

GREEN'S FUNCTIONS

DR K M UDAYANANDAN, NAS COLLEGE

Introduction

Green's functions are named after the British mathematician George Green, who developed the concept in the 1830s. Green's function methods enable the solution of a differential equation containing an inhomogeneous term (often called a source term) to be related to an integral operator. It can be used to solve both partial and exact differential equations.

George Green

He (14 July 1793 to 31 May 1841) was a British mathematical physicist who wrote

'An Essay on the Application of Mathematical Analysis to the Theories of Electricity and Magnetism'. The essay introduced several important concepts, among them a theorem similar to the modern Green's theorem, the idea of potential functions as currently used in physics, and the concept of what are now called Green's functions.

Green was the first person to create a mathematical theory of electricity and magnetism and his theory formed the foundation for the work of other scientists such as James Clerk Maxwell, William Thomson, and others.

Simple homogeneous differential equations

Consider the differential equation

$$\frac{d^2y}{dx^2} = 0$$

This can be solved very easily and we will get the solution as

$$y = Ax + B$$

which is the equation for a straight line. The constants can be found if boundary conditions are given. Similarly consider another homogeneous equation

$$\frac{d^2y}{dx^2} + k^2y = 0$$



Figure 1: This are the writings on Green's tomb



Figure 2: George Green



Figure 3: George Green's house and his father's mill

This can be solved to get

$$y = A \sin kx + B \cos kx$$

. Thus there are simple techniques available to solve homogeneous equations. But if we replace them with source terms like

$$\frac{d^2y}{dx^2} = \ln x$$

$$\frac{d^2y}{dx^2} + k^2y = \tan x$$

then the problem become difficult to solve. Before thinking of solving such nonhomogeneous equations let us look at different types of differential operators.

Sturm Liouville operator

Sturm Liouville operator is the most general form of second order differential operator which can be written in the equation form as

$$\mathcal{L}y = \frac{d}{dx} \left(p(x) \frac{dy}{dx} \right) + q(x)y = 0$$

For

$$\frac{d^2y}{dx^2} = 0$$

$p(x) = 1$ and $q(x) = 0$ and for

$$\frac{d^2y}{dx^2} + k^2y = 0$$

$p(x) = 1$ and $q(x) = k^2$. Any differential operator can be changed into SL operator form.

Dirac delta function

While studying GF techniques we will encounter some properties of Dirac delta function. They are

$$\int_{all\ space} \delta(x - t) dx = 1$$

$$\int \delta(x - t) f(t) dt = f(x)$$

Greens function technique

Suppose SL operator operating on a function $y(x)$ gives as

$$\mathcal{L} y(x) = f(x) \tag{1}$$

which is a non homogeneous equation. To solve this NHE let us define

$$\mathcal{L} G(x, t) = \delta(x - t) \tag{2}$$

so that we can show that if we define $y(x) = \int G(x, t)f(t)dt$ we will get equation (1). The proof of this argument is given below

Proof:

$$\mathcal{L} y(x) = \mathcal{L} \int G(x, t)f(t)dt$$

Interchanging integral and differential

$$= \int \mathcal{L} G(x, t)f(x)dt$$

Using the definition of Greens function

$$= \int \delta(x - t)f(t)dt$$

and using the property Dirac delta function that $\int \delta(x - t)f(t)dt = f(x)$ we get

$$\mathcal{L} y(x) = f(x)$$

Definition

Generally speaking, a Green's function is an integral kernel that can be used to solve differential equations from a large number of families including simpler examples such as ordinary differential equations with initial or boundary value conditions, as well as more difficult examples such as inhomogeneous partial differential equations (PDE) with boundary conditions.

One dimensional Greens function and its properties

So let us start with

$$\mathcal{L}G(x, t) = \delta(x - t)$$

Taking the SL operator

$$\frac{d}{dx} \left(p(x) \frac{d}{dx} G(x, t) \right) + q(x)G(x, t) = \delta(x - t)$$

Integrating over x for a small interval $t - \epsilon$ to $t + \epsilon$

$$\int_{t-\epsilon}^{t+\epsilon} \frac{d}{dx} \left(p(x) \frac{d}{dx} G(x, t) \right) dx + \int_{t-\epsilon}^{t+\epsilon} q(x)G(x, t) dx = \int_{t-\epsilon}^{t+\epsilon} \delta(x - t) dx$$

ϵ is a small quantity. Hence RHS is 1. Taking the second part as zero

$$p(t + \epsilon) \frac{dG}{dx}_{t+\epsilon} - p(t - \epsilon) \frac{dG}{dx}_{t-\epsilon} = 1$$

In the limit $\epsilon \rightarrow 0$

$$\begin{aligned} p(t) \left(\frac{dG}{dx}_{t+\epsilon} - \frac{dG}{dx}_{t-\epsilon} \right) &= 1 \\ \frac{dG}{dx}_{t+\epsilon} - \frac{dG}{dx}_{t-\epsilon} &= \frac{1}{p(t)} \\ \frac{dG_2}{dx} - \frac{dG_1}{dx} &= \frac{1}{p(t)} \end{aligned}$$

This property shows that the values of GF must be different for x less than t and x greater than t . So let label GF before t as $G_1(x, t)$ and GF after t as $G_2(x, t)$. We had taken the second integral as zero which means that

$$G_2(x, t + \epsilon) - G_1(x, t - \epsilon) = 0$$

At $x = t$ $G_1 = G_2$ or Greens function is

1. Continuous at boundary and
2. Derivative of the Greens function is discontinuous.

These are the two properties of one dimensional Green's function.

Form of Greens function

Next is to find G_1 and G_2 . Assume

$$G_1(x, t) = C_1 u_1(x)$$

and

$$G_2(x, t) = C_2 u_2(x)$$

where C_1 and C_2 which are functions of t are to be determined. The Greens functions are determined using the two properties we got. The continuity of Greens function demands that

$$C_2 u_2(t) - C_1 u_1(t) = 0$$

Discontinuity of Greens function demands that

$$C_2 u_2'(t) - C_1 u_1'(t) = \frac{1}{p(t)}$$

Multiplying the first equation by $u_2'(t)$ and second by $u_1'(t)$

$$C_2 u_2'(t) u_2(t) - C_1 u_2'(t) u_1(t) = 0$$

$$C_2 u_2'(t) u_2(t) - C_1 u_1'(t) u_2(t) = \frac{-1}{p(t)} u_2(t)$$

Subtracting gives

$$C_1 u_1'(t) u_2(t) - C_1 u_2'(t) u_1(t) = \frac{u_2(t)}{p(t)}$$

$$C_1 (u_2 u_1' - u_1 u_2') = \frac{u_2(t)}{p(t)}$$

If $W = u_1 u_2' - u_2 u_1'$, (is also called Wronskian) Then

$$C_1 = \frac{u_2(t)}{W p(t)}$$

$$C_2 = \frac{u_1(t)}{W p(t)}$$

Hence

$$G_1(x, t) = \frac{u_1(x) u_2(t)}{W p(t)}$$

$$G_2(x, t) = \frac{u_2(x) u_1(t)}{W p(t)}$$

Then we get the solution as

$$y(x) = \int_a^t G_1(x, t) f(t) dt + \int_t^b G_2(x, t) f(t) dt$$

Solved Problems

Case 1 Finite initial and final boundary values given

1. Derive the Green's function for the operator $\frac{d^2}{dx^2}$ with the boundary conditions $y(0) = 0$ and $y(1) = 0$.

Solution Here it is given that

$$\frac{d^2 y}{dx^2} = f(x)$$

For the homogeneous equation

$$\frac{d^2 y}{dx^2} = 0$$

$\frac{d}{dx} \frac{dy}{dx} = 0$, $\frac{dy}{dx} = \text{constant}$. Integrating

$$y = Ax + B$$

First bc implies $y(0) = 0 \Rightarrow B = 0$, $u_1(x) = Ax$, $u_1(t) = At$, $u_1'(x) = A$. Second bc implies $y(1) = 0 \Rightarrow 0 = A + B$, $B = -A$, $u_2(x) = Ax - A$, $u_2(t) = At - A$, $u_2'(x) = A$. Then Wronskian,

$$W = u_1(t) u_2'(t) - u_1'(t) u_2(t) = A^2$$

For $x < t$

$$G_1(x, t) = \frac{u_1(x)u_2(t)}{A}$$

$$G_1(x, t) = x(t - 1)$$

For $x > t$

$$G_2(x, t) = \frac{u_2(x)u_1(t)}{A}$$

$$G_2(x, t) = t(x - 1)$$

2. Derive the Green's function for the operator $\frac{d^2}{dx^2}$ with the boundary conditions $y(0) = 0$ and $y(a) = 0$.

Solution

$$\frac{d^2y}{dx^2} = 0$$

$$y = Ax + B$$

$$y(0) = 0 \Rightarrow$$

$$B = 0$$

$$u_1(x) = Ax$$

$$u_1(t) = At$$

$$u_1'(x) = A$$

$$y(a) = 0 \Rightarrow$$

$$0 = Aa + B$$

$$B = -Aa$$

$$u_2(x) = Ax - Aa$$

$$u_2(t) = At - Aa$$

$$u_2'(x) = A$$

Then Wronskian,

$$\begin{aligned} W &= u_1u_2' - u_1'u_2 \\ &= A_1x A_2 - A_1(A_2x - A_2a) \\ &= A_1A_2x - A_1A_2x + A_1A_2a \\ W &= A^2a \end{aligned}$$

For $x < t$

$$G_1(x, t) = \frac{u_1(x)u_2(t)}{W p(t)}$$

$$G_1(x, t) = \frac{x(t - a)}{a}$$

For $x > t$

$$G_2(x, t) = \frac{u_2(x)u_1(t)}{W p(t)}$$

$$G_2(x, t) = \frac{t(x - a)}{a}$$

3. Obtain the Green's function for the operator $\frac{d^2}{dx^2}$ corresponding to the boundary conditions $y(0) = 0$; $y'(a) = 0$.

Answer:

$$\frac{d^2 y}{dx^2} = 0$$

Its solutions is,

$$y = Ax + B$$

The first BC gives $y(0) = 0$ implies $0 = A \times 0 + B$ and hence $B = 0$.

$$u_1(x) = Ax$$

$$u_1(t) = At$$

$$u_1'(x) = A$$

The second BC gives $y(a) = 0$ implies

$$A = 0$$

$$u_2(x) = B$$

$$u_2(t) = B$$

$$u_2'(x) = 0$$

Then Wronskian,

$$W = u_1 u_2' - u_1' u_2$$

$$W = -AB$$

For $x < t$

$$G_1(x, t) = \frac{u_1(x)u_2(t)}{Wp(t)}$$

$$G_1(x, t) = -x$$

For $x > t$

$$G_2(x, t) = \frac{u_2(x)u_1(t)}{Wp(t)}$$

$$G_2(x, t) = -t$$

4. Obtain the Green's function for the operator $\frac{d^2}{dx^2}$ corresponding to the boundary conditions $y(0) = 0$; $y'(1) = 0$.

Answer:

Solution is same as in the above problem with same answer

5. Deduce Green's function of the operator $(\frac{d^2}{dx^2} + k^2)$ with boundary condition $y(0) = 0$, $y(L) = 0$

$$\frac{d^2 y}{dx^2} + k^2 y = f(x)$$

its solution is

$$y = A \sin kx + B \cos kx$$

$u_1(x)$ is defined as the value of y at applying first boundary condition and $u_2(x)$ is defined as the value of y at applying second boundary condition.

$$y(0) = 0 \Rightarrow$$

$$B = 0$$

$$u_1(x) = A \sin kx$$

$$u_1(t) = A \sin kt$$

$$u_1'(x) = A k \cos kx$$

Similarly $y(L) = 0 \Rightarrow$

$$0 = A \sin kL + B \cos kL$$

$$B = \frac{-A \sin kL}{\cos kL}$$

$$u_2(x) = A \sin kx - \frac{a_2 \cos kL}{\cos kL} \cos kx = A \left(\frac{\sin kx \cos kL - \sin kL \cos kx}{\cos kL} \right)$$

$$u_2(x) = A \frac{\sin k(x-L)}{\cos kL}$$

$$u_2(t) = A \frac{\sin k(t-L)}{\cos kL}$$

$$u_2'(t) = A \frac{k \cos k(x-L)}{\cos kL}$$

Then Wronskian, $W = \frac{A^2 k}{\cos kL} (\sin kL)$ For $x < t$

$$G_1(x, t) = \frac{\sin kx \sin k(t-L)}{k \sin kL}$$

For $x > t$

$$G_2(x, t) = \frac{\sin k(x-L) \sin kt}{k \sin kL}$$

6. Find an appropriate Green's function for the equation $y'' + 1/4y = f(x)$ with boundary condition $y(0) = y(\pi) = 0$.

$$\frac{d^2y}{dx^2} + \frac{1}{4}y = 0$$

$$\frac{d^2y}{dx^2} + \left(\frac{1}{2}\right)^2 y = 0$$

So it's solution is

$$y = A \sin \frac{1}{2}x + B \cos \frac{1}{2}x$$

$$y(0) = 0 \Rightarrow$$

$$B = 0$$

$$u_1(x) = A \sin \frac{1}{2}x$$

$$u_1(t) = A \sin \frac{1}{2}t$$

$$y(\pi) = 0 \Rightarrow$$

$$u_1'(x) = \frac{1}{2}A \cos \frac{1}{2}x$$

$$A = 0$$

$$u_2(x) = B \cos \frac{1}{2}x$$

$$u_2(t) = B \cos \frac{1}{2}t$$

$$u_2'(x) = -\frac{1}{2}B \sin \frac{1}{2}x$$

Then Wronskian,

$$W = u_1 u_2' - u_1' u_2$$

$$W = \frac{-AB}{2}$$

For $x < t$

$$G_1(x, t) = -2 \sin \frac{1}{2}x \cos \frac{1}{2}t$$

For $x > t$

$$G_2(x, t) = -2 \cos \frac{1}{2}x \sin \frac{1}{2}t$$

Case 2: Initial bc given, final bc not given

Solved Problem

1. Derive the Green's function for the differential equation $\frac{d^2}{dx^2} = 0$ with the boundary conditions $y(0) = 0 = y'(0)$.

Solution:

$$\frac{d^2 y}{dx^2} = 0$$

Its solution is

$$y = Ax + B$$

$$y(0) = 0 \Rightarrow$$

$$B = 0$$

Then $u_1(x) = Ax$, $u_1(t) = At$, $u_1'(x) = A$. $y'(0) = 0 \Rightarrow$

$$u_2(x) = A$$

$$u_2(t) = A$$

$$u_2'(x) = 0$$

Then Wronskian, $W = -A^2$. For $x < t$

$$G_1(x, t) = -x$$

For $x > t$

$$G_2(x, t) = -t$$

Case 3, Boundaries at infinity

Solved Problem

1. Find Greens function for $\frac{d^2y}{dx^2} - k^2y = f(x)$; $y(\pm\infty) = 0$

It's solution is

$$y = Ae^{kx} + Be^{-kx}$$

then, first boundary condition $y(+\infty) = 0 \Rightarrow$

$$0 = Ae^\infty + Be^\infty$$

$Ae^\infty = 0, A = 0$. So $u_1(x) = Be^{-kx}$, $u_1(t) = Be^{-kt}$, $u_1'(t) = -kBe^{-kt}$. Second boundary condition $y(-\infty) = 0 \Rightarrow$

$$0 = Ae^{-\infty} + Be^{+\infty}$$

$$0 = 0 + Be^{+\infty}$$

$$Be^{+\infty} = 0$$

$$B = 0$$

So $u_2(x) = Ae^{kx}$, $u_2(t) = Ae^{kt}$, $u_2'(t) = kAe^{kt}$. Then Wronskian, $W = 2kAB$ For $x < t$

$$G_1(x, t) = \frac{e^{k(t-x)}}{2k}$$

For $x > t$

$$G_2(x, t) = \frac{-e^{k(x-t)}}{2k}$$

Green's Function for Poisson's Equation

We've by definitions

$$\mathcal{L}y(x) = f(x)$$

$$\mathcal{L}G(x, t) = \delta(x - t)$$

$$y(x) = \int G(x, t)f(t)dt$$

Poisson's equation says

$$\nabla^2\phi = \frac{-\rho}{\epsilon_0}$$

$$\nabla^2G = \delta(\vec{r}_1 - \vec{r}_2)$$

Then we get using definitions

$$\phi(\vec{r}_2) = \int G(\vec{r}_1, \vec{r}_2) \frac{\rho(r_1)}{\epsilon_0} d^3 r_1$$

But from electrodynamics we know

$$\phi(\vec{r}_2) = \int \frac{\rho(r_1) d^3 r_1}{4\pi\epsilon_0|\vec{r}_2 - \vec{r}_1|}$$

Comparing we get

$$G(\vec{r}_1, \vec{r}_2) = \frac{1}{4\pi|\vec{r}_2 - \vec{r}_1|}$$

This is the Greens function for Poisson's equation.

Green's Function as a series of Eigen functions

Readers are requested to read the chapter on Sturm Liouville operator in Arfkan 7th Edition before reading this section. Let

$$\mathcal{L}\phi_n(x) = \lambda_n\phi_n(x)$$

where $\phi_n(x)$ is the eigen function and λ_n is the eigen value. Usually we have

$$\mathcal{L}y(x) = f(x)$$

and the solution is always written in terms of Greens function. Here we assume Greens function in terms of eigen functions.

$$G(x, t) = \sum_n c_n(t)\phi_n(x)$$

where $\phi_n(x)$ are orthogonal eigen functions and $c_n(t)$ is unknown which is to be found out.

$$\mathcal{L}G(x, t) = \delta(x - t)$$

Substituting

$$\mathcal{L}\sum_n c_n\phi_n(x) = \sum_n c_n(t)\mathcal{L}\phi_n(x)$$

$$\delta(x - t) = \sum_n c_n(t)\lambda_n\phi_n(x)$$

Multiplying with $\phi_m^*(x)$ and integrating over x

$$\int \phi_m^*(x)\delta(x - t)dx = \sum_n c_n(t)\lambda_n \int \phi_m^*(x)\phi_n(x)dx$$

$$\phi_m^*(t) = \sum_n c_n(t)\lambda_n\delta_{mn}$$

$$\phi_m^*(t) = c_m(t)\lambda_m$$

Thus

$$c_m(t) = \frac{\phi_m^*(t)}{\lambda_m}$$

or

$$c_n(t) = \frac{\phi_n^*(t)}{\lambda_n}$$

Then,

$$G(x, t) = \sum_n \frac{\phi_n^*(t)\phi_n(x)}{\lambda_n}$$

This is the eigen function expansion of Green's function.

University questions

1. By the method of Green's function $G(\vec{r}, \vec{r}')$, solve the Poisson's equation in electro-statistics $\nabla^2 u(\vec{r}) = -4\pi\rho(\vec{r})$, $\rho(\vec{r})$ is the charge density.
2. Define Green's function in one and three dimensions.
3. Explain how the method of Green's function is useful in obtaining the solutions of Poisson's equation.

4. Deduce Green's function of the Helmholtz operator $(\nabla^2 + k^2)$ for an outgoing spherical wave.
5. a) Define Green's function b) Expand the Green's function as a series in eigen functions of the corresponding homogeneous equation c) Hence prove the symmetry of the Green's function.
6. What are the important properties of Green's function?
7. Develop the Green's function for the Laplace equation $\nabla^2\phi = 0$ in an infinite region with the boundary condition $\phi \rightarrow 0$ as $r \rightarrow \infty$.
8. Explain how the Green's function can be expressed as an eigen function expansion.
9. Show that $\frac{\exp(ik|\vec{r}_1 + \vec{r}_2|)}{4\pi(\vec{r}_1 + \vec{r}_2)}$ is the Green's function of the Helmholtz operator $(\nabla^2 + k^2)$
is the Green's function for $\frac{d}{dx} [x \frac{dy}{dx}]$