

# The Foundations: Logic and Proofs (Mathematical Logic) Propositional Logic:

## INTRODUCTION

### PROPOSITION

### CONNECTIVES: Negation, Conjunction, Disjunction, Conditional and Bi-Conditional CONVERSE, INVERSE, AND CONTRAPOSITIVE

### SOME MORE CONNECTIVES: Exclusive or

### Truth Tables of Compound Propositions

### Propositional Equivalence

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### Logical Equivalence:

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## *INTRODUCTION:*

Logic is concerned with studying arguments and conclusions.

Logic is used in computer science to verify the correctness of programs.

The rules of logic give precise meaning to mathematical statements. The rules are used to distinguish between valid and invalid mathematical arguments.

Sentences are usually classified as declarative, exclamatory, interrogative and imperative.

In our study of logic, we will confine ourselves to declarative sentences only.

A primary statement is a declarative sentence which cannot be further broken down or analyzed into simpler sentences.

**Proposition:** A **proposition** or **statement** is a declarative sentence which is either true or false but not both. The truth or falsity of a proposition is called its **truth-value**.

**Example 1: All the following declarative sentences are propositions.**

- (a) The Sun rises in the East
- (b) Delhi is the capital of India.
- (c) Kolkata is a country.
- (d) Washington, D. C., is the capital of United states of America.
- (e) 5 is a prime number.
- (f)  $2 + 3 = 6$ .

Propositions a, b, d and e are true, c and f are false

These are propositions (or statements) because they are either true or false.

Some sentences that are not propositions.

Consider the following sentences:

- (g) What time is it ?  
Do you speak Hindi?,  
How beautiful are you?
- (h) Take one book,  
Close the door,

Read this carefully.

- (i)  $4-x=9, x+1=2$   
 $x + y = z$ .
- (j) Balaji is a good boy.
- (k) What a beautiful day!
- (l) May god bless you,  
Wish you a happy new year

(g) is a question(Interrogative), not a declarative sentence, hence it is not a statement.

(h) is not a statement, it is a command.

(i) is a declarative sentence, but not a statement, since it is true or false depends on the variable of  $x$ ,  $x,y$  and  $z$ .

(j) is a declarative sentence, but not a statement, since it is true or false (depends on his character).

(k) is exclamatory, not a declarative sentence.

(l) is imperative, not a declarative sentence.

These are not propositions as they are not declarative in nature, that is, they do not declare a definite truth value  $T$  or  $F$ .

Propositional variables (or statement variables) are denoted by letter  $p, q, r, \dots$

The **truth value** of a proposition is **true**, denoted by **T**, and if it is **false**, is denoted by **F**.

The area of logic that deals with propositions is called the **Propositional Calculus or propositional logic** ( it is also known as **statement calculus**).

It is the branch of mathematics that is used to describe a logical system or structure.

A logical system consists of (1) a universe of propositions, (2) truth tables (as axioms) for the logical operators and (3) definitions that explain equivalence and implication of propositions.

**Till** now, we considered atomic or primary statements. But in practice, we often combine simple (primary) statements to form compound statements by certain connecting words known as sentential connectives or simply connectives.

### **Connectives:**

The words or phrases or symbols which are used to make a proposition by two or more propositions are called **logical connectives** or **simply connectives**.

There are five basic connectives called **negation, conjunction, disjunction, conditional and bi conditional**.

### **Negation:**

The **negation** of a statement is generally formed by writing the word 'not' at a proper place in the statement (proposition) or by prefixing the statement with the phrase 'It is not the case that'.

Def: Let  $p$  be a proposition. The negation of  $p$ , denoted by  $\neg p$  ( $\sim p$ ), is the statement "It is not the case that  $p$ ".

If  $p$  denotes a statement then the negation of  $p$  is written as  $\neg p$  and read as 'not  $p$ '. If the truth value of  $p$  is  $T$  then the truth value of  $\neg p$  is  $F$ . Also if the truth value of  $p$  is  $F$  then the truth value of  $\neg p$  is  $T$ .

**Table 1.** Truth table for negation of a proposition.

$p$	$\neg p$
T	F
F	T

**Example 2:** Consider the statement  $p$ : Kolkata is a city. Then  $\neg p$ : Kolkata is not a city.

Although the two statements ‘Kolkata is not a city’ and ‘It is not the case that Kolkata is a city’ are not identical, we have translated both of them by  $p$ . The reason is that both these statements have the same meaning.

Ex 3:  $q$ : “Today is Friday”

Its negation is  $\neg q$ : “Today is not Friday”

Ex 4:  $P$ : London is a city.

Then  $\neg P$  is the statement  $\neg P$ : It is not the case that London is a city.

$\neg P$ : London is not a city.

Ex: Find the negation of the proposition “At least 10 inches of rain fell today in Miami.”

Sol: The negation is “ It is not the case that at least 10 inches of rain fell today in Miami.”

i.e., “Less than 10 inches of rain fell today in Miami.”

### Conjunction:

Def: Let  $p$  and  $q$  be two propositions. The conjunction  $p$  and  $q$ , denoted by statement  $p \wedge q$  which is read as ‘ $p$  and  $q$ ’.

i.e., The **conjunction** of two statements (or propositions)  $p$  and  $q$  is the statement  $p \wedge q$  which is read as ‘ $p$  and  $q$ ’. The statement  $p \wedge q$  has the truth value  $T$  whenever both  $p$  and  $q$  have the truth value  $T$ . Otherwise it has truth value  $F$ .

**Table 2.** Truth table for conjunction

$p$	$q$	$p \wedge q$
$T$	$T$	$T$
$T$	$F$	$F$
$F$	$T$	$F$
$F$	$F$	$F$

**Example 3:** Consider the following statements

$p$  : Today is Friday

$q$  : “It is raining today”.

Then  $p \wedge q$  is the proposition: “Today is Friday and it is raining today”.

**Note:** Usually, in our everyday language the conjunction ‘and’ is used between two statements which have some kind of relation. Thus a statement ‘It is raining today and  $1 + 1 = 2$ ’ sounds odd, but in logic it is a perfectly acceptable statement formed from the statements ‘It is raining today’ and ‘ $1 + 1 = 2$ ’.

**Example 4:** Translate the following statement:

‘Jack and Jill went up the hill’ into symbolic form using conjunction.

**Solution:** Let  $p$  : Jack went up the hill,  $q$  : Jill went up the hill.

Then the given statement can be written in symbolic form as  $p \wedge q$ .

**Disjunction:**

Let  $p$  and  $q$  be two propositions. The disjunction  $p$  and  $q$ , denoted by statement  $p \vee q$  is the proposition 'p or q'.

i.e., The **disjunction** of two statements  $p$  and  $q$  is the statement  $p \vee q$  which is read as 'p or q'. The statement  $p \vee q$  has the truth value F only when both  $p$  and  $q$  have the truth value F. Otherwise it has truth value T.

Thus, a disjunction is true when at least one of the two propositions in it is true.

**Table 3:** Truth table for disjunction

$p$	$q$	$p \vee q$
$T$	$T$	$T$
$T$	$F$	$T$
$F$	$T$	$T$
$F$	$F$	$F$

**Example 5:** Consider the following statements  $p$  : I shall go to the game.

$q$  : I shall watch the game on television.

Then  $p \vee q$  : I shall go to the game or watch the game on television.

**Ex:**  $p$  : Today is Friday

$q$  : "It is raining today".

Then  $p \vee q$  is the proposition: "Today is Friday or it is raining today".

**Exclusive or:**

**Def:** Let  $p$  and  $q$  be propositions. The exclusive or of  $p$  and  $q$ , denoted by  $p \oplus q$ , is the proposition that is true when exactly one of  $p$  and  $q$  is true and is false otherwise.

The truth table for the Exclusive or of two propositions.

$p$	$q$	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

**Ex:** Ramu will take M.P.C or Bi. P. C in intermediate.

Here "or" is used in the sense "one or the other, but not both".

Ex: "I will buy a laptop or a car in the next year".

Here "or" used in the sense "one or other or the both" (**INCLUSIVE OR**)

**P:** I will buy a laptop

**Q:** I will buy a car

$P \vee Q$ : I will buy a laptop or I will buy a car.

**Conditional proposition:**

If  $p$  and  $q$  are any two statements (or propositions) then the statement  $p \rightarrow q$  which is read as, 'If  $p$ , then  $q$ ' is called a **conditional statement** (or **proposition**) or **implication** and the connective is the **conditional connective**.

The statement  $p \rightarrow q$  has a truth value F, when  $q$  has truth value F and  $p$  has truth value T, otherwise it has the truth value T.

The conditional is defined by the following table:

**Table 4.** Truth table for conditional

$p$	$q$	$p \rightarrow q$
$T$	$T$	$T$
$T$	$F$	$F$
$F$	$T$	$T$
$F$	$F$	$T$

In this conditional statement,  $p$  is called the **hypothesis** or **premise** or **antecedent** and  $q$  is called the **consequence** or **conclusion**.

To understand better, this connective can be looked as a conditional promise. If the promise is violated (broken), the conditional (implication) is false. Otherwise it is true. For this reason, the only circumstances under which the conditional  $p \rightarrow q$  is false is when  $p$  is true and  $q$  is false.

The statement  $p \rightarrow q$  is called a conditional statement because  $p \rightarrow q$  asserts that  $q$  is true on the condition that  $p$  holds. A conditional statement is also called an **implication**.

**Conditional Statements:**

- “If  $p$  then  $q$ ”    “ $p$  implies  $q$ ”
- “if  $p$ ,  $q$ ”    “ $p$  only if  $q$ ”
- “ $p$  is sufficient for  $q$ ”    “ a sufficient condition for  $q$  is  $p$ ”
- “ $q$  if  $p$ ”     “ $q$  whenever  $p$ ”
- “ $q$  when  $p$ ”    “ $q$  is necessary for  $p$ ”
- “ a necessary condition for  $p$  is  $q$ ”                              “ $q$  follows from  $p$ ”
- “ $q$  unless  $\sim p$ ”

Ex: Let  $p$  be the statement “ Harika learns discrete mathematics” and  $q$  the statement “Harika will find a good job”.

From the definition of conditional statements, we see that when  $p$  is the statement “ Harika learns discrete mathematics” and  $q$  is the statement “Harika will find a good job”,  $p \rightarrow q$  represents the statement If Harika learns discrete mathematics, then she will find a good job. There are many other ways to express this conditional statement in English.

- “Harika will find a good job when she learns discrete mathematics”
- “ For Harika to get a good job, it is sufficient for her to learn discrete mathematics”

And

“Harika will find a good job unless she doesnot learn discrete mathematics.”

Ex:

“If it is Sunday, then we will go to the R. K. beach.”

The statements are used in normal language where there is a relationship between the hypothesis and the conclusion.

Further, the first statement is true unless harika learns discrete mathematics, but she does not get a good job, and the second is true unless it is indeed Sunday today, but we do not go to the R K beach.

**Example 6:** *Translate the following statement:*

*‘The crop will be destroyed if there is a flood’ into symbolic form using conditional connective.*

**Solution:** Let  $c$  : the crop will be destroyed;  $f$ : there is a flood.

Let us rewrite the given statement as

‘If there is a flood, then the crop will be destroyed’. So, the symbolic form of the given statement is  $f \rightarrow c$ .

**Example 7:** Let  $p$  and  $q$  denote the statements:

$p$  : You drive over 70 km per hour.

$q$  : You get a speeding ticket.

Write the following statements into symbolic forms.

(i) You will get a speeding ticket if you drive over 70 km per hour.

(ii) Driving over 70 km per hour is sufficient for getting a speeding ticket.

(iii) If you do not drive over 70 km per hour then you will not get a speeding ticket. (iv) Whenever you get a speeding ticket, you drive over 70 km per hour.

**Solution:** (i)  $p \rightarrow q$  (ii)  $p \rightarrow q$  (iii)  $\neg p \rightarrow \neg q$  (iv)  $q \rightarrow p$ .