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Centurion University of Technology And Management

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The Laplace Transform

Time Integration:

The property is:

$$L\left[\int_0^{\infty} f(t)dt\right] = \int_0^{\infty} \left[\int_0^t f(x)dx\right] e^{-st} dt$$

Integrate by parts:

$$\text{Let } u = \int_0^t f(x)dx, \quad du = f(t)dt$$

and

$$dv = e^{-st} dt, \quad v = -\frac{1}{s} e^{-st}$$



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Home Task:

For Inverse Laplace
transform material go
through Arfken
available in your library.



Real Integration Theorem

This theorem establishes the relationship between the Laplace transform of a function and that of its integral. It states that

$$\mathcal{L} \left[\int_0^t f(t) dt \right] = \frac{1}{s} F(s) \quad (2-1.7)$$

The proof of this theorem is carried out by integrating the definition of the Laplace transform by parts. This proof is similar to that of the real differentiation theorem and is left as an exercise. The Laplace transform of the n th integral of a function is the transform of the function divided by s^n .



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Laplace Transforms of Common Functions

Function	Laplace Transform	Restrictions
1	$\frac{1}{s}$	$s > 0$
e^{at}	$\frac{1}{s-a}$	$s > a$
t^n	$\frac{n!}{s^{n+1}}$	$s > 0, n$ an integer > 0
$\cos at$	$\frac{s}{s^2 + a^2}$	$s > 0$
$\sin at$	$\frac{a}{s^2 + a^2}$	$s > 0$
$\cosh at$	$\frac{s}{s^2 - a^2}$	$s > a $
$\sinh at$	$\frac{a}{s^2 - a^2}$	$s > a $
$e^{at} \cos bt$	$\frac{s-a}{(s-a)^2 + b^2}$	$s > a$
$e^{at} \sin bt$	$\frac{b}{(s-a)^2 + b^2}$	$s > a$
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$	$s > a, n$ an integer > 0
$f(ct)$	$\frac{1}{c} \mathcal{L}\{f(s/c)\}$	$c > 0$
$f^{(n)}(t)$	$s^n \mathcal{L}\{f(t)\} - s^{n-1}f(0) - \dots - sf^{(n-2)}(0) - f^{(n-1)}(0)$	



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Table of Inverse Laplace Transform

$$1. \mathcal{L}^{-1} \left\{ \frac{1}{s} \right\} = 1$$

$$2. \mathcal{L}^{-1} \left\{ \frac{1}{s - a} \right\} = e^{at}$$

$$3. \mathcal{L}^{-1} \left\{ \frac{1}{s^n} \right\} = \frac{t^{n-1}}{(n-1)!}; \quad n = 1, 2, 3, \dots$$

$$4. \mathcal{L}^{-1} \left\{ \frac{1}{(s - a)^n} \right\} = \frac{e^{at} t^{n-1}}{(n-1)!}$$

$$5. \mathcal{L}^{-1} \left\{ \frac{1}{s^2 + b^2} \right\} = \frac{1}{b} \sin bt$$

$$6. \mathcal{L}^{-1} \left\{ \frac{s}{s^2 + b^2} \right\} = \cos bt$$

$$7. \mathcal{L}^{-1} \left\{ \frac{1}{s^2 - a^2} \right\} = \frac{1}{a} \sinh at$$

$$8. \mathcal{L}^{-1} \left\{ \frac{s}{s^2 - a^2} \right\} = \cosh at$$

$$9. \mathcal{L}^{-1} \left\{ \frac{1}{(s - a)^2 + b^2} \right\} = \frac{1}{b} e^{at} \sin bt$$

$$10. \mathcal{L}^{-1} \left\{ \frac{s - a}{(s - a)^2 + b^2} \right\} = e^{at} \cos bt$$

$$11. \mathcal{L}^{-1} \left\{ \frac{s}{(s^2 + b^2)^2} \right\} = \frac{1}{2b} t \sin bt$$



Laplace Transform

■ Theorems

<i>Theorem</i>	<i>Description</i>
Definition of Laplace Transform	$\mathcal{L}\{f(t)\} = \int_0^{\infty} e^{-st} f(t) dt = F(s)$
Linear Property	$\mathcal{L}\{af(t) + bg(t)\} = aF(s) + bG(s)$
Derivatives	$\mathcal{L}\{f^{(n)}(t)\} = s^n F(s) - s^{n-1} f(0) - \dots - f^{(n-1)}(0)$
Integrals	$\mathcal{L}\left\{\int_0^{\infty} f(\tau) d\tau\right\} = \frac{1}{s} F(s)$
First Shifting Property	$\mathcal{L}\{e^{at} f(t)\} = F(s - a)$
Second Shifting Property	$\mathcal{L}\{f(t - a)u(t - a)\} = e^{-as} F(s)$



Transform of derivatives

- ▶ **Laplace transform of the derivative f^n of any order :**

Let $f, f', \dots, f^{(n-1)}$ be continuous for all $t \geq 0$ and f^n be piecewise continuous on every finite interval on the semi-axis $t \geq 0$ then transform of f^n given by,

$$\mathbf{L}\left(f^n\right) = s^n \mathbf{L}(f) - s^{n-1} f(0) - s^{n-2} f'(0) - \dots - f^{(n-1)}(0)$$



For first-order derivative:

$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)$$

For second-order derivative:

$$\mathcal{L}\{f''(t)\} = s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0)$$

For third-order derivative:

$$\mathcal{L}\{f'''(t)\} = s^3\mathcal{L}\{f(t)\} - s^2f(0) - sf'(0) - f''(0)$$

For n^{th} order derivative:

$$\mathcal{L}\{f^n(t)\} = s^n\mathcal{L}\{f(t)\} - s^{n-1}f(0) - s^{n-2}f'(0) - \dots - f^{n-1}(0)$$



The Laplace Transform of the **First** Derivative

$$L[f'(t)] = sL[f(t)] - f(0)$$

$$L[f'(t)] = \int_0^{\infty} \underbrace{e^{-st}}_u \underbrace{f(t) dt}_{dv}$$

$$\int u dv = uv - \int v du$$

$$L[f'(t)] = \int u dv = e^{-st} f(t) \Big|_0^{\infty} - \int_0^{\infty} f(t)(-s) e^{-st} dt$$



The Laplace Transform of the **Third** Derivative

$$\mathcal{L}\left[f'(t)\right] = s\mathcal{L}\left[f(t)\right] - f(0)$$

$$\mathcal{L}\left[f''(t)\right] = s^2\mathcal{L}\left[f(t)\right] - sf(0) - f'(0)$$

$$\mathcal{L}\left[f'''(t)\right] = s\mathcal{L}\left[f''(t)\right] - f''(0)$$

$$= s\left[s^2\mathcal{L}\left[f(t)\right] - f(0) - f'(0)\right] - f''(0)$$

$$\mathcal{L}\left[f'''(t)\right] = s^3\mathcal{L}\left[f(t)\right] - s^2f(0) - sf'(0) - f''(0)$$



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Unit Step Function

$$u(t - a) = 0, t < a$$

$$1, t \geq a$$

$$\begin{aligned} L\{u(t - a)\} &= \int_0^{\infty} e^{-st} u(t - a) dt \\ &= \int_0^a e^{-st} (0) dt + \int_a^{\infty} e^{-st} (1) dt \\ &= \left[\frac{e^{-st}}{-s} \right]_a^{\infty} = \frac{1}{s} e^{-as} \end{aligned}$$

if $a = 0$

$$L\{u(t)\} = \frac{1}{s}$$



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Application to linear differential equations

We illustrate the Laplace transform technique by a simple example of harmonic oscillator with a time-dependent external force $f(t) = t$.

$$\ddot{x} + \omega^2 x = t, \quad (15)$$

$$x(0) = a, \quad \dot{x}(0) = b. \quad (16)$$

The starting point is to Laplace-transform the equation:

$$\int_0^{\infty} dt e^{-pt} [\ddot{x} + \omega^2 x] = \int_0^{\infty} dt e^{-pt} t. \quad (17)$$

For the derivatives in the l.h.s. of (17) we use (4), with (16) taken into account. This yields

$$\int_0^{\infty} dt e^{-pt} [\ddot{x} + \omega^2 x] = (p^2 + \omega^2)X(p) - pa - b, \quad (18)$$

where

$$X(p) = \int_0^{\infty} dt e^{-pt} x(t). \quad (19)$$

The integral in the r.h.s. of (17) needs to be done explicitly:

$$\int_0^{\infty} dt e^{-pt} t = -\frac{\partial}{\partial p} \int_0^{\infty} dt e^{-pt} = -\frac{\partial}{\partial p} \frac{1}{p} = \frac{1}{p^2}. \quad (20)$$

We thus have

$$(p^2 + \omega^2)X(p) - pa - b = \frac{1}{p^2}, \quad (21)$$

and find

$$X(p) = \frac{1 + p^2(pa + b)}{p^2(p^2 + \omega^2)} = \frac{1 + p^2(pa + b)}{p^2(p - i\omega)(p + i\omega)}. \quad (22)$$



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Now we perform the inverse transform

$$x(t) = \frac{1}{2\pi i} \int_C \frac{[1 + p^2(pa + b)] e^{pt}}{p^2(p - i\omega)(p + i\omega)} dp . \quad (23)$$

That is we sum up all the residues of the function

$$\frac{[1 + p^2(pa + b)] e^{pt}}{p^2(p - i\omega)(p + i\omega)} . \quad (24)$$

There are three poles. The two simple poles $p = \pm i\omega$ come from the l.h.s. of the original differential equation, while the second-order pole at $p = 0$ is totally due to the particular form of the external force. The residue calculus routine leads to the final answer

$$x(t) = t/\omega^2 + a \cos \omega t + (b/\omega - 1/\omega^3) \sin \omega t . \quad (25)$$



Problem 45. Use the Laplace transform method to solve for the evolution of a damped harmonic oscillator with a time-independent external force:

$$\ddot{x} + \omega^2 x + \gamma \dot{x} = f_0, \quad (26)$$

$$x(0) = 0, \quad \dot{x}(0) = 0, \quad (27)$$

Find $x(t)$; make sure it is real and does satisfy the equation and the initial conditions. Compare two cases: $0 < \gamma < 2\omega$ and $\gamma \geq 2\omega$.

If we are dealing with a system of linear differential equations, the procedure is the same. We Laplace-transform each of the equations and arrive at a system of *algebraic* equations for the Laplace transforms of the unknown functions. From this system we find each of the Laplace transforms and then perform the inverse transforms.



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Problem 46. Use the Laplace transform method to solve for the evolution of one of two coupled harmonic oscillators (the coupling force is proportional to the relative displacement of the oscillators, the proportionality coefficient $\lambda > 0$):

$$\ddot{x}_1 + \omega_1^2 x_1 = \lambda(x_2 - x_1), \quad (28)$$

$$\ddot{x}_2 + \omega_2^2 x_2 = \lambda(x_1 - x_2). \quad (29)$$

Find $x_1(t)$ for $t > 0$ under the following initial conditions:

$$x_1(0) = 0, \quad \dot{x}_1(0) = 0, \quad x_2(0) = 0, \quad \dot{x}_2(0) = 1. \quad (30)$$

Make sure $x_1(t)$ is real and does satisfy the initial conditions.



Laplace Transform

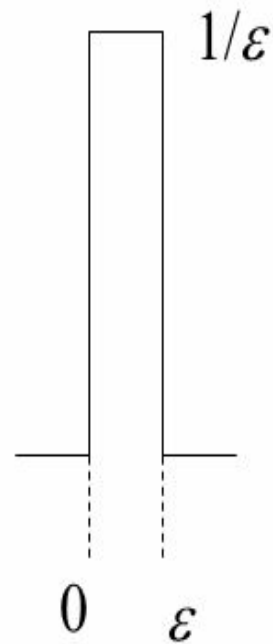
- Applications of the Laplace transform
 - solve differential equations (both ordinary and partial)
 - application to RLC circuit analysis
- Laplace transform converts differential equations in the time domain to algebraic equations in the frequency domain, thus 3 important processes:
 - (1) transformation from the time to frequency domain
 - (2) manipulate the algebraic equations to form a solution
 - (3) inverse transformation from the frequency to time domain



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Laplace transform of an impulse 'function' (Dirac delta distribution)



$$L(\delta(t)) = \lim_{\epsilon \rightarrow 0} \int_0^{\epsilon} \frac{1}{\epsilon} e^{-st} dt = \frac{-1}{s\epsilon} e^{-st} \Big|_0^{\epsilon} =$$

$$\lim_{\epsilon \rightarrow 0} \left(\frac{[e^{-s\epsilon} - 1]}{-s\epsilon} \right)$$

Writing e^x as a McLaurin series

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 h.o.t.$$

Writing $e^{-s\epsilon}$ as a McLaurin series

$$e^{-s\epsilon} = 1 + (-s\epsilon) + \frac{1}{2!}(-s\epsilon)^2 + \frac{1}{3!}(-s\epsilon)^3 h.o.t.$$



Laplace transform of impulse (Dirac delta distribution)

$$e^{-s\varepsilon} - 1 = (-s\varepsilon) + \frac{1}{2!}(-s\varepsilon)^2 + \frac{1}{3!}(-s\varepsilon)^3 + h.o.t.$$

$$\frac{e^{-s\varepsilon} - 1}{-s\varepsilon} = 1 + \frac{1}{2!}(-s\varepsilon) + \frac{1}{3!}(-s\varepsilon)^2 + h.o.t.$$

$$L(\delta(t)) = \lim_{\varepsilon \rightarrow 0} \left[\frac{e^{-s\varepsilon} - 1}{-s\varepsilon} \right] = \lim_{\varepsilon \rightarrow 0} \left[1 + \frac{1}{2!}(-s\varepsilon) + \frac{1}{3!}(-s\varepsilon)^2 + h.o.t. \right] = 1$$

$$L(\delta(t)) = 1$$



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Solutions to two topics
are left as an
assignment.

Submit it before last
working day of the
Semester.



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Thank you!
All the best for Exams !