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Centurion University of Technology And Management

De• nitions

De• nition 1.1 *Complex numbers are de• ned as ordered pairs* (x, y)

Points on a complex plane. Real axis, imaginary axis, purely imaginary numbers. Real and imaginary parts of complex number. Equality of two complex numbers.

De• nition 1.2 *The sum and product of two complex numbers are de• ned as follows:*

$$\begin{aligned}(x_1, y_1) + (x_2, y_2) &= (x_1 + x_2, y_1 + y_2)\\ (x_1, y_1) \cdot (x_2, y_2) &= (x_1x_2 - y_1y_2, x_1y_2 + x_2y_1)\end{aligned}$$

In the rest of the chapter use $z, z_1, z_2, \ldots$ for complex numbers and x, y for real numbers. introduce i and $z = x + iy$ notation.

Algebraic Properties

1. Commutativity
$$z_1 + z_2 = z_2 + z_1, \qquad z_1 z_2 = z_2 z_1$$
2. Associativity
$$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3), \qquad (z_1 z_2) z_3 = z_1 (z_2 z_3)$$
3. Distributive Law
$$z(z_1 + z_2) = zz_1 + zz_2$$
4. Additive and Multiplicative Indentity
$$z + 0 = z, \qquad z \cdot 1 = z$$
5. Additive and Multiplicative Inverse
$$\begin{aligned}-z &= (-x, -y)\\ z^{-1} &= \left(\frac{x}{x^2 + y^2}, \frac{-y}{x^2 + y^2}\right), \qquad z \neq 0\end{aligned}$$

6. Subtraction and Division

$$z_1 - z_2 = z_1 + (-z_2), \qquad \frac{z_1}{z_2} = z_1 z_2^{-1}$$

7. Modulus or Absolute Value

$$|z| = \sqrt{x^2 + y^2}$$

8. Conjugates and properties

$$\begin{aligned} \overline{z} &= x - iy = (x, -y) \\ \overline{z_1 \pm z_2} &= \overline{z_1} \pm \overline{z_2} \\ \overline{z_1 z_2} &= \overline{z_1}\,\overline{z_2} \\ \overline{\left(\frac{z_1}{z_2}\right)} &= \frac{z_1}{z_2} \end{aligned}$$

9.

$$\begin{aligned} |z|^2 &= z\overline{z} \\ \operatorname{Re} z &= \frac{z + \overline{z}}{2}, \operatorname{Im} z = \frac{z - \overline{z}}{2i} \end{aligned}$$

10. Triangle Inequality

$$|z_1 + z_2| \leq |z_1| + |z_2|$$

Polar Coordinates and Euler Formula

1. Polar Form: for $z \neq 0$,

$$z = r(\cos\theta + i\sin\theta)$$

where $r = |z|$ and $\tan\theta = y/x$. θ is called the argument of z. Since $\theta + 2n\pi$ is also an argument of z,the principle value of argument of z is take such that $-\pi < \theta \leq \pi$. For $z = 0$ the $\arg z$ is unde• ned.

2. Euler formula: Symbollically,

$$e^{i\theta} = (\cos\theta + i\sin\theta)$$

3.

$$\begin{aligned} z_1 z_2 &= r_1 r_2 e^{i(\theta_1 + \theta_2)} \\ \frac{z_1}{z_2} &= \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)} \\ z^n &= r^n e^{n\theta} \end{aligned}$$

4. de Moivre's Formula

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

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Roots of Complex Numbers

Let $z = re^{i\theta}$ then

$$z^{1/n} = r^{1/n} \exp\left(i\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right)\right)$$

There are only n distinct roots which can be given by $k = 0, 1, \ldots, n-1$. If θ is a principle value of $\arg z$ then θ/n is called the principle root.

Example 1.1 *The three possible roots of* $\left(\frac{1+i}{\sqrt{2}}\right)^{1/3} = \left(e^{i\pi/4}\right)^{1/3}$ *are* $e^{i\pi/12}$, $e^{i\pi/12+i2\pi/}$

Regions in Complex Plane

1. $\epsilon - nbd$ of z_0 is de• ned as a set of all points z which satisfy
$$|z - z_0| < \epsilon$$
2. *Deleted nbd* of z_0 is a nbd of z_0 excluding point z_0.
3. Interior Point, Exterior Point, Boundary Point, Open set and closed set.
4. Domain, Region, Bounded sets, Limit Points.

Limits

A function f is de• ned in a deleted nbd of z_0.

De• nition 3.1 *The limit of the function $f(z)$ as $z \to z_0$ is a number w_0 if, for any given $\epsilon > 0$ there exists a $\delta > 0$ such that*

$$|z - z_0| < \delta \Rightarrow |f(z) - w_0| < \epsilon.$$

Example 3.1 $f(z) = 5z$. *Show that* $\lim_{z \to z_0} f(z) = 5z_0$.

Example 3.2 $f(z) = z^2$. *Show that* $\lim_{z \to z_0} f(z) = z_0^2$.

Example 3.3 $f(z) = z/\bar{z}$. *Show that the limit of f does not exist as* $z \to 0$.

Theorem 3.1 *Let* $f(z) = u(x, y) + iv(x, y)$ *and* $w_0 = u_0 + iv_0$. $\lim_{z \to z_0} f(z) = w_0$ *if and only if* $\lim_{(x,y) \to (x_0, y_0)} u = u_0$ *and* $\lim_{(x,y) \to (x_0, y_0)} v = v_0$.

Example 3.4 $f(z) = \sin z$. *Show that the* $\lim_{z \to z_0} f(z) = \sin z_0$

Example 3.5 $f(z) = 2x + iy^2$. *Show that the* $\lim_{z \to 2i} f(z) = 4i$.

Theorem 3.2 *If* $\lim_{z \to z_0} f(z) = w_0$ *and* $\lim_{z \to z_0} F(z) = W_0$,

$$\begin{aligned} \lim_{z \to z_0} [f(z) + F(z)] &= w_0 + W_0; \\ \lim_{z \to z_0} f(z) F(z) &= w_0 W_0; \\ \lim_{z \to z_0} f(z) / F(z) &= \frac{w_0}{W_0} \qquad W_0 \neq 0. \end{aligned}$$

This theorem immediately makes available the entire machinery and tools used for real analysis to be applied to complex analysis. The rules for • nding limits then can be listed as follows:

1. $\lim_{z \to z_0} c = c.$
2. $\lim_{z \to z_0} z^n = z_0^n.$
3. $\lim_{z \to z_0} P(z) = P(z_0)$ if P is a polynomial in z.
4. $\lim_{z \to z_0} \exp(z) = \exp(z_0).$
5. $\lim_{z \to z_0} \sin(z) = \sin z_0.$

Continuity

Definition 3.2 *A function f, defined in some nbd of z_0 is continuous at z_0 if*

$$\lim_{z \to z_0} f(z) = f(z_0).$$

This definition clearly assumes that the function is defined at z_0 and the limit on the LHS exists. The function f is continuous in a region if it is continuous at all points in that region.

If funtions f and g are continuous at z_0 then $f + g$, fg and f/g $(g(z_0) \neq 0)$ are also continuous at z_0.

If a function $f(z) = u(x, y) + iv(x, y)$ is continuous at z_0 then the component functions u and v are also continuous at (x_0, y_0).

Derivative

Definition 3.3 *A function f, defined in some nbd of z_0 is differentiable at z_0 if*

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists. The limit is called the derivative of f at z_0 and is denoted by $f'(z_0)$ or $\frac{df}{dz}(z_0)$.

Example 3.6 $f(z) = z^2$. *Show that* $f'(z) = 2z$.

Example 3.7 $f(z) = |z|^2$. *Show that this function is differentiable only at* $z = 0$. *In real analysis* $|x|$ *is not differentiable but* $|x|^2$ *is.*

If a function is differentiable at z, then it is continuous at z.

Analytic Functions

Definition 3.4 *A function is analytic in an open set if it has a derivative at each point in that set.*

Definition 3.5 *A function is analytic at a point z_0 if it is analytic in some nbd of z_0.*

Definition 3.6 *A function is an entire function if it is analytic at all points of C.*

Example 3.13 *$f(z) = 1/z$ is analytic at all nonzero points.*

Example 3.14 *$f(z) = |z|^2$ is not analytic anywhere.*

A function is not analytic at a point z_0, but is analytic at some point in each nbd of z_0 then z_0 is called the singular point of the function f.

Harmonic Functions

Definition 3.7 *A real valued function $H(x, y)$ is said to be* harmonic *in a domain of xy plane if it has continuous partial derivatives of the first and second order and satisfies* Laplace equation:

$$H_{xx}(x, y) + H_y(x, y) = 0.$$

Cauchy-Riemann Equations

Theorem 3.3 *If $f'(z_0)$ exists, then all the first order partial derivatives of component function $u(x, y)$ and $v(x, y)$ exist and satisfy Cauchy-Riemann Conditions:*

$$\begin{aligned} u_x &= v_y \\ u_y &= -v_x. \end{aligned}$$

Example 3.8 *$f(z) = z^2 = x^2 - y^2 + i2xy$. Show that Cauchy-Riemann Condtions are satisfied.*

Example 3.9 *$f(z) = |z|^2 = x^2 + y^2$. Show that the Cauchy-Riemann Condtions are satisfied only at $z = 0$.*

Theorem 3.4 *Let $f(z) = u(x, y) + iv(x, y)$ be defined in some nbd of the point z_0. If the first partial derivatives of u and v exist and are continuous at z_0 and satisfy Cauchy-Riemann equations at z_0, then f is differentiable at z_0 and*

$$f'(z) = u_x + iv_x = v_y - iu_y.$$

Example 3.10 *$f(z) = \exp(z)$. Show that $f'(z) = \exp(z)$.*

Example 3.11 *$f(z) = \sin(z)$. Show that $f'(z) = \cos(z)$.*

Example 3.12 *$f(z) = \frac{z^2}{z}$. Show that the CR conditions are satisfied at $z = 0$ but the function f is not differentiable at 0.*

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1. The *Cauchy–Riemann equations* for two functions $u(x,y)$ and $v(x,y)$ are

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

(a) Show that $u(x,y) = x^2 - y^2$ and $v(x,y) = 2xy$ satisfy the Cauchy–Riemann equations.

(b) The *Laplace equation* for a function $f(x,y)$ is

$$f_{xx} + f_{yy} = 0.$$

Suppose that a and b satisfy the Cauchy–Riemann equations and have equal mixed partial derivatives (i.e. $a_{xy} = a_{yx}$, etc.). Show that $a + b$ satisfies the Laplace equation.

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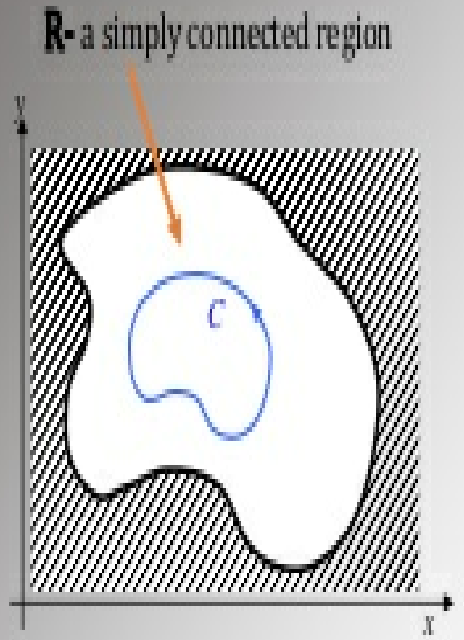
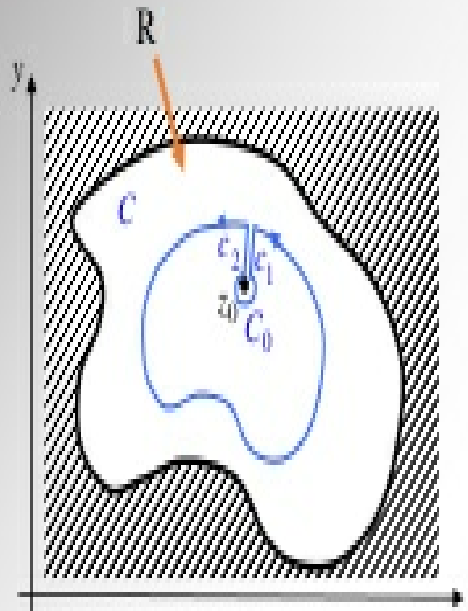
Cauchy Integral Formula



#. $f(z)$ is assumed analytic in R but we multiply by a factor $\frac{1}{(z-z_0)}$ that is analytic except at z_0 and consider the integral around C

$$\int_C \frac{f(z)}{(z-z_0)}dz$$

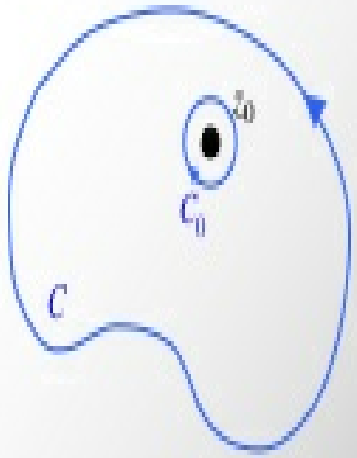


#. To evaluate, consider the path $C + c_1 + c_2 + C_0$ shown that encloses a simply – connected region for which the integrand is analytic on and inside the path:

$$\int_{C+c_1+c_2+C_0} \frac{f(z)}{(z-z_0)}dz = 0 \Rightarrow \int_C \frac{f(z)}{(z-z_0)}dz = -\int_{C_0} \frac{f(z)}{(z-z_0)}dz$$

Cauchy Integral Formula, cont'd

$$\int_C \frac{f(z)}{z-z_0}dz = -\int_{C_0} \frac{f(z)}{z-z_0}dz$$



Evaluate the C_0 integral on a circular path, $z - z_0 = re^{i\theta}$, $dz = rie^{i\theta}d\theta$

$$\int_{C_0} \frac{f(z)}{(z-z_0)}dz \overset{r\to 0}{=} f(z_0)\int_{2\pi}^{0} \frac{f\,i\,e^{i\theta}\,d\theta}{f\,e^{i\theta}} = -2\pi i f(z_0)$$

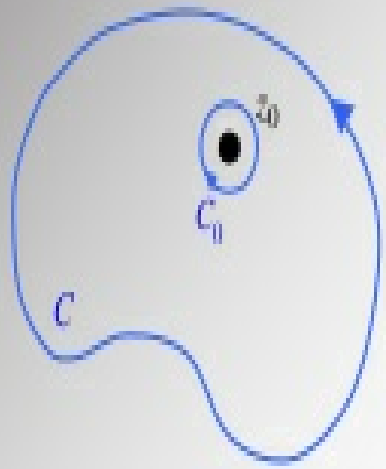
for $r \to 0$

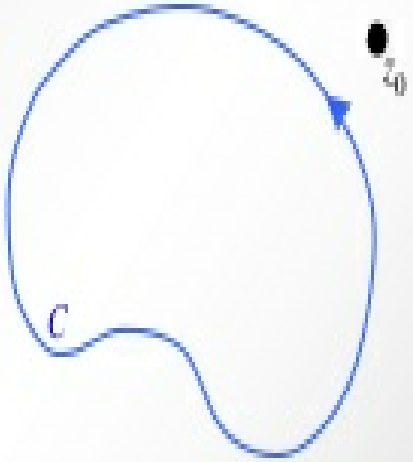
$$\Rightarrow \int_C \frac{f(z)}{z-z_0}dz = 2\pi i f(z_0) \Rightarrow f(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-z_0}dz$$

Cauchy Integral Formula

The value of $f(z)$ at z_0 is completely determined by its values on C!

Cauchy Integral Formula, cont'd





Note that if z_0 is outside C, the integrand is analytic inside C, hence by the Cauchy Integral Theorem,

$$\frac{1}{2\pi i}\oint_C \frac{f(z)}{z - z_0}\,dz = 0$$

In summary,

$$\oint_C \frac{f(z)}{z - z_0}\,dz = \begin{cases} 2\pi i\, f(z_0), & z_0 \text{ inside } C \\ 0, & z_0 \text{ outside } C \end{cases}$$

Contour Integral

If C is a contour in complex plane de• ned by $z(t) = x(t) + iy(t)$ and a function $f(z) = u(x, y) + iv(x, y)$ is de• ned on it. The integral of $f(z)$ along the contour C is denoted and de• ned as follows:

$$\begin{aligned}\int_C f(z)\,dz &= \int_a^b f(z)\,z'(t)\,dt \\ &= \int_a^b (ux' - vy')dt + i\int_a^b (uy' + vx')dt \\ &= \int (udx - vdy) + i\int (udy + vdx)\end{aligned}$$

The component integrals are usual real integrals and are well de• ned. In the last form appropriate limits must placed in the integrals.

Some very straightforward rules of integration are given below:

1. $\int_C wf(z)\,dz = w\int_C f(z)\,dz$ where w is a complex constant.
2. $\int_C (f(z) + g(z))dz = \int_C f(z)\,dz + \int_C g(z)\,dz$.
3. $\int_{C_1+C_2} f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz$.
4. $\left|\int_C f(z)\,dz\right| \leq \int_C |f(z(t))\,z'(t)|\,dt$.
5. If $|f(z)| \leq M$ for all $z \in C$ then $\left|\int_C f(z)\,dz\right| \leq ML$, where L is length of the countour C.

Example 4.4 *$f(z) = z^2$. Find integral of f from $(0, 0)$ to $(2, 1)$ along a straight line and also along st line path from $(0, 0)$ to $(2, 0)$ and from $(2, 0)$ to $(2, 1)$.*

Example 4.5 *$f(z) = 1/z$. Find the integral from $(2, 0)$ to $(-2, 0)$ along a semicircular path in upper plane given by $|z| = 2$.*

Example 4.6 *Show that $\left|\int_C f(z)\,dz\right| \leq \frac{\pi}{3}$ for $f(z) = 1/\left(z^2 - 1\right)$ and $C : |z| = 2$ from 2 to $2i$.*

Cauchy-Goursat Theorem

Theorem 4.1 (Jordan Curve Theorem) *Every simple and closed contour in complex plane splits the entire plane into two domains one of which is bounded. The bounded domain is called the interior of the countour and the other one is called the exterior of the contour.*

De• ne a sense direction for a contour.

Theorem 4.2 *Let C be a simple closed contour with positive orientation and let D be the interior of C. If P and Q are continuous and have continuous partial derivatives P_x, P_y, Q_x and Q_y at all points on C and D, then*

$$\int_C (P(x,y)dx + Q(x,y)\,dy) = \iint [Q_x(x,y) - P_y(x,y)]\,dxdy$$

Theorem 4.3 (Cauchy-Goursat Theorem) *Let f be analytic in a simply connected domain D. If C is any simple closed contour in D, then*

$$\int_C f(z)\,dz = 0.$$

Example 4.7 *$f(z) = z^2, \exp(z), \cos(z)$ etc are entire functions so integral about any loop is zero.*

Theorem 4.4 *Let C_1 and C_2 be two simple closed positively oriented contours such that C_2 lies entirely in the interior of C_1. If f is an analytic function in a domain D that contains C_1 and C_2 both and the region between them, then*

$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz.$$

Example 4.8 *$f(z) = 1/z$. Find $\int_C f(z)\,dz$ if C is any contour containing origin. Choose a circular contour inside C.*

Example 4.9 $\int_C \frac{1}{z-z_0}dz = 2\pi i$ *if C contains z_0.*

Example 4.10 *Find $\int_C \frac{2zdz}{z^2+2}$ where $C : |z| = 2$. Extend the Cauchy Goursat theorem to multiply connected domains.*

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Singularities

De• nition 6.1 *If a function f fails to be analytic at z_0 but is analytic at some point in each neighbourhood of z_0, then z_0 is a* singular point *of f.*

De• nition 6.2 *If a function f fails to be analyitc at z_0 but is analytic at each z in $0 < |z - z_0| < \delta$ for some δ, then f is said to be an* isolated singular point *of f.*

Example 6.1 *$f(z) = 1/z$ has an isolated singularity at 0.*

Example 6.2 *$f(z) = 1/\sin(\pi z)$ has isolated singularities at $z = 0, \pm 1, \ldots$.*

Example 6.3 *$f(z) = 1/\sin(\pi/z)$ has isolated singularities at $z = 1/n$ for integral n, also has a singularity at $z = 0$.*

Example 6.4 *$f(z) = \log z$ all points of negative x-axis are singular.*

Types of singularities

If a function f has an isolated singularity at z_0 then $\exists$ a δ such that f is analytic at all points in $0 < |z - z_0| < \delta$. Then f must have a Laurent series expansion about z_0. The part $\sum_{n=1}^{\infty} b_1 (z - z_0)^{-n}$ is called the principal part of f.

1. If there are in• nite nonzero b_i in the prinipal part then z_0 is called an essential singularity of f.
2. If for some integer m, $b_m \neq 0$ but $b_i = 0$ for all $i > m$ then z_0 is called a pole of order m of f. If $m = 1$ then it is called a simple pole.
3. If all bs are zero then z_0 is called a removable singularity.

Residues

Suppose a function f has an isolated singularity at z_0, then there exists a $\delta > 0$ such that f is analytic for all z in deleted nbd $0 < |z - z_0| < \delta$. Then f has a Laurent series representation

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}.$$

The coef• cient

$$b_1 = \frac{1}{2\pi i} \int_C f(z)\, dz$$

where C is any contour in the deleted nbd, is called the residue of f at z_0.

Example 6.6 $f(z) = \frac{1}{z - z_0}$. *Then* $\int_C f(z)\, dz = 2\pi i b_1 = 2\pi i$ *if* C *contains* z_0, *otherwise* 0.

Example 6.7 $f(z) = \frac{1}{z(z-z_0)^4}$. *Show* $\int_C f(z)\, dz = -\frac{\pi i}{8}$ *if* $C : |z - 2| = 1$.

Example 6.8 $f(z) = z \exp\left(\frac{1}{z}\right)$. *Show* $\int_C f(z)\, dz = \pi i$ *if* $C : |z| = 1$.

Example 6.9 $f(z) = \exp\left(\frac{1}{z^2}\right)$. *Show* $\int_C f(z)\, dz = 0$ *if* $C : |z| = 1$ *even though it has a singularity at* $z = 0$.

Theorem 6.1 *If a function* f *is analytic on and inside a positively oriented countour* C, *except for a •nite number of points* $z_1, z_2, \ldots, z_k$ *inside* C, *then*

$$\int_C f(z)\, dz = 2\pi i \sum_{i=1}^{k} Res f(z_i).$$

Example 6.10 *Show that* $\int \frac{5z-2}{z(z-1)} dz = 10\pi i$

Residues of Poles

Theorem 6.2 *If a function* f *has a pole of order* m *at* z_0 *then*

$$Res f(z_0) = \lim_{z \to z_0} \left[\frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \left((z - z_0)^m f(z) \right) \right].$$

Example 6.11 $f(z) = \frac{1}{z - z_0}$. *Simple pole Res* $f(z_0) = 1$.

Example 6.12 $f(z) = \frac{1}{z(z-z_0)^4}$. *Simple pole at* $z = 0$. *Res* $f(0) = 1/16$. *Pole of order* 4 *at* $z = 2$. *Res* $f(2) = -1/16$.

Example 6.13 $f(z) = \frac{\cos z}{z^2(z-\pi)^3}$. *Res* $f(0) = -3/\pi^4$. *Res* $f(\pi) = -\left(6 - \pi^2\right)/2\pi^4$.

Quotients of Analytic Functions

Theorem 6.3 *If a function* $f(z) = \frac{P(z)}{Q(z)}$, *where* P *and* Q *are analytic at* z_0, *then*

1. f is singular at z_0 iff $Q(z_0) = 0$.
2. f has a simple pole at z_0 if $Q'(z_0) \neq 0$. Then residue of f at z_0 is $P(z_0)/Q'(z_0)$.

Singularities

Isolated singularities

Removable singularities	Poles of finite order	Isolated essential singularities (pole of infinite order)
$\frac{\sin(z)}{z}, \frac{1-\cos(z)}{z^2}$	$\frac{1}{z}, \frac{1}{z^2}, \frac{1}{(z-1)^m}, \frac{2z+3}{(z-1)^2(z+2)}$	$\sin\left(\frac{1}{z}\right), e^{1/z}$

These are each discussed in more detail next.

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Essential singularities can be distinguished from poles by the fact that they cannot be removed by multiplying by a factor of finite value.

Example:

$$w = e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!\,z^2} + \ldots + \frac{1}{n!\,z^n} + \ldots$$

← infinite at the origin

We try to remove the singularity of the function at the origin by multiplying z^p

$$z^p w = z^p + z^{p-1} + \frac{z^{p-2}}{2!} + \ldots + \frac{z^{p-n}}{n!} + \ldots$$

It consists of a finite number of positive powers of z, followed by an infinite number of negative powers of z.

All terms are positive

$$As \quad z \to 0, \quad z^p w \to \infty$$

It is impossible to find a finite value of p which will remove the singularity in $e^{1/z}$ at the origin.
The singularity is "essential".

Singularities

Examples of singularities:

(These will be discussed in more detail later.)

If expanded about the singularity, we can have:
T = Taylor
L = Laurent
N = Neither

T $\dfrac{\sin(z)}{z}$ removable singularity at $z = 0$

L $\dfrac{1}{(z - z_s)^p}$ pole of order p at $z = z_s$ (if $p = 1$, pole is a *simple pole*)

L $e^{1/z}$ isolated essential singularity at $z = z_0$ (pole of infinite order)

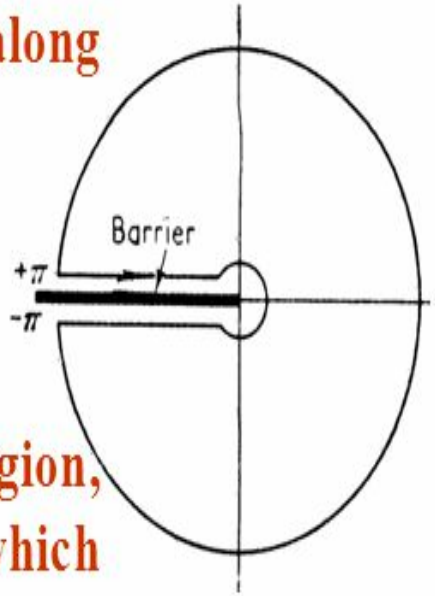
N $\dfrac{1}{\sin\left(\dfrac{1}{z}\right)}$ non-isolated essential singularity $z = 0$

N $z^{1/2}$ branch point (*not* an isolated singularity)

Barrier - Branch Cut

The barrier must start from the branch point but it can go to infinity in any direction in the z plane, and may be either curved or straight.

In most normal applications, the barrier is drawn along the negative real axis.



-The branch is termed the "principle branch".
-The barrier is termed the "branch cut".
-For the example given in the previous slide, the region, the barrier confines the function to the region in which the argument of z is within the range $-\pi < \theta < \pi$.

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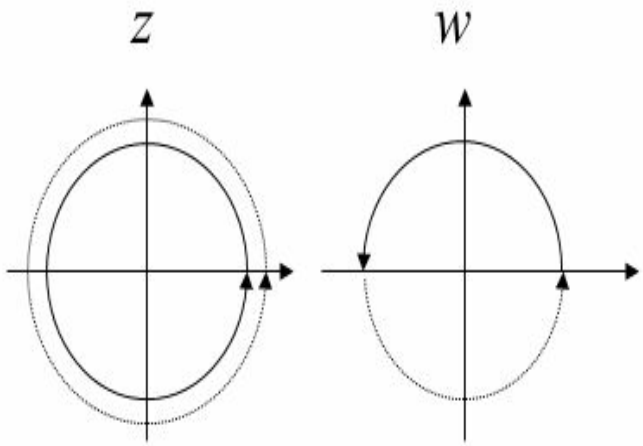
Branch points

- The singularities described above arise from the non-analytic behaviour of single-valued functions.
- However, multi-valued functions frequently arise in the solution of engineering problems.
- For example:

$$w = z^{\frac{1}{2}} \xrightarrow{z = re^{i\theta}} w = r^{\frac{1}{2}} e^{\frac{1}{2}i\theta}$$



For any value of z represented by a point on the circumference of the circle in the z plane, there will be two corresponding values of w represented by points in the w plane.

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Laurent's Theorem

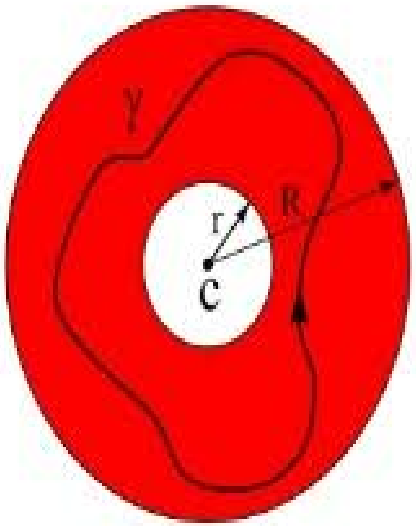
Laurent series of a complex function $f(z)$ is a representation of that function as a power series which includes terms of negative degree. It may be used to express complex functions in cases where a Taylor series expansion cannot be applied.

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-c)^n$$

C = z_0
K = n
A = Zo

Also say that ,

$$f(z) = \sum_{k=0}^{\infty} a_k (z-a)^k + \sum_{k=1}^{\infty} b_k (z-a)^{-k},$$



$$a_n = \frac{1}{2\pi i} \oint_\gamma \frac{f(z)\,dz}{(z-c)^{n+1}}.$$

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Taylor's series

- **Taylor series** is a representation of a function as an infinite sum of terms that are calculated from the values of the function's derivatives at a single point.

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \ldots$$

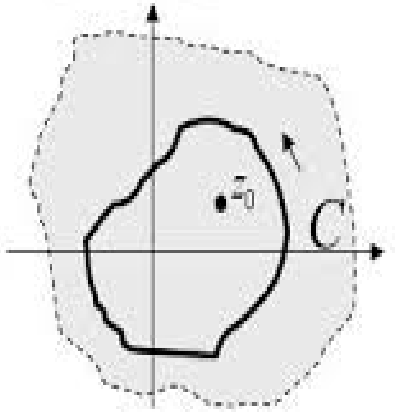
Here, $a = z_0$, center

x = Z inside circle C

The Residue Theorem

We saw that the integral of an analytic function $f(z)$ over a closed curve C when $f(z)$ has one singular point inside C is

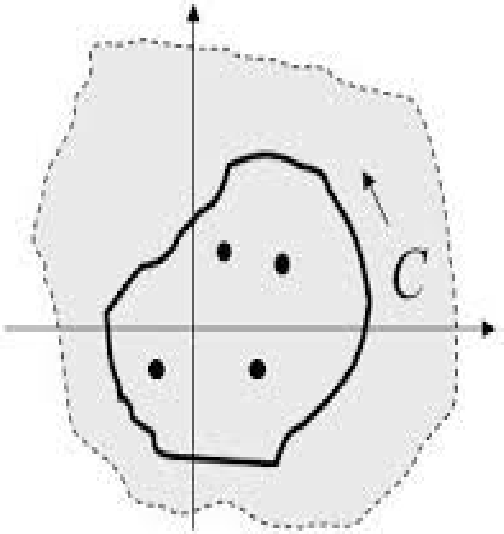
$$\oint_C f(z)dz = 2\pi j b_1$$



b_1 is the residue of $f(z)$ at z_0

Residue Theorem: Let $f(z)$ be an analytic function inside and on a closed path C except for at k singular points inside C. Then

$$\oint_C f(z)dz = 2\pi j \sum_{i=1}^{k} \operatorname*{Res}_{z=z_i} f(z)$$



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Numericals from various sources

Integral Transforms

Fourier Integral

If $f(x)$ and $f'(x)$ are piecewise continuous in every finite interval, and $f(x)$ is absolutely integrable on R, i.e.

converges, then

$$\frac{1}{2}[f(x-)+f(x+)]=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{iwx}\left(\int_{-\infty}^{\infty}e^{-iwt}f(t)dt\right)dw$$

Remark: the above conditions are sufficient, but not necessary.

$$\mathcal{F}_c = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos kx \, dx = F_c(k)$$

$$\mathcal{F}_c^{-1} = \sqrt{\frac{2}{\pi}} \int_0^\infty F_c(k) \cos kx \, dk = f(x)$$

} Fourier cosine transform

Fourier sine Transform:

$$\mathcal{F}_s = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin kx \, dx = F_s(k)$$

$$\mathcal{F}_s^{-1} = \sqrt{\frac{2}{\pi}} \int_0^\infty F_s(k) \sin kx \, dk = f(x)$$

} Fourier sine transform.

13.1.3 The Dirac δ-function

Before going on to consider further properties of Fourier transforms we make a digression to discuss the Dirac δ-function and its relation to Fourier transforms. The δ-function is different from most functions encountered in the physical sciences but we will see that a rigorous mathematical definition exists; the utility of the δ-function will be demonstrated throughout the remainder of this chapter. It can be visualised as a very sharp narrow pulse (in space, time, density, etc.) which produces an integrated effect having a definite magnitude. The formal properties of the δ-function may be summarised as follows.

The Dirac δ-function has the property that

$$\delta(t) = 0 \quad \text{for } t \neq 0, \tag{13.11}$$

but its fundamental defining property is

$$\int f(t)\delta(t-a)\,dt = f(a), \tag{13.12}$$

provided the range of integration includes the point $t = a$; otherwise the integral

► *Find the Fourier transform of the normalised Gaussian distribution*

$$f(t) = \frac{1}{\tau\sqrt{2\pi}} \exp\left(-\frac{t^2}{2\tau^2}\right), \qquad -\infty < t < \infty.$$

This Gaussian distribution is centred on $t = 0$ and has a root mean square deviation $\Delta t = \tau$. (Any reader who is unfamiliar with this interpretation of the distribution should refer to chapter 30.)

Using the definition (13.5), the Fourier transform of $f(t)$ is given by

$$\begin{aligned}\tilde{f}(\omega) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\tau\sqrt{2\pi}} \exp\left(-\frac{t^2}{2\tau^2}\right) \exp(-i\omega t)\, dt \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\tau\sqrt{2\pi}} \exp\left\{-\frac{1}{2\tau^2}\left[t^2 + 2\tau^2 i\omega t + (\tau^2 i\omega)^2 - (\tau^2 i\omega)^2\right]\right\} dt,\end{aligned}$$

where the quantity $-(\tau^2 i\omega)^2/(2\tau^2)$ has been both added and subtracted in the exponent in order to allow the factors involving the variable of integration t to be expressed as a complete square. Hence the expression can be written

$$\tilde{f}(\omega) = \frac{\exp(-\frac{1}{2}\tau^2\omega^2)}{\sqrt{2\pi}} \left\{\frac{1}{\tau\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left[-\frac{(t + i\tau^2\omega)^2}{2\tau^2}\right] dt\right\}.$$

The quantity inside the braces is the normalisation integral for the Gaussian and equals unity, although to show this strictly needs results from complex variable theory (chapter 24). That it is equal to unity can be made plausible by changing the variable to $s = t + i\tau^2\omega$ and assuming that the imaginary parts introduced into the integration path and limits (where the integrand goes rapidly to zero anyway) make no difference.

We are left with the result that

$$\tilde{f}(\omega) = \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-\tau^2\omega^2}{2}\right), \tag{13.7}$$

which is another Gaussian distribution, centred on zero and with a root mean square deviation $\Delta\omega = 1/\tau$. It is interesting to note, and an important property, that the Fourier transform of a Gaussian is another Gaussian. ◄

In the above example the root mean square deviation in t was τ, and so it is seen that the deviations or 'spreads' in t and in ω are inversely related:

$$\Delta\omega\, \Delta t = 1,$$

independently of the value of τ. In physical terms, the narrower in time is, say, an electrical impulse the greater the spread of frequency components it must contain. Similar physical statements are valid for other pairs of Fourier-related variables, such as spatial position and wave number. In an obvious notation, $\Delta k \Delta x = 1$ for a Gaussian wave packet.

The uncertainty relations as usually expressed in quantum mechanics can be related to this if the de Broglie and Einstein relationships for momentum and energy are introduced; they are

$$p = \hbar k \qquad \text{and} \qquad E = \hbar\omega.$$

equals zero. This leads immediately to two further useful results:

$$\int_{-a}^{b} \delta(t)\,dt = 1 \quad \text{for all } a, b > 0 \tag{13.13}$$

and

$$\int \delta(t-a)\,dt = 1, \tag{13.14}$$

provided the range of integration includes $t = a$.

Equation (13.12) can be used to derive further useful properties of the Dirac δ-function:

$$\delta(t) = \delta(-t), \tag{13.15}$$

$$\delta(at) = \frac{1}{|a|}\delta(t), \tag{13.16}$$

$$t\delta(t) = 0. \tag{13.17}$$

▶ *Prove that* $\delta(bt) = \delta(t)/|b|$.

Let us first consider the case where $b > 0$. It follows that

$$\int_{-\infty}^{\infty} f(t)\delta(bt)\,dt = \int_{-\infty}^{\infty} f\left(\frac{t'}{b}\right)\delta(t')\frac{dt'}{b} = \frac{1}{b}f(0) = \frac{1}{b}\int_{-\infty}^{\infty} f(t)\delta(t)\,dt,$$

where we have made the substitution $t' = bt$. But $f(t)$ is arbitrary and so we immediately see that $\delta(bt) = \delta(t)/b = \delta(t)/|b|$ for $b > 0$.

Now consider the case where $b = -c < 0$. It follows that

$$\int_{-\infty}^{\infty} f(t)\delta(bt)\,dt = \int_{\infty}^{-\infty} f\left(\frac{t'}{-c}\right)\delta(t')\left(\frac{dt'}{-c}\right) = \int_{-\infty}^{\infty}\frac{1}{c}f\left(\frac{t'}{-c}\right)\delta(t')\,dt'$$

$$= \frac{1}{c}f(0) = \frac{1}{|b|}f(0) = \frac{1}{|b|}\int_{-\infty}^{\infty} f(t)\delta(t)\,dt,$$

where we have made the substitution $t' = bt = -ct$. But $f(t)$ is arbitrary and so

$$\delta(bt) = \frac{1}{|b|}\delta(t),$$

for all b, which establishes the result. ◀

(i) Differentiation:

$$\mathcal{F}[f'(t)] = i\omega\tilde{f}(\omega). \tag{13.28}$$

This may be extended to higher derivatives, so that

$$\mathcal{F}[f''(t)] = i\omega\mathcal{F}[f'(t)] = -\omega^2\tilde{f}(\omega),$$

and so on.

(ii) Integration:

$$\mathcal{F}\left[\int^t f(s)\,ds\right] = \frac{1}{i\omega}\tilde{f}(\omega) + 2\pi c\delta(\omega), \tag{13.29}$$

where the term $2\pi c\delta(\omega)$ represents the Fourier transform of the constant of integration associated with the indefinite integral.

(iii) Scaling:

$$\mathcal{F}[f(at)] = \frac{1}{a}\tilde{f}\left(\frac{\omega}{a}\right). \tag{13.30}$$

(iv) Translation:

$$\mathcal{F}[f(t+a)] = e^{ja\omega}\tilde{f}(\omega). \tag{13.31}$$

(v) Exponential multiplication:

$$\mathcal{F}\left[e^{\alpha t}f(t)\right] = \tilde{f}(\omega + i\alpha), \tag{13.32}$$

where α may be real, imaginary or complex.

▶ *Prove relation (13.28).*

Calculating the Fourier transform of $f'(t)$ directly, we obtain

$$\begin{aligned}\mathcal{F}[f'(t)] &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f'(t)\,e^{-i\omega t}\,dt \\ &= \frac{1}{\sqrt{2\pi}}\left[e^{-i\omega t}f(t)\right]_{-\infty}^{\infty} + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} i\omega\, e^{-i\omega t}f(t)\,dt \\ &= i\omega\tilde{f}(\omega),\end{aligned}$$

if $f(t) \to 0$ at $t = \pm\infty$, as it must since $\int_{-\infty}^{\infty} |f(t)|\,dt$ is finite. ◀

Example 15.4.2 HEAT FLOW PDE

To illustrate another transformation of a PDE into an ODE, let us Fourier transform the heat flow partial differential equation

$$\frac{\partial \psi}{\partial t} = a^2 \frac{\partial^2 \psi}{\partial x^2},$$

where the solution $\psi(x,t)$ is the temperature in space as a function of time. By taking the Fourier transform of both sides of this equation (note that here only ω is the transform variable conjugate to x because t is the time in the heat flow PDE), where

$$\Psi(\omega,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi(x,t) e^{i\omega x}\, dx,$$

this yields an ODE for the Fourier transform Ψ of ψ in the time variable t,

$$\frac{\partial \Psi(\omega,t)}{\partial t} = -a^2\omega^2 \Psi(\omega,t).$$

Integrating we obtain

$$\ln \Psi = -a^2\omega^2 t + \ln C, \qquad \text{or} \qquad \Psi = Ce^{-a^2\omega^2 t},$$

where the integration constant C may still depend on ω and, in general, is determined by initial conditions. In fact, $C = \Psi(\omega,0)$ is the initial spatial distribution of Ψ, so it is given by the transform (in x) of the initial distribution of ψ, namely, $\psi(x,0)$. Putting this solution back into our inverse Fourier transform, this yields

$$\psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} C(\omega) e^{-i\omega x} e^{-a^2\omega^2 t}\, d\omega.$$

For simplicity, we here take C ω-independent (assuming a delta-function initial temperature distribution) and integrate by completing the square in ω, as in Example 15.1.1,

making appropriate changes of variables and parameters ($a^2 \to a^2 t,\ \omega \to x, t \to -\omega$). This yields the particular solution of the heat flow PDE,

$$\psi(x,t) = \frac{C}{a\sqrt{2t}} \exp\left(-\frac{x^2}{4a^2 t}\right),$$

which appears as a clever guess in Chapter 8. In effect, we have shown that ψ is the inverse Fourier transform of $C \exp(-a^2\omega^2 t)$. ■

Example 15.4.3 INVERSION OF PDE

15.5 CONVOLUTION THEOREM

We shall employ convolutions to solve differential equations, to normalize momentum wave functions (Section 15.6), and to investigate transfer functions (Section 15.7).

Let us consider two functions $f(x)$ and $g(x)$ with Fourier transforms $F(t)$ and $G(t)$, respectively. We define the operation

$$f * g \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(y) f(x-y)\, dy \tag{15.52}$$

as the **convolution** of the two functions f and g over the interval $(-\infty, \infty)$. This form of an integral appears in probability theory in the determination of the probability density of two random, independent variables. Our solution of Poisson's equation, Eq. (9.148), may be interpreted as a convolution of a charge distribution, $\rho(\mathbf{r}_2)$, and a weighting function, $(4\pi\varepsilon_0|\mathbf{r}_1 - \mathbf{r}_2|)^{-1}$. In other works this is sometimes referred to as the **Faltung**, to use the

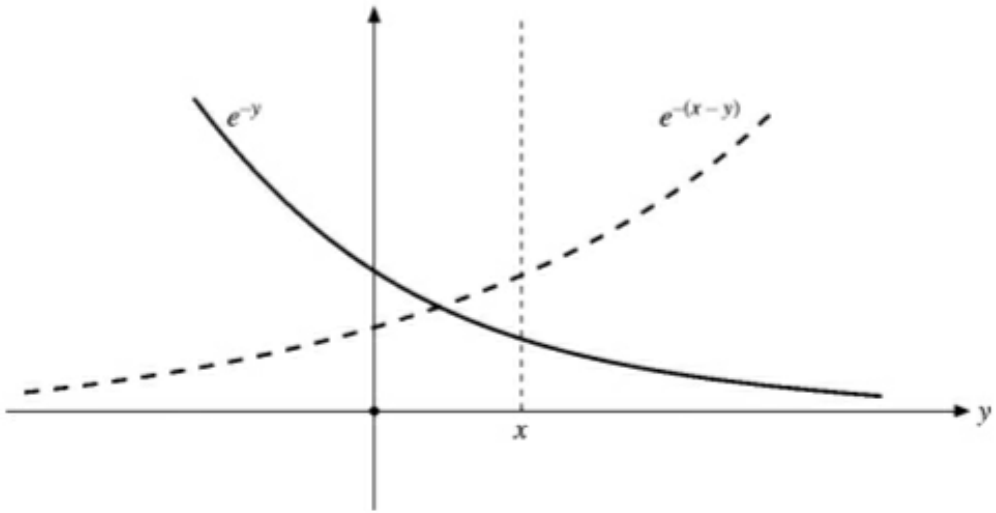

FIGURE 15.5

German term for "folding."[5] We now transform the integral in Eq. (15.52) by introducing the Fourier transforms:

$$\begin{aligned}\int_{-\infty}^{\infty} g(y)f(x-y)dy &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(y)\int_{-\infty}^{\infty} F(t)e^{-it(x-y)}\,dt\,dy \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(t)\left[\int_{-\infty}^{\infty} g(y)e^{ity}dy\right]e^{-itx}\,dt \\ &= \int_{-\infty}^{\infty} F(t)G(t)e^{-itx}\,dt, \qquad (15.53)\end{aligned}$$

interchanging the order of integration and transforming $g(y)$. This result may be interpreted as follows: The Fourier inverse transform of a **product** of Fourier transforms is the convolution of the original functions, $f * g$.

For the special case $x = 0$ we have

$$\int_{-\infty}^{\infty} F(t)G(t)\,dt = \int_{-\infty}^{\infty} f(-y)g(y)\,dy. \qquad (15.54)$$

The minus sign in $-y$ suggests that modifications be tried. We now do this with g^* instead of g using a different technique.

The Laplace Transform

Time Integration:

The property is:

$$L\left[\int_0^{\infty} f(t)dt\right] = \int_0^{\infty}\left[\int_0^{t} f(x)dx\right]e^{-st}dt$$

Integrate by parts:

$$Let\ u = \int_0^{t} f(x)dx, \quad du = f(t)dt$$

and

$$dv = e^{-st}dt, \quad v = -\frac{1}{s}e^{-st}$$

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Home Task:

For Inverse Laplace transform material go through Arfken available in your library.

Real Integration Theorem

This theorem establishes the relationship between the **Laplace** transform of a function and **that** of its integral. It states that

$$\mathcal{L}\left[\int_0^t f(t)\,dt\right] = \frac{1}{s}F(s) \qquad (2\text{-}1.7)$$

The proof of this theorem is carried out by integrating the definition of the **Laplace** transform by parts. This proof is similar to that of the real differentiation theorem and is left as an exercise. The **Laplace** transform of the nth integral of a function is the transform of the function divided by s^n.

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Laplace Transforms of Common Functions

Function	Laplace Transform	Restrictions
1	$\frac{1}{s}$	$s > 0$
e^{at}	$\frac{1}{s-a}$	$s > a$
t^n	$\frac{n!}{s^{n+1}}$	$s > 0$, n an interger > 0
$\cos at$	$\frac{s}{s^2+a^2}$	$s > 0$
$\sin at$	$\frac{a}{s^2+a^2}$	$s > 0$
$\cosh at$	$\frac{s}{s^2-a^2}$	$s > \lvert a \rvert$
$\sinh at$	$\frac{a}{s^2-a^2}$	$s > \lvert a \rvert$
$e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}$	$s > a$
$e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}$	$s > a$
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$	$s > a$, n an integer > 0
$f(ct)$	$\frac{1}{c}\mathcal{L}\{f(s/c)\}$	$c > 0$
$f^{(n)}(t)$	$s^n\mathcal{L}\{f(t)\} - s^{n-1}f(0) - \ldots - sf^{(n-2)}(0) - f^{(n-1)}(0)$	

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Table of Inverse Laplace Transform

1. $\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 1$

2. $\mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$

3. $\mathcal{L}^{-1}\left\{\frac{1}{s^n}\right\} = \frac{t^{n-1}}{(n-1)!}; \quad n = 1,\ 2,\ 3, \ldots$

4. $\mathcal{L}^{-1}\left\{\frac{1}{(s-a)^n}\right\} = \frac{e^{at}t^{n-1}}{(n-1)!}$

5. $\mathcal{L}^{-1}\left\{\frac{1}{s^2+b^2}\right\} = \frac{1}{b}\sin bt$

6. $\mathcal{L}^{-1}\left\{\frac{s}{s^2+b^2}\right\} = \cos bt$

7. $\mathcal{L}^{-1}\left\{\frac{1}{s^2-a^2}\right\} = \frac{1}{a}\sinh at$

8. $\mathcal{L}^{-1}\left\{\frac{s}{s^2-a^2}\right\} = \cosh at$

9. $\mathcal{L}^{-1}\left\{\frac{1}{(s-a)^2+b^2}\right\} = \frac{1}{b}e^{at}\sin bt$

10. $\mathcal{L}^{-1}\left\{\frac{s-a}{(s-a)^2+b^2}\right\} = e^{at}\cos bt$

11. $\mathcal{L}^{-1}\left\{\frac{s}{(s^2+b^2)^2}\right\} = \frac{1}{2b}t\sin bt$

Laplace Transform

- Theorems

Theorem	*Description*
Definition of Laplace Transform	$\mathcal{L}\{f(t)\}=\int_0^{\infty} e^{-st} f(t)dt = F(s)$
Linear Property	$\mathcal{L}\{af(t)+bg(t)\}=aF(s)+bG(s)$
Derivatives	$\mathcal{L}\{f^{(n)}(t)\}=s^n F(s)-s^{n-1}f(0)-\cdots-f^{(n-1)}(0)$
Integrals	$\mathcal{L}\{\int_0^{\infty} f(\tau)d\tau\}=\frac{1}{s}F(s)$
First Shifting Property	$\mathcal{L}\{e^{at}f(t)\}=F(s-a)$
Second Shifting Property	$\mathcal{L}\{f(t-a)u(t-a)\}=e^{-as}F(s)$

Transform of derivatives

- **Laplace transform of the derivative f^n of any order :**

Let $f, f', \ldots, f^{(n-1)}$ be continuous for all $t \geq 0$ and f^n be piecewise continuous on every finite interval on the semi-axis $t \geq 0$ then transform of f^n given by,

$$L\left(f^n\right) = s^n L\left(f\right) - s^{n-1} f\left(0\right) - s^{n-2} f'\left(0\right) - \ldots - f^{n-1}\left(0\right)$$

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For first-order derivative:

$$\mathcal{L}\{f'(t)\} = s\,\mathcal{L}\{f(t)\} - f(0)$$

For second-order derivative:

$$\mathcal{L}\{f''(t)\} = s^2\,\mathcal{L}\{f(t)\} - s\,f(0) - f'(0)$$

For third-order derivative:

$$\mathcal{L}\{f'''(t)\} = s^3\,\mathcal{L}\{f(t)\} - s^2 f(0) - s\,f'(0) - f''(0)$$

For n^{th} order derivative:

$$\mathcal{L}\{f^n(t)\} = s^n\,\mathcal{L}\{f(t)\} - s^{n-1} f(0) - s^{n-2}\,f'(0) - \cdots - f^{n-1}(0)$$

The Laplace Transform of the First Derivative

$$L\left[f'(t)\right] = sL\left[f(t)\right] - f(0)$$

$$\int udv = uv - \int vdu$$

$$L\left[f'(t)\right] = \int_0^{\infty} \underbrace{e^{-st}}_{u} \underbrace{f(t)dt}_{dv}$$

$$L\left[f'(t)\right] = \int udv = e^{-st} f(t) \Big|_0^{\infty} - \int_0^{\infty} f(t)(-s) e^{-st} dt$$

The Laplace Transform of the Third Derivative

$$L\left[f'(t)\right] = sL\left[f(t)\right] - f(0)$$

$$L\left[f''(t)\right] = s^2 L\left[f(t)\right] - sf(0) - f'(0)$$

$$L\left[f'''(t)\right] = sL\left[f''(t)\right] - f''(0)$$

$$= s\left[s^2 L\left[f(t)\right] - f(0) - f'(0)\right] - f''(0)$$

$$L\left[f'''(t)\right] = s^3 L\left[f(t)\right] - s^2 f(0) - s f'(0) - f''(0)$$

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Unit Step Function

$$u(t-a) = 0, t < a$$

$$1, t \geq a$$

$$L\{u(t-a)\} = \int_0^\infty e^{-st} u(t-a)\,dt$$

$$= \int_0^a e^{-st}(0)\,dt + \int_a^\infty e^{-st}(1)\,dt$$

$$= \left[\frac{e^{-st}}{-s}\right]_a^\infty = \frac{1}{s}e^{-as}$$

if $a = 0$

$$L\{u(t)\} = \frac{1}{s}$$

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Application to linear differential equations

We illustrate the Laplace transform technique by a simple example of harmonic oscillator with a time-dependent external force $f(t) = t$.

$$\ddot{x} + \omega^2 x = t \,, \tag{15}$$

$$x(0) = a \,, \qquad \dot{x}(0) = b \,. \tag{16}$$

The starting point is to Laplace-transform the equation:

$$\int_0^\infty dt\, \mathrm{e}^{-pt} \left[\ddot{x} + \omega^2 x\right] = \int_0^\infty dt\, \mathrm{e}^{-pt}\, t \,. \tag{17}$$

For the derivatives in the l.h.s. of (17) we use (4), with (16) taken into account. This yields

$$\int_0^\infty dt\, \mathrm{e}^{-pt} \left[\ddot{x} + \omega^2 x\right] = (p^2 + \omega^2) X(p) - pa - b \,, \tag{18}$$

where

$$X(p) = \int_0^\infty dt\, \mathrm{e}^{-pt}\, x(t) \,. \tag{19}$$

The integral in the r.h.s. of (17) needs to be done explicitly:

$$\int_0^\infty dt\, \mathrm{e}^{-pt}\, t = -\frac{\partial}{\partial p} \int_0^\infty dt\, \mathrm{e}^{-pt} = -\frac{\partial}{\partial p} \frac{1}{p} = \frac{1}{p^2} \,. \tag{20}$$

We thus have

$$(p^2 + \omega^2) X(p) - pa - b = \frac{1}{p^2} \,, \tag{21}$$

and find

$$X(p) = \frac{1 + p^2(pa + b)}{p^2(p^2 + \omega^2)} = \frac{1 + p^2(pa + b)}{p^2(p - i\omega)(p + i\omega)} \,. \tag{22}$$

Now we perform the inverse transform

$$x(t) = \frac{1}{2\pi i}\int_C \frac{[1+p^2(pa+b)]\,e^{pt}}{p^2(p-i\omega)(p+i\omega)}\,dp\,. \tag{23}$$

That is we sum up all the residues of the function

$$\frac{[1+p^2(pa+b)]\,e^{pt}}{p^2(p-i\omega)(p+i\omega)}\,. \tag{24}$$

There are three poles. The two simple poles $p = \pm i\omega$ come from the l.h.s. of the original differential equation, while the second-order pole at $p = 0$ is totally due to the particular form of the external force. The residue calculus routine leads to the final answer

$$x(t) = t/\omega^2 + a\cos\omega t + (b/\omega - 1/\omega^3)\sin\omega t\,. \tag{25}$$

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Problem 45. Use the Laplace transform method to solve for the evolution of a damped harmonic oscillator with a time-independent external force:

$$\ddot{x} + \omega^2 x + \gamma \dot{x} = f_0 \,, \tag{26}$$

$$x(0) = 0 \,, \qquad \dot{x}(0) = 0 \,, \tag{27}$$

Find $x(t)$; make sure it is real and does satisfy the equation and the initial conditions. Compare two cases: $0 < \gamma < 2\omega$ and $\gamma \geq 2\omega$.

If we are dealing with a system of linear differential equations, the procedure is the same. We Laplace-transform each of the equations and arrive at a system of *algebraic* equations for the Laplace transforms of the unknown functions. From this system we find each of the Laplace transforms and then perform the inverse transforms.

Problem 46. Use the Laplace transform method to solve for the evolution of one of two coupled harmonic oscillators (the coupling force is proportional to the relative displacement of the oscillators, the proportionality coefficient $\lambda > 0$):

$$\ddot{x}_1 + \omega_1^2 x_1 = \lambda(x_2 - x_1)\,, \tag{28}$$

$$\ddot{x}_2 + \omega_2^2 x_2 = \lambda(x_1 - x_2)\,. \tag{29}$$

Find $x_1(t)$ for $t > 0$ under the following initial conditions:

$$x_1(0) = 0\,, \qquad \dot{x}_1(0) = 0\,, \qquad x_2(0) = 0\,, \qquad \dot{x}_2(0) = 1. \tag{30}$$

Make sure $x_1(t)$ is real and does satisfy the initial conditions.

Laplace Transform

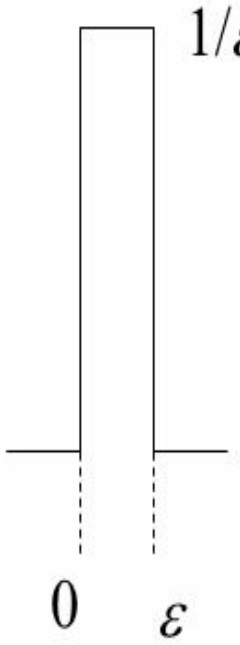
- Applications of the Laplace transform
  - solve differential equations (both ordinary and partial)
  - application to RLC circuit analysis
- Laplace transform converts differential equations in the time domain to algebraic equations in the frequency domain, thus 3 important processes:

  (1) transformation from the time to frequency domain

  (2) manipulate the algebraic equations to form a solution

  (3) inverse transformation from the frequency to time domain

Laplace transform of an impulse 'function' (Dirac delta distribution)



$$L\left(\delta(t)\right) = \lim_{\varepsilon \to 0} \int_0^{\varepsilon} \frac{1}{\varepsilon} e^{-st}\, dt = \frac{-1}{s\varepsilon} e^{-st}\Big|_0^{\varepsilon} =$$

$$\lim_{\varepsilon \to 0} \left(\frac{\left[e^{-s\varepsilon} - 1 \right]}{-s\varepsilon} \right)$$

Writing e^x as a McLaurin series

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 h.o.t.$$

Writing $e^{-s\varepsilon}$ as a McLaurin series

$$e^{-s\varepsilon} = 1 + (-s\varepsilon) + \frac{1}{2!}(-s\varepsilon)^2 + \frac{1}{3!}(-s\varepsilon)^3 h.o.t.$$

Laplace transform of impulse (Dirac delta distribution)

$$e^{-s\varepsilon}-1=(-s\varepsilon)+\frac{1}{2!}(-s\varepsilon)^2+\frac{1}{3!}(-s\varepsilon)^3+h.o.t.$$

$$\frac{e^{-s\varepsilon}-1}{-s\varepsilon}=1+\frac{1}{2!}(-s\varepsilon)+\frac{1}{3!}(-s\varepsilon)^2+h.o.t.$$

$$L(\delta(t))=\lim_{\varepsilon\to 0}\left[\frac{e^{-s\varepsilon}-1}{-s\varepsilon}\right]=\lim_{\varepsilon\to 0}\left[1+\frac{1}{2!}(-s\varepsilon)+\frac{1}{3!}(-s\varepsilon)^2+h.o.t.\right]=1$$

$$\boxed{L(\delta(t))=1}$$

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Thank you!
All the best for Exams !