



Digital Signal & Image Processing

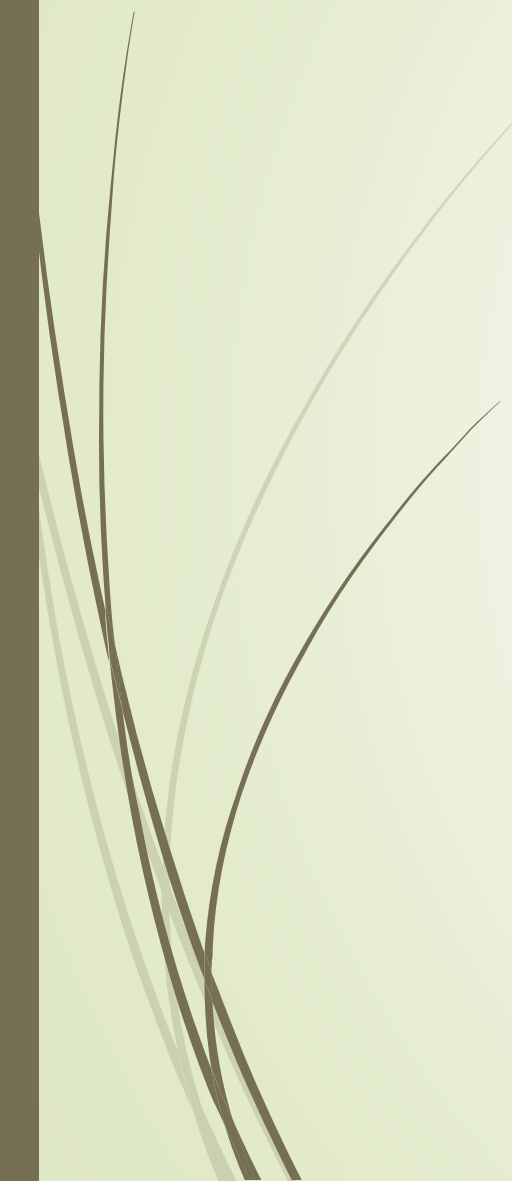
Lecture-2

Dr Prasanthi Rathnala

Department of ECE



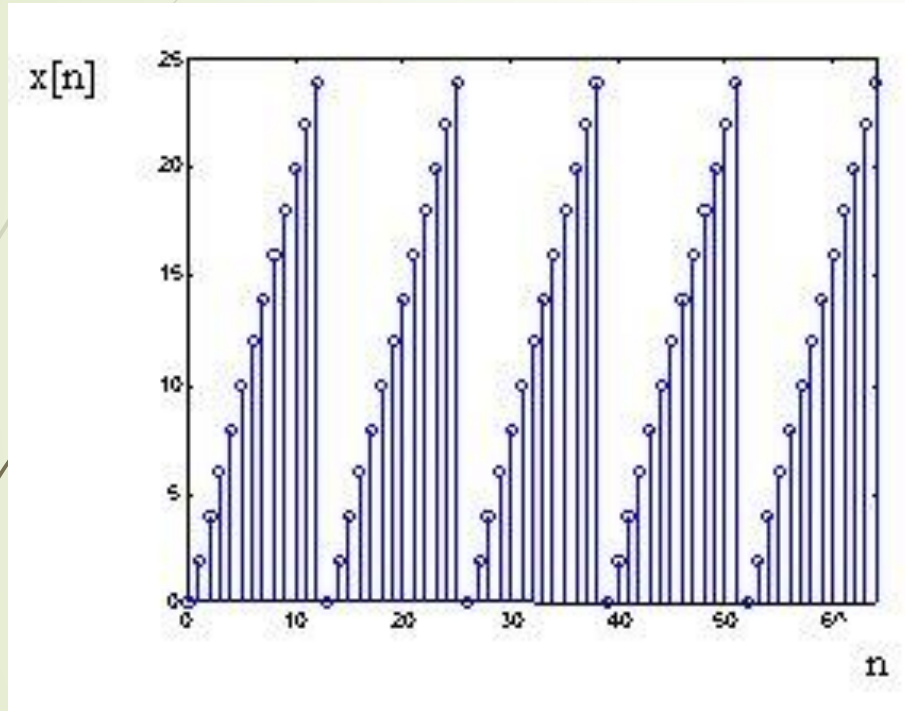
Overview

- Signal Spectrum
 - Periodic vs. Non-Periodic Signal
 - Spectra of Non-Periodic Signals
 - Properties of DTFT
 - Spectra of Periodic Signals
- 

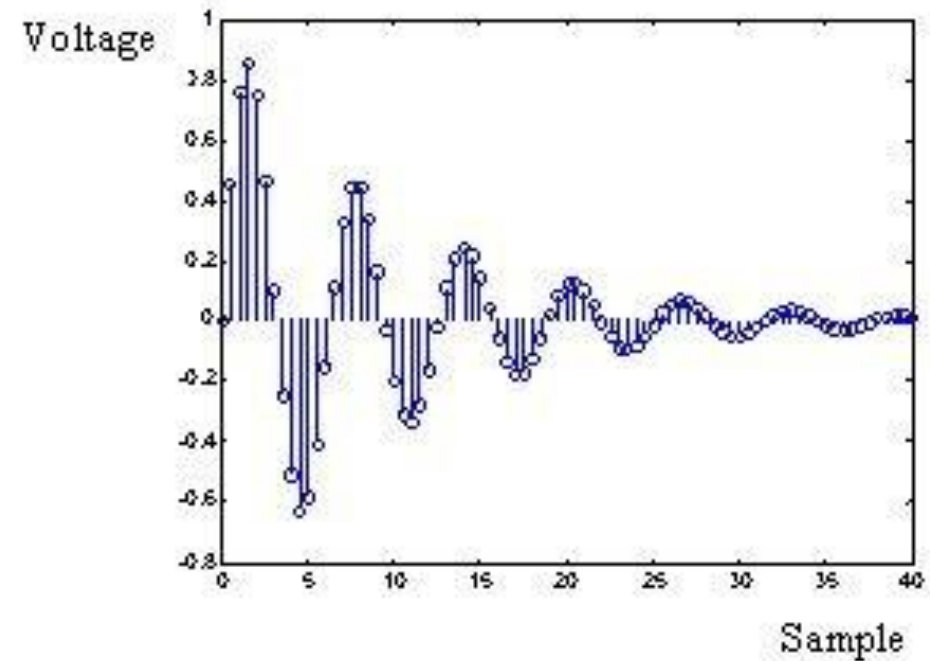
Signal Spectrum

- ▶ The **spectrum** of a signal describes its frequency content.
- ▶ For example, a sinusoid contains a single frequency, while white noise contains all frequencies.
- ▶ The tool used to calculate an accurate spectrum depends on the nature of the signal i.e., (Periodic or Non-Periodic)

Periodic vs. Non-Periodic Signal



- Periodic signals are those that repeat at regular intervals for all time.



- Non-periodic signals do not repeat at regular intervals.

Spectra of Non-Periodic Signals

- ▶ If the signal is non-periodic, the discrete time Fourier transform (DTFT) is used.
- ▶ The DTFT for a non-periodic signal gives signal's spectrum as

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-jn\Omega}$$

- ▶ The DTFT spectrum $X(\Omega)$ is a complex number and may be expressed as

$$X(\Omega) = |X(\Omega)|e^{j\theta(\Omega)}$$

Properties of DTFT

- The spectrum for both non-periodic and periodic signals furnishes a magnitude spectrum and a phase spectrum.
- The Magnitude Spectrum- relates to the size or amplitude of the components at each frequency.
- The Phase Spectrum- gives the phase relationships between the components at different frequencies.
- For non-periodic signals, $X(\Omega) = |X(\Omega)| e^{j\theta(\Omega)}$
- The magnitude spectrum is even and the phase spectrum is odd.
- Both magnitude and phase spectra are continuous, smooth and periodic with period 2π .

Properties of DTFT

- ▶ The **magnitude spectrum** may be plotted as
 $|X(\Omega)|$ versus digital frequency Ω
or
 $|X(f)|$ versus analog frequency f
- ▶ The **phase spectrum** may be plotted as
 $\theta(\Omega)$ versus digital frequency Ω
or
 $\theta(f)$ versus analog frequency f

Spectra of Non-Periodic Signals

- ▶ The calculation of the DTFT requires all samples of the non-periodic signal.
- ▶ When signal has infinite number of non-zero samples that decrease in size, the DTFT may be approximated by truncating the signal where its amplitude drops below some suitably low threshold.
- ▶ It is easier to interpret the spectra by converting the digital frequencies into analog frequencies.

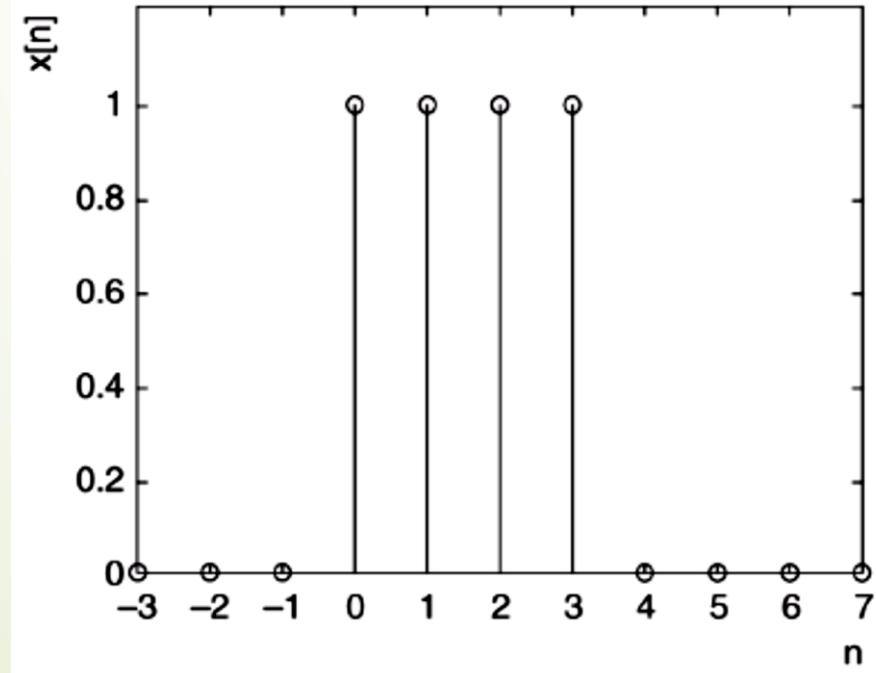
$$f = \Omega \frac{f_s}{2\pi}$$

Spectra of Non-Periodic Signals

Example 1: Find the magnitude and phase spectra for the rectangular pulse $x[n] = u[n] - u[n-4]$ as function of Ω . Plot linear gains and phases in radians.

Solution

► The rectangular pulse is plotted in following figure.



Spectra of Non-Periodic Signals

- ▶ The spectrum may be computed as

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-jn\Omega} = 1 + e^{-j\Omega} + e^{-j2\Omega} + e^{-j3\Omega}$$

- ▶ we can compute the spectrum by substituting the values of Ω
- ▶ The computation method is analogous to that used in the DTFT.

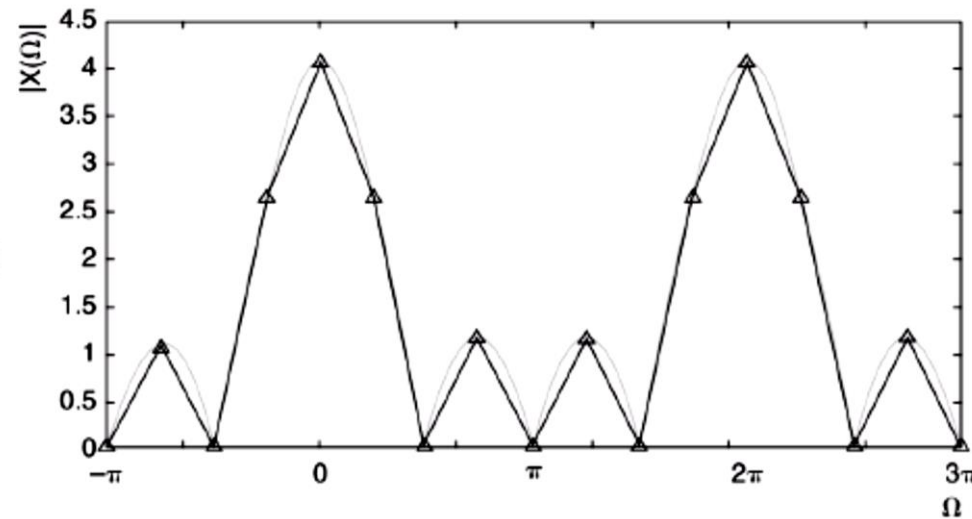
$$\begin{aligned} X(-\pi) &= 0.00 \angle 0 \\ X\left(-\frac{3\pi}{4}\right) &= 1.0824 \angle 0.3927 \\ &\vdots \\ X\left(\frac{11\pi}{4}\right) &= 1.0824 \angle -0.3927 \\ X(3\pi) &= 0.00 \angle 0 \end{aligned}$$

Ω	$X(\Omega)$	$ X(\Omega) $	$\theta(\Omega)$ (rads)
$-\pi$	$1 - 1 \left \frac{\pi}{4} + 1 \right \left 2\pi + 1 \right \left 3\pi \right $	0.0000	0.0000
$-\frac{3\pi}{4}$	$1 - 1 \left \frac{3\pi}{4} + 1 \right \left \frac{3\pi}{2} + 1 \right \left \frac{9\pi}{4} \right $	1.0824	0.3927
$-\frac{\pi}{2}$	$1 + 1 \left \frac{\pi}{2} + 1 \right \left \pi + 1 \right \left \frac{3\pi}{2} \right $	0.0000	0.0000
$-\frac{\pi}{4}$	$1 - 1 \left \frac{\pi}{4} + 1 \right \left \frac{\pi}{2} + 1 \right \left \frac{3\pi}{4} \right $	2.6131	1.1781
0	$1 + 1 \left 0 + 1 \right \left 0 + 1 \right \left 0 \right $	4.0000	0.0000
$\frac{\pi}{4}$	$1 - 1 \left -\frac{\pi}{4} + 1 \right \left -\frac{\pi}{2} + 1 \right \left -\frac{3\pi}{4} \right $	2.6131	-1.1781
$\frac{\pi}{2}$	$1 - 1 \left -\frac{\pi}{2} + 1 \right \left -\pi + 1 \right \left -\frac{3\pi}{2} \right $	0.0000	0.0000
$\frac{3\pi}{4}$	$1 + 1 \left -\frac{3\pi}{4} + 1 \right \left -\frac{3\pi}{2} + 1 \right \left -\frac{9\pi}{4} \right $	1.0824	-0.3927
π	$1 + 1 \left -\pi + 1 \right \left -2\pi + 1 \right \left -3\pi \right $	0.0000	0.0000
$\frac{5\pi}{4}$	$1 + 1 \left -\frac{5\pi}{4} + 1 \right \left -\frac{5\pi}{2} + 1 \right \left -\frac{15\pi}{4} \right $	1.0824	0.3927
$\frac{3\pi}{2}$	$1 + 1 \left -\frac{3\pi}{2} + 1 \right \left -3\pi + 1 \right \left -\frac{9\pi}{2} \right $	0.0000	0.0000
$\frac{7\pi}{4}$	$1 + 1 \left -\frac{7\pi}{4} + 1 \right \left -\frac{7\pi}{2} + 1 \right \left -\frac{21\pi}{4} \right $	2.6131	1.1781
2π	$1 + 1 \left -2\pi + 1 \right \left -4\pi + 1 \right \left -6\pi \right $	4.0000	0.0000
$\frac{9\pi}{4}$	$1 + 1 \left -\frac{9\pi}{4} + 1 \right \left -\frac{9\pi}{2} + 1 \right \left -\frac{27\pi}{4} \right $	2.6131	-1.1781
$\frac{5\pi}{2}$	$1 + 1 \left -\frac{5\pi}{2} + 1 \right \left -5\pi + 1 \right \left -\frac{15\pi}{2} \right $	0.0000	0.0000
$\frac{11\pi}{4}$	$1 + 1 \left -\frac{11\pi}{4} + 1 \right \left -\frac{11\pi}{2} + 1 \right \left -\frac{33\pi}{4} \right $	1.0824	-0.3927
3π	$1 + 1 \left -3\pi + 1 \right \left -6\pi + 1 \right \left -9\pi \right $	0.0000	0.0000

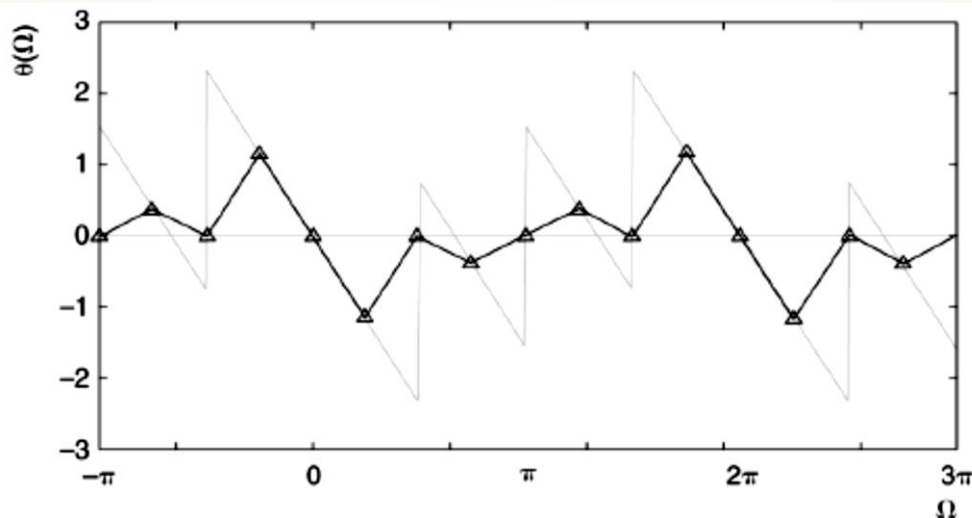
Spectra of Non-Periodic Signals

The magnitude and phase spectra are plotted for the range $-\pi \leq \Omega \leq 3\pi$

(a) Magnitude Spectrum

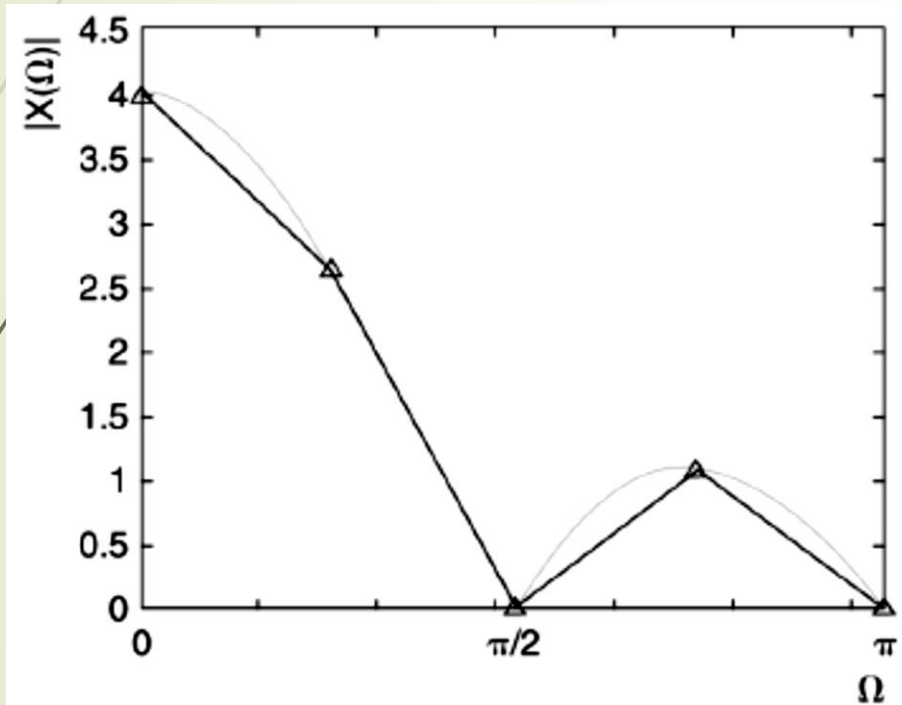


(b) Phase Spectrum

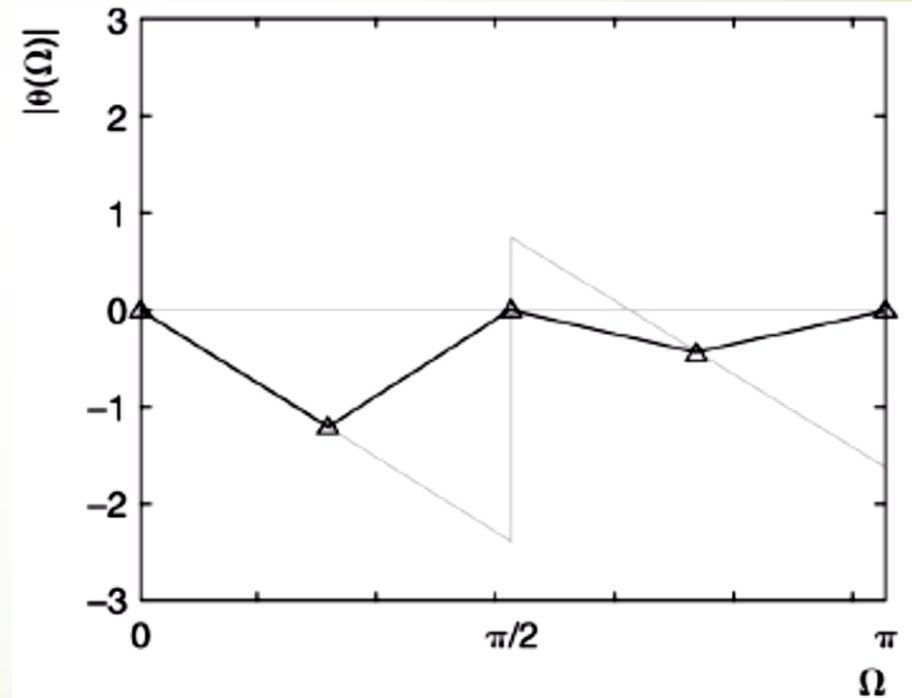


Spectra of Non-Periodic Signals

The magnitude and phase spectra are plotted for the range $0 \leq \Omega \leq \pi$



(a) Magnitude Spectrum

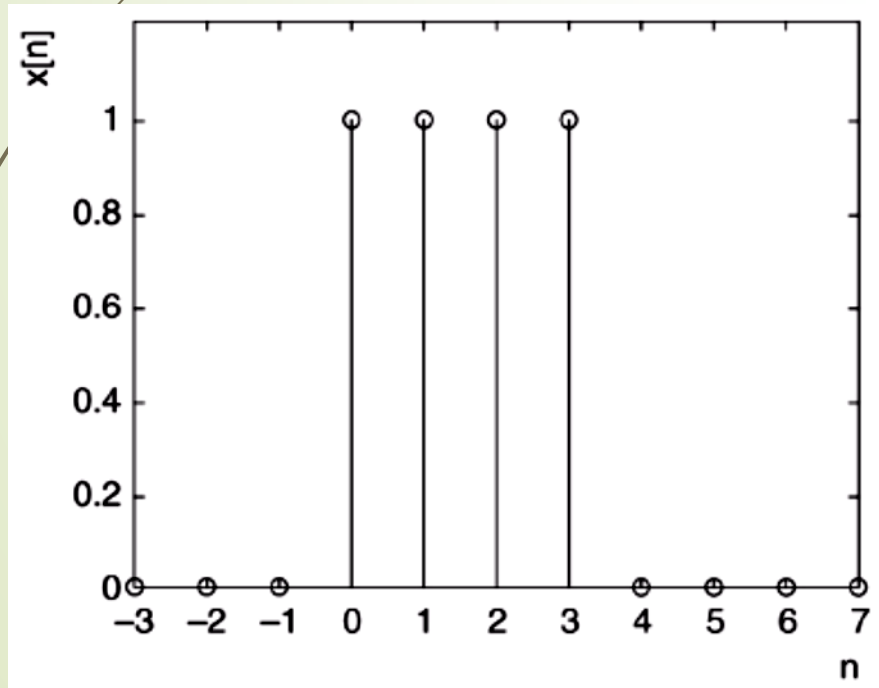


(b) Phase Spectrum

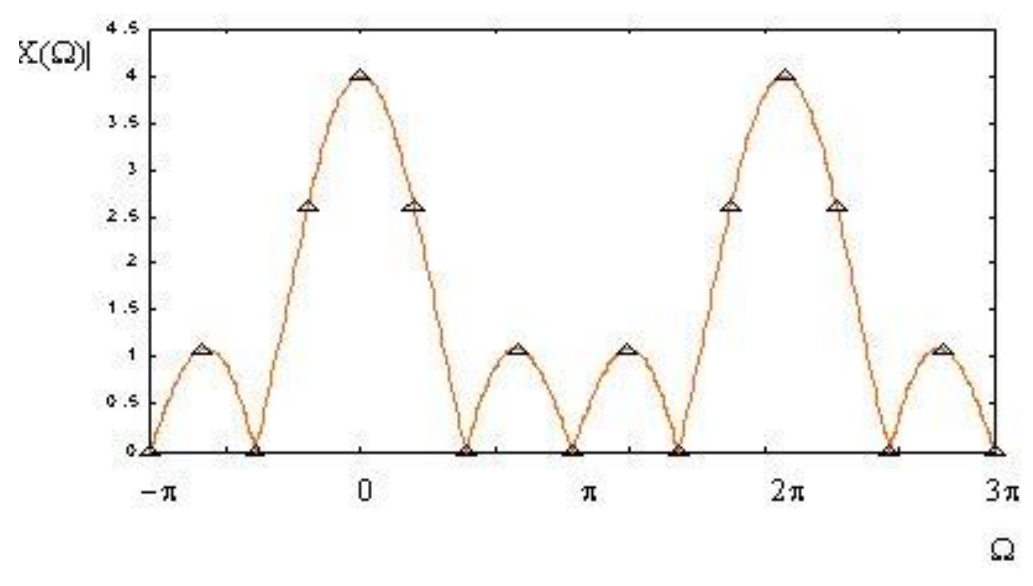
Spectra of Non-Periodic Signals

- Magnitude spectrum has a shape that is characteristic for all rectangular pulses, called a **sinc function**

Rectangular Pulse Signal



Magnitude Spectrum



Spectra of Non-Periodic Signals

Example 2: Find the magnitude and phase spectra for the signal $x[n] = (0.1)^n u[n]$, sampled at 15kHz. Plot the magnitude in dB and phase in degrees.

Solution

$$x[n] = \{1 \quad 0.1 \quad 0.01 \quad 0.001 \quad 0.0001 \quad \dots\}$$

- ▶ The signal is
- ▶ Since the sample amplitudes drop off quickly, the first three samples are sufficient to obtain good approximation to the DTFT spectrum.

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-jn\Omega} = 1 + 0.1e^{-j\Omega} + 0.01e^{-j2\Omega}$$

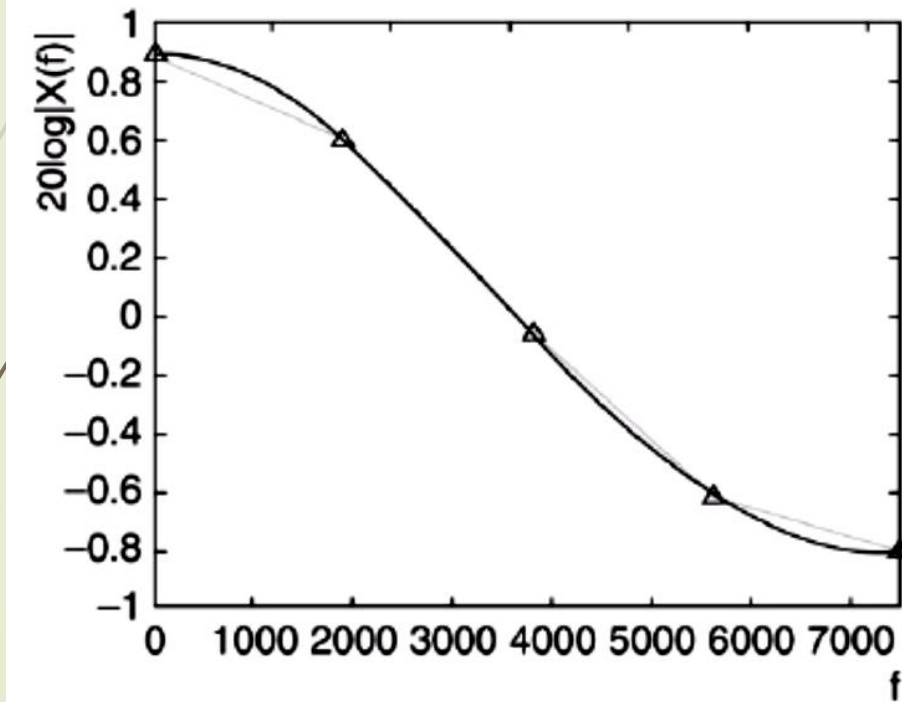
Spectra of Non-Periodic Signals

TABLE 8.3
Coefficients for Example 8.2

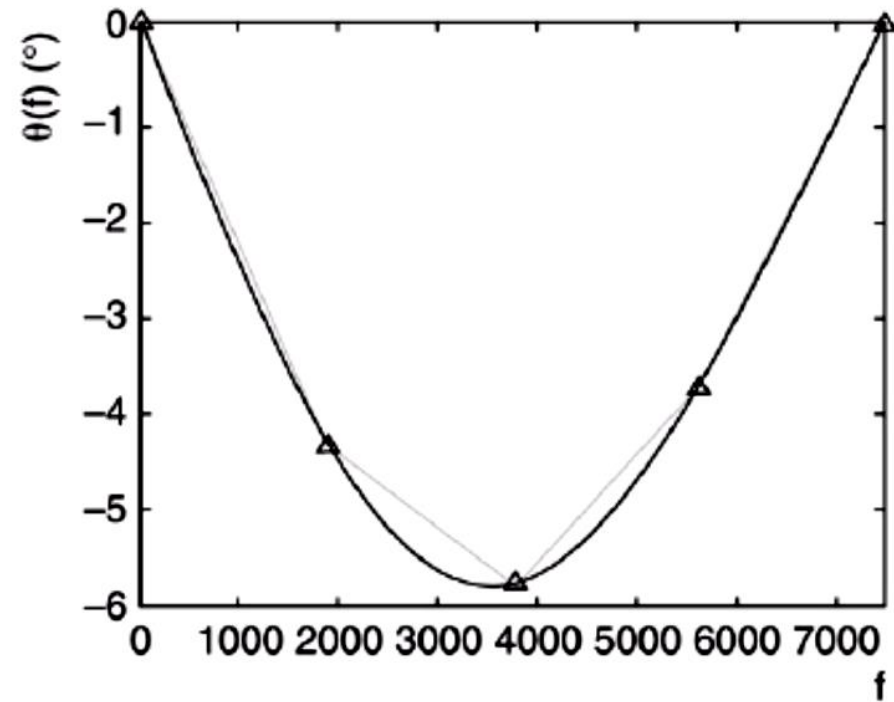
n	$X(n)$	$ X(n) $	$\theta(n)$ (rads)	$20 \log_{10} X(n) $ (dB)	$\theta(n)$ ($^{\circ}$)	f (Hz)
0	$1 - 0.1 0 + 0.01 0$	1.1100	0.0000	0.9065	0.0000	0
4	$1 + 0.1 \left[\frac{\pi + 0.01}{4} - \frac{\pi}{2} \right]$	1.0737	-0.0752	0.6181	-4.3108	1875
8	$1 - 0.1 \left[\frac{\pi + 0.01}{2} - \pi \right]$	0.9950	-0.1007	-0.0432	-5.7679	3750
12	$1 + 0.1 \left[\frac{3\pi + 0.01}{4} - \frac{3\pi}{2} \right]$	0.9313	-0.0652	-0.6185	-3.7378	5625
16	$1 + 0.1 \left[\pi + 0.01 \right] - 2\pi$	0.9100	0.0000	-0.8192	0.0000	7500

Spectra of Non-Periodic Signals

Magnitude Spectrum

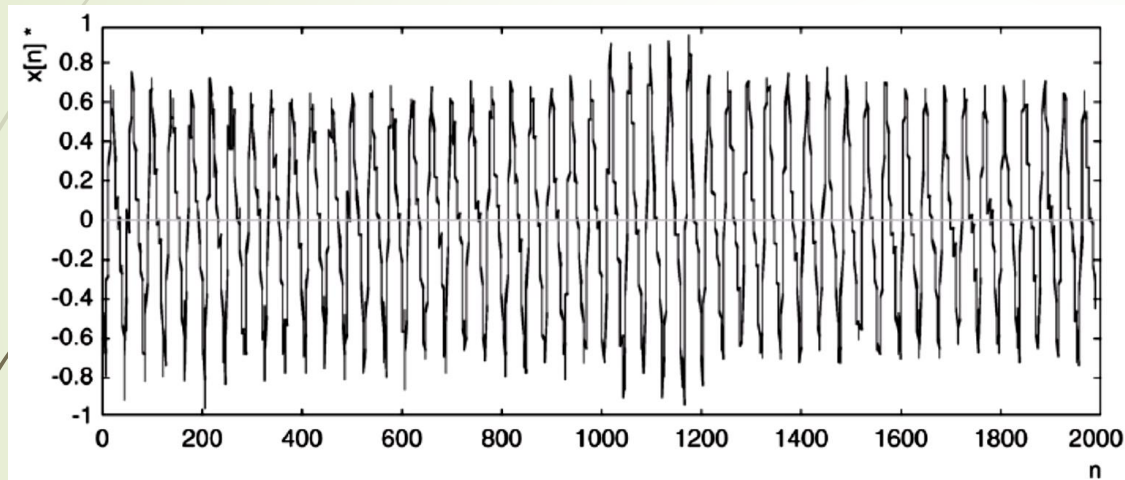


Phase Spectrum

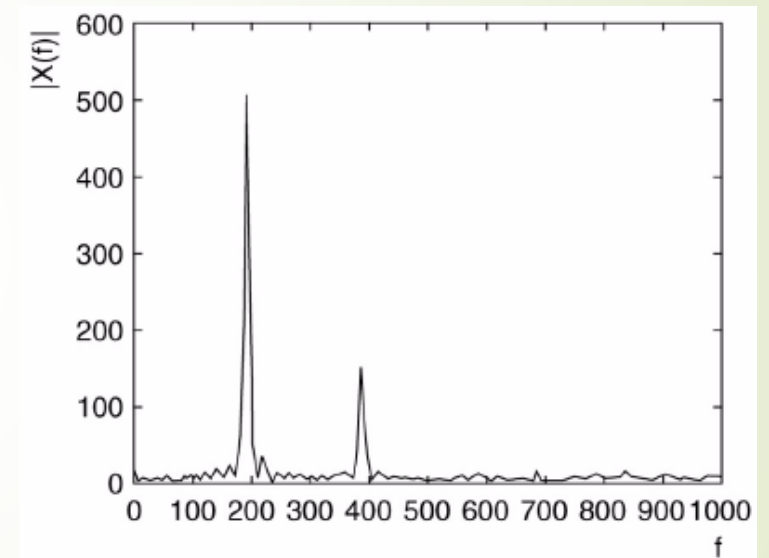


Spectra of Non-Periodic Signals

Example 3: A piece of the spoken vowel “eee” sampled at 8 kHz, and their spectrum are shown in the figures. What are the main frequency components of $x[n]$?



* For clarity, the envelope rather than the individual samples of the signal is plotted.

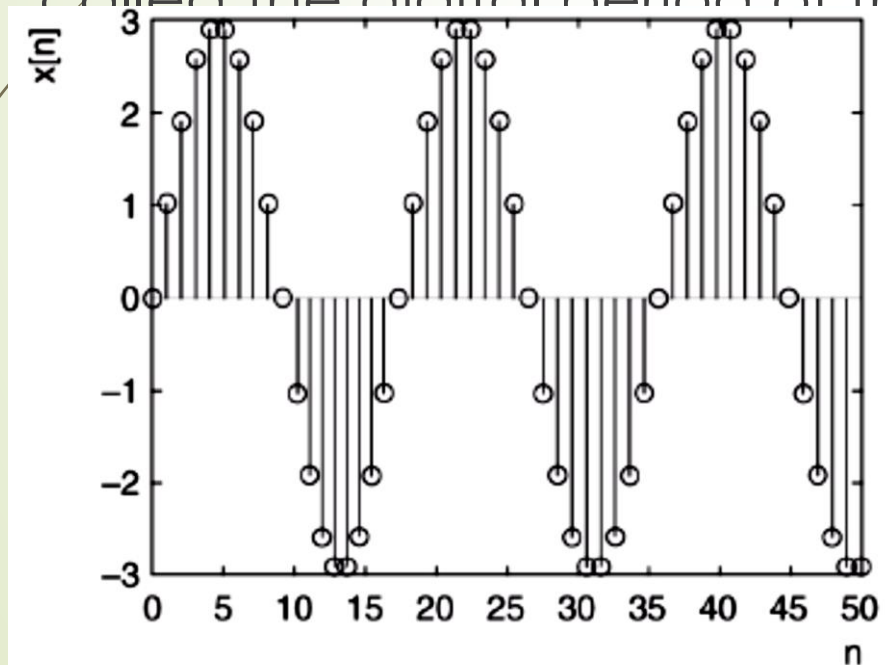


Solution

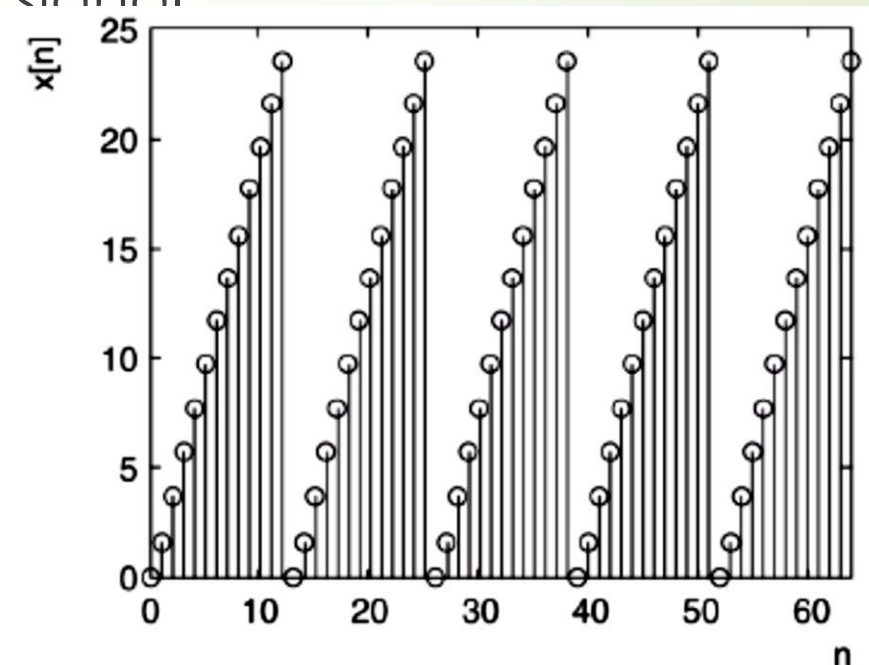
- The vowel “eee” sound is quite regular but not perfectly periodic.
- As magnitude spectrum shows the vowel “eee” consists almost exclusively of 200 Hz and 400 Hz frequency components.

Spectra of Periodic Signals

- ▶ **Periodic signals** are those that repeat at regular intervals for all time.
- ▶ The number of samples that occur in each interval is called the digital period of the signal



(a) Sinusoidal Signal



(b) Sawtooth Signal

Spectra of Periodic Signals

- ▶ In periodic signals, the same sequence repeats over all time, the DTFT is not an appropriate tool for calculating the spectrum.
- ▶ The infinite sum that is part of the DTFT would give an infinite result.
- ▶ The tool needed to find the spectrum of a periodic signals is the Discrete Fourier Series (DFS).

Spectra of Periodic Signals

- ▶ If the signal is non-periodic, the discrete time Fourier transform (DTFT) is used:

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-jn\Omega}$$

- ▶ If the signal is periodic, the discrete Fourier series (DFS) is used:

$$C_k = \sum_{n=0}^{N-1} x[n]e^{-j2\pi\frac{k}{N}n}$$

Spectra of Periodic Signals

- ▶ According to Fourier theory, every periodic signal can be expressed as the sum of sines and cosines or compactly, as the sum of complex exponentials.
- ▶ The Fourier series representation for a periodic signal $x[n]$ with period N is

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} c_k e^{j2\pi \frac{k}{N}n}$$

Where

n is the sample number

k is the coefficient number

$1/N$ is the scaling factor to recover $x[n]$ from its Fourier expansion

Spectra of Periodic Signals

- ▶ The Fourier coefficient c_k are calculated from the signal samples as

$$c_k = \sum_{n=0}^{N-1} x[n] e^{-j2\pi \frac{k}{N} n}$$

- ▶ Since $x[n]$ has a period of N samples therefore only N samples of the signal need to be used to find the coefficient c_k for all k .

Spectra of Periodic Signals

- ▶ The Fourier coefficient \mathbf{c}_k are complex numbers and may be written in the polar form as

$$\mathbf{c}_k = |\mathbf{c}_k|e^{j\varphi_k}$$

- ▶ The magnitude spectrum is plotted as $|\mathbf{c}_k|$ versus \mathbf{k}
- ▶ The phase spectrum is plotted as φ_k versus \mathbf{k}
- ▶ The DFS is **periodic** with period N
- ▶ The magnitude spectrum of DFS is always **Even**
- ▶ The phase spectrum of DFS is **Odd**
- ▶ Both magnitude and phase spectra for periodic signals are **line function**, with line spacing \mathbf{f}_s/N Hz, and contributions only at DC and harmonic frequencies.

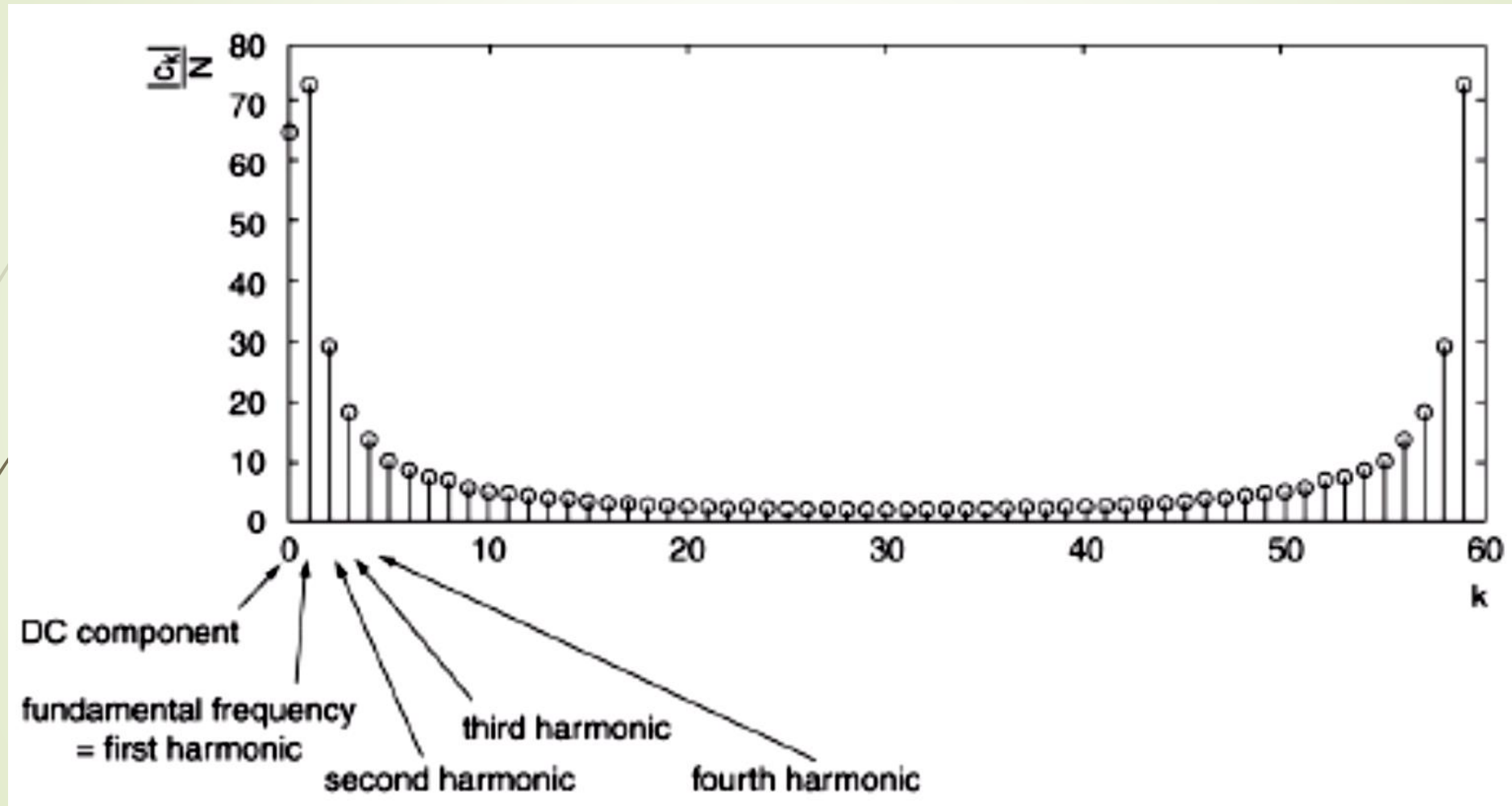
Spectra of Periodic Signals

- ▶ The DFS index k corresponds to the analog frequency

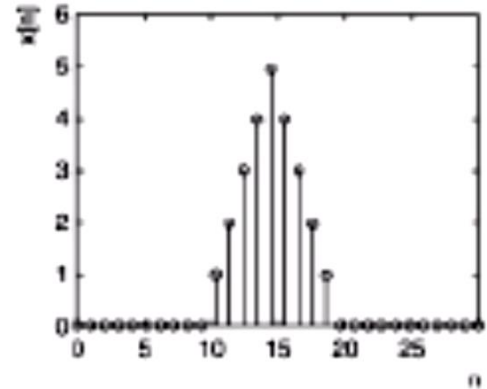
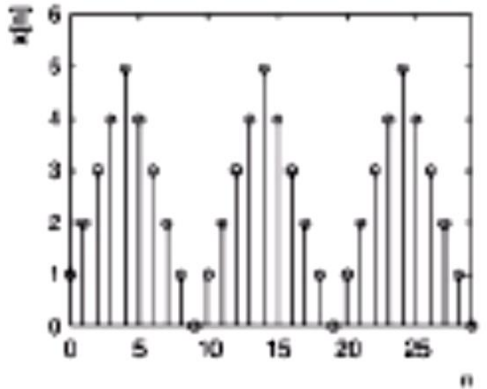
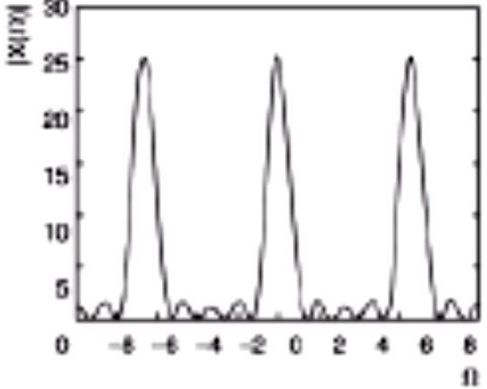
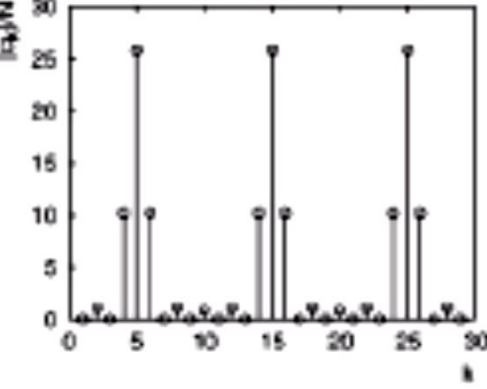
$$f = k \frac{f_s}{N}$$

- ▶ The $k = 0$ coefficient gives the DC component of the signal
- ▶ The $k = 1$ coefficient gives the Fundamental frequency f_s/N or first harmonic of the signal
- ▶ The reciprocal of the fundamental frequency f_s/N gives the one full cycle of the signal in time domain that is NT_s , where T_s is the sampling interval.
- ▶ Harmonics are the integer multiple of the fundamental frequency
- ▶ The $k > 1$ coefficient gives the higher harmonic of the signal

Spectra of Periodic Signals



DTFT vs. DFS

	Discrete Time Fourier Transform	Discrete Fourier Series
Type of Signal	nonperiodic	periodic
TIME DOMAIN		
Type of Spectrum	continuous, periodic	line, periodic
FREQUENCY DOMAIN		

Complex Numbers

- **Rectangular Form:** $x + jy$
- **Polar Form:** $r\angle\theta$
- **Euler Form:** $e^{j\theta} = \cos\theta + j\sin\theta$
- **Order Pair Form:** $(\cos\theta, \sin\theta)$

Rectangular to Polar Conversion

- $r = \sqrt{x^2 + y^2}$
- $\theta = \tan^{-1} \frac{y}{x}$

Polar to Rectangular Conversion

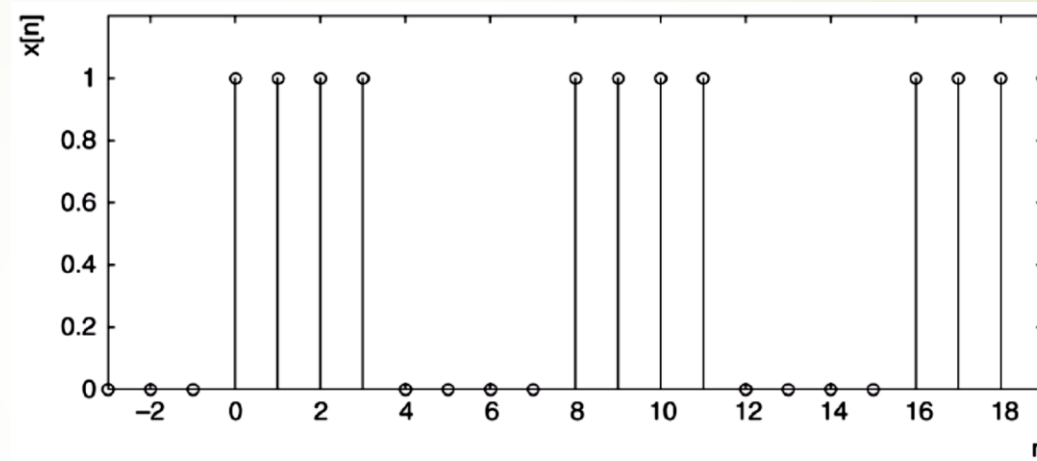
- $x = r\cos\theta$
- $y = r\sin\theta$

Note:

- for Addition (+) and Subtraction (-) use the Rectangular form
- for Multiplication (\times) and Division ($/$) use the Polar form

Spectra of Periodic Signals

Example 4: Find the magnitude and phase spectra for the periodic signal shown in the figure.



Solution

- The signal is periodic with period $N = 8$.

$$c_k = \sum_{n=0}^{N-1} x[n] e^{-j2\pi \frac{k}{N} n}$$

$$c_k = \sum_{n=0}^{8-1} x[n] e^{-j2\pi \frac{k}{8} n}$$

Spectra of Periodic Signals

$$c_k = 1 + e^{-j\pi\frac{k}{4}} + e^{-j\pi\frac{k}{2}} + e^{-j3\pi\frac{k}{4}}$$

When $k = 0$ $c_0 = 1 + e^{-j\pi\frac{0}{4}} + e^{-j\pi\frac{0}{2}} + e^{-j3\pi\frac{0}{4}} = 1 + 1 + 1 + 1 = 4$

$$c_0 = 1 + 1 + 1 + 1 = 4$$

$$c_0 = |4| \angle 0 \text{ rad}$$

$$\frac{|c_0|}{8} = \frac{|4|}{8} = 0.5$$

Spectra of Periodic Signals

$$c_k = 1 + e^{-j\pi\frac{k}{4}} + e^{-j\pi\frac{k}{2}} + e^{-j3\pi\frac{k}{4}}$$

When $k = 1$ $c_1 = 1 + e^{-j\pi\frac{1}{4}} + e^{-j\pi\frac{1}{2}} + e^{-j3\pi\frac{1}{4}}$

$$c_1 = 1 + [\cos(\pi/4) - j\sin(\pi/4)] + [\cos(\pi/2) - j\sin(\pi/2)] + [\cos(3\pi/4) - j\sin(3\pi/4)]$$

$$c_1 = 1 + [0.707 - j0.707] + [0 - j] + [-0.707 - j0.707]$$

$$c_1 = 1 - j2.414$$

$$c_1 = |2.6129| \angle -1.1781 \text{ rad}$$

$$\frac{|c_1|}{8} = \frac{|2.6129|}{8} = 0.3266$$

Note: compute the other values of C_k for $k = 2, 3, 4, 5, 6,$ and 7 by following the above procedure.

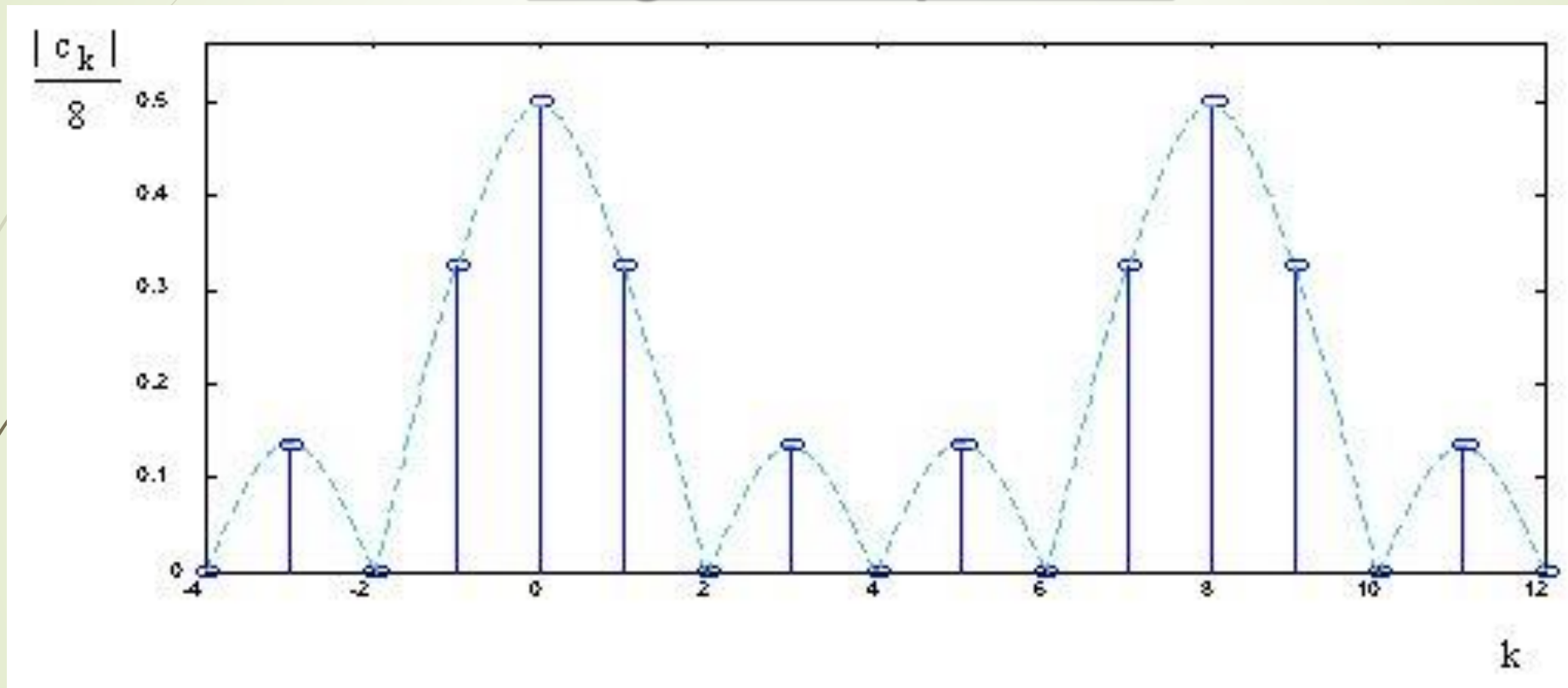
Spectra of Periodic Signals

$$c_k = 1 + e^{-j\pi\frac{k}{4}} + e^{-j\pi\frac{k}{2}} + e^{-j3\pi\frac{k}{4}}$$

k	c_k	$ c_k /8$	ϕ_k
0	$1 + 1 _0 + 1 _0 + 1 _0$	0.5000	0.0
1	$1 + 1 _{-\frac{\pi}{4}} + 1 _{-\frac{\pi}{2}} + 1 _{-\frac{3\pi}{4}}$	0.3266	-1.1781
2	$1 + 1 _{-\frac{\pi}{2}} + 1 _{-\pi} + 1 _{-\frac{3\pi}{2}}$	0.0	0.0
3	$1 + 1 _{-\frac{3\pi}{4}} + 1 _{-\frac{3\pi}{2}} + 1 _{-\frac{3\pi}{2}}$	0.1353	-0.3927
4	$1 + 1 _{-\pi} + 1 _{-2\pi} + 1 _{-3\pi}$	0.0	0.0
5	$1 + 1 _{-\frac{5\pi}{4}} + 1 _{-\frac{5\pi}{2}} + 1 _{-\frac{15\pi}{4}}$	0.1353	0.3927
6	$1 + 1 _{-\frac{3\pi}{2}} + 1 _{-3\pi} + 1 _{-\frac{9\pi}{2}}$	0.0	0.0
7	$1 + 1 _{-\frac{7\pi}{4}} + 1 _{-\frac{7\pi}{2}} + 1 _{-\frac{21\pi}{4}}$	0.3266	1.1781

Spectra of Periodic Signals

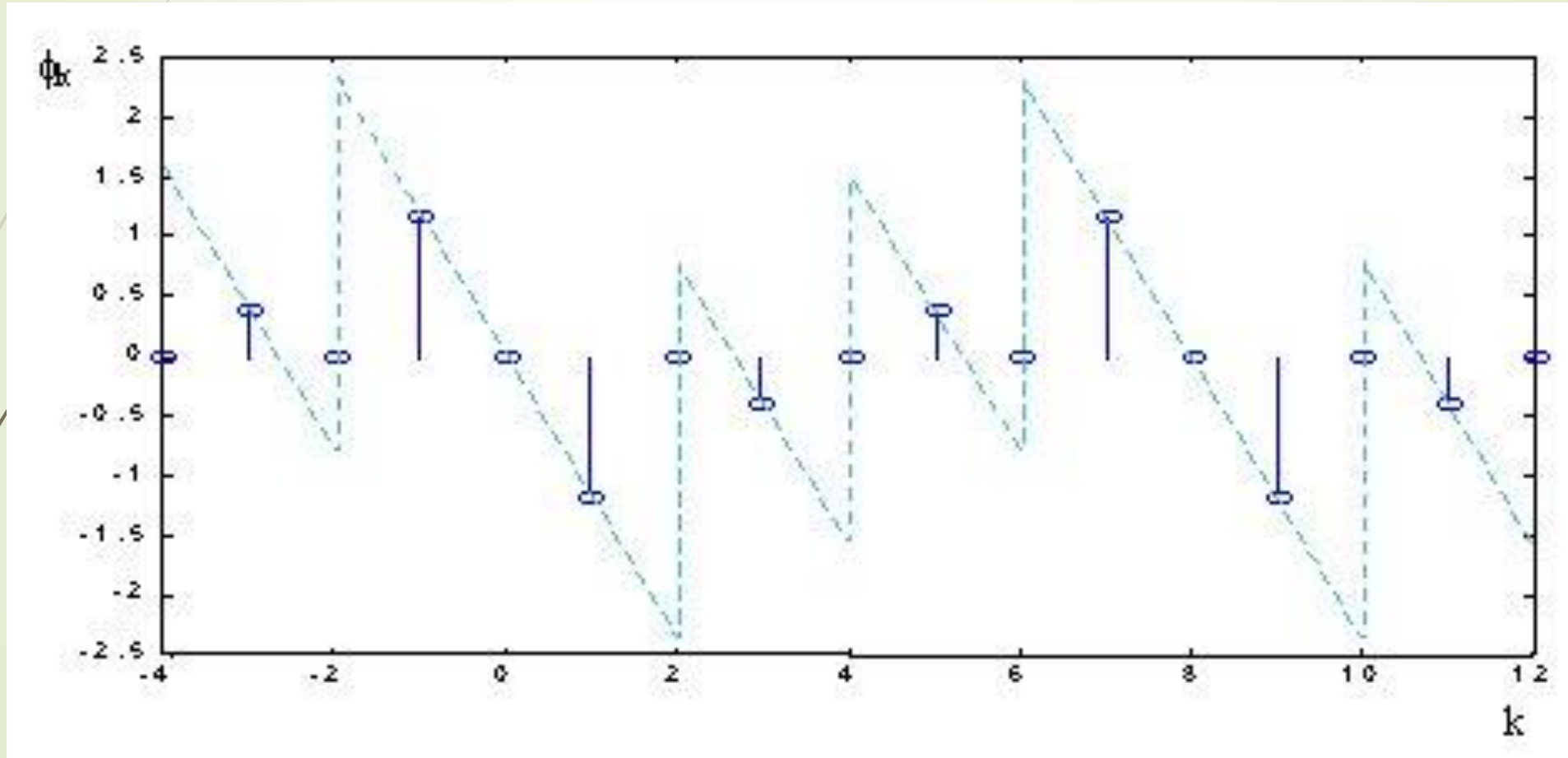
Magnitude Spectrum



- The dashed line in the magnitude spectrum for this square wave shows that its envelope has the shape of the absolute value of a **sinc function**.
- All square and rectangular periodic signals have this

Spectra of Periodic Signals

Phase Spectrum

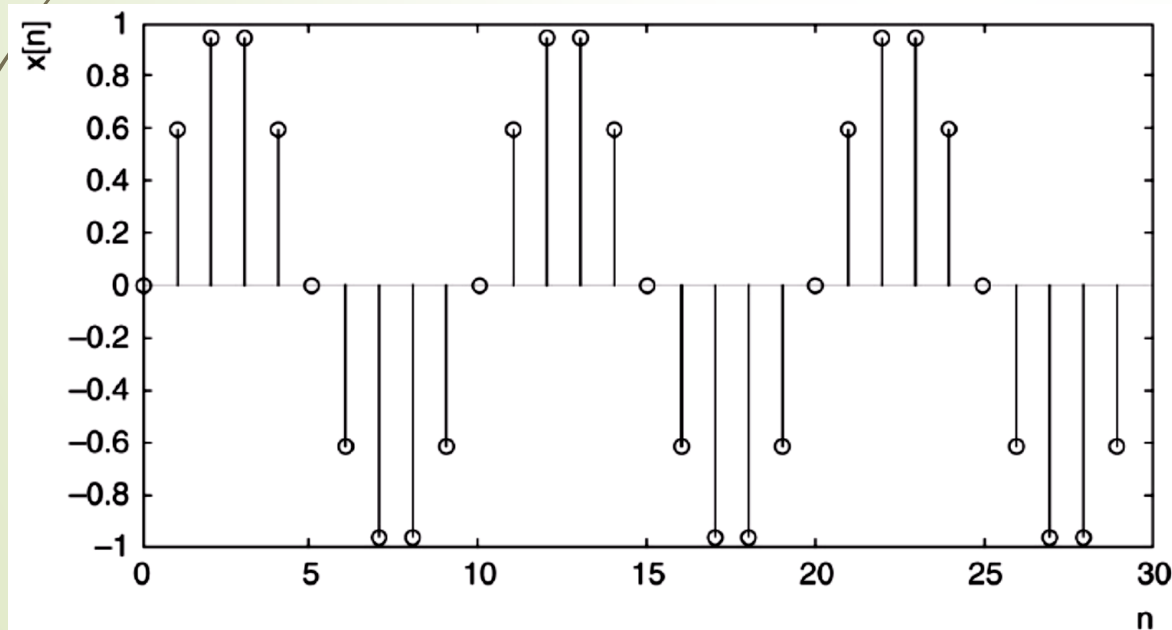


Spectra of Periodic Signals

Example 5: Find the magnitude and phase spectra for the periodic signal $x[n] = \sin(n\pi/5)$ with sampling rate is 1 kHz.

Solution

- The sample values are listed in the Table.
- The signal is plotted in the figure.
- The signal is periodic with period $N = 10$



n	$x[n]$
0	0.0000
1	0.5878
2	0.9511
3	0.9511
4	0.5878
5	0.0000
6	-0.5878
7	-0.9511
8	-0.9511
9	-0.5878

Spectra of Periodic Signals

36

$$c_k = \sum_{n=0}^{10-1} x[n]e^{-j2\pi\frac{k}{10}n}$$

k	c_k	$ c_k /V$	ϕ_k
0	0.0000	0.0000	0.0000
1	$-j5.0000$	0.50000	-1.5708
2	0.0000	0.0000	0.0000
3	0.0000	0.0000	0.0000
4	0.0000	0.0000	0.0000
5	0.0000	0.0000	0.0000
6	0.0000	0.0000	0.0000
7	0.0000	0.0000	0.0000
8	0.0000	0.0000	0.0000
9	$j5.0000$	0.50000	1.5708

Spectra of Periodic Signals

Magnitude Spectrum

