \alpha), s > \alpha + a$
$e^{\alpha t} h(t)$	$\frac{1}{s-\alpha}, s > \alpha$
$e^{\alpha t} t^n, n = 1, 2, \dots$	$\frac{n!}{(s-\alpha)^{n+1}}, s > \alpha$
$e^{\alpha t} \sin(\omega t)$	$\frac{\omega}{(s-\alpha)^2+\omega^2}, s > \alpha$
$e^{\alpha t} \cos(\omega t)$	$\frac{s-\alpha}{(s-\alpha)^2+\omega^2}, s > \alpha$
$f(t - \alpha)h(t - \alpha), \alpha \geq 0$ with $ f(t)  \leq M e^{\alpha t}$	$e^{-\alpha s} F(s), s > a$

$f(t)$	$F(s)$ (continued)
$h(t - \alpha), \alpha \geq 0$	$\frac{e^{-\alpha s}}{s}, s > 0$
$tf(t)$	$-F'(s)$
$\frac{t}{2\omega} \sin \omega t$	$\frac{s}{(s^2 + \omega^2)^2}, s > 0$
$\frac{1}{2\omega^3} [\sin \omega t - \omega t \cos \omega t]$	$\frac{1}{(s^2 + \omega^2)^2}, s > 0$
$f'(t)$ , with $f(t)$ continuous and $ f'(t)  \leq Me^{at}$	$sF(s) - f(0)$ $s > \max\{a, 0\} + 1$
$f''(t)$ , with $f'(t)$ continuous and $ f''(t)  \leq Me^{at}$	$s^2F(s) - sf(0) - f'(0)$ $s > \max\{a, 0\} + 1$
$f^{(n)}(t)$ , with $f^{(n-1)}(t)$ continuous and $ f^{(n)}(t)  \leq Me^{at}$	$s^n F(s) - s^{n-1} f(0) - \dots - sf^{(n-2)}(0) - f^{(n-1)}(0)$ $s > \max\{a, 0\} + 1$
$\int_0^t f(u)du$ , with $ f(t)  \leq Me^{at}$	$\frac{F(s)}{s}, s > \max\{a, 0\} + 1$

Table  $\mathcal{L}$

## Practice Problems

### Problem 44.1

Use Table  $\mathcal{L}$  to find  $\mathcal{L}[2e^t + 5]$ .

### Problem 44.2

Use Table  $\mathcal{L}$  to find  $\mathcal{L}[e^{3t-3}h(t-1)]$ .

### Problem 44.3

Use Table  $\mathcal{L}$  to find  $\mathcal{L}[\sin^2 \omega t]$ .

### Problem 44.4

Use Table  $\mathcal{L}$  to find  $\mathcal{L}[\sin 3t \cos 3t]$ .

### Problem 44.5

Use Table  $\mathcal{L}$  to find  $\mathcal{L}[e^{2t} \cos 3t]$ .

### Problem 44.6

Use Table  $\mathcal{L}$  to find  $\mathcal{L}[e^{4t}(t^2 + 3t + 5)]$ .

### Problem 44.7

Use Table  $\mathcal{L}$  to find  $\mathcal{L}^{-1}[\frac{10}{s^2+25} + \frac{4}{s-3}]$ .

### Problem 44.8

Use Table  $\mathcal{L}$  to find  $\mathcal{L}^{-1}[\frac{5}{(s-3)^4}]$ .

### Problem 44.9

Use Table  $\mathcal{L}$  to find  $\mathcal{L}^{-1}[\frac{e^{-2s}}{s-9}]$ .

### Problem 44.10

Use Table  $\mathcal{L}$  to find  $\mathcal{L}^{-1}[\frac{e^{-3s}(2s+7)}{s^2+16}]$ .

### Problem 44.11

Graph the function  $f(t) = h(t-1) + h(t-3)$  for  $t \geq 0$ , where  $h(t)$  is the Heaviside step function, and use Table  $\mathcal{L}$  to find  $\mathcal{L}[f(t)]$ .

### Problem 44.12

Graph the function  $f(t) = t[h(t-1) - h(t-3)]$  for  $t \geq 0$ , where  $h(t)$  is the Heaviside step function, and use Table  $\mathcal{L}$  to find  $\mathcal{L}[f(t)]$ .