

## Module-III ( Relations and their properties)

**Set:** A set is collection of well defined objects.

In the above definition the words set and collection for all practical purposes are Synonymous.

We have really used the word set to define itself.

Each of the objects in the set is called a member of an element of the set. The objects themselves can be almost anything. Books, cities, numbers, animals, flowers, etc. Elements of a set are usually denoted by lower-case letters. While sets are denoted by capital letters of English language.

The symbol  $\in$  indicates the membership in a set.

If “ $a$  is an element of the set  $A$ ”, then we write  $a \in A$ .

The symbol  $\in$  is read “is a member of” or “is an element of”.

The symbol  $\notin$  is used to indicate that an object is not in the given set.

The symbol  $\notin$  is read “is not a member of” or “is not an element of”.

If  $x$  is not an element of the set  $A$  then we write  $x \notin A$ .

### **Subset:**

A set  $A$  is a subset of the set  $B$  if and only if every element of  $A$  is also an element of  $B$ . We also say that  $A$  is contained in  $B$ , and use the notation  $A \subseteq B$ .

### **Proper Subset:**

A set  $A$  is called proper subset of the set  $B$ . If (i)  $A$  is subset of  $B$  and (ii)  $B$  is not a subset  $A$  i.e.,  $A$  is said to be a proper subset of  $B$  if every element of  $A$  belongs to the set  $B$ , but there is at least one element of  $B$ , which is not in  $A$ . If  $A$  is a proper subset of  $B$ , then we denote it by  $A \subset B$ .

**Super set:** If  $A$  is subset of  $B$ , then  $B$  is called a superset of  $A$ .

**Null set:** The set with no elements is called an empty set or null set.

A Null set is designated by the symbol  $\phi$ .

The null set is a subset of every set, i.e., If  $A$  is any set then  $\phi \subset A$ .

### **Universal set:**

In many discussions all the sets are considered to be subsets of one particular set. This set is called the universal set for that discussion. The Universal set is often designated by the script letter  $\mu$ . Universal set is not unique and it may change from one discussion to another.

### **Power set:**

The set of all subsets of a set  $A$  is called the power set of  $A$ .

The power set of  $A$  is denoted by  $P(A)$ . If  $A$  has  $n$  elements in it, then  $P(A)$  has  $2^n$  elements:

### **Disjoint sets:**

Two sets are said to be disjoint if they have no element in common.

### **Union of two sets:**

The union of two sets  $A$  and  $B$  is the set whose elements are all of the elements in  $A$  or in  $B$  or in both. The union of sets  $A$  and  $B$  denoted by  $A \cup B$  is read as “ $A$  union  $B$ ”.

### **Intersection of two sets:**

The intersection of two sets  $A$  and  $B$  is the set whose elements are all of the elements common to both  $A$  and  $B$ . The intersection of the sets of “ $A$ ” and “ $B$ ” is denoted by  $A \cap B$  and is read as “ $A$  intersection  $B$ ”

### **Difference of sets:**

If  $A$  and  $B$  are subsets of the universal set  $U$ , then the relative complement of  $B$  in  $A$  is the set of all elements in  $A$  which are not in  $B$ . It is denoted by  $A - B$  thus:  $A - B = \{x \mid x \in A \text{ and } x \notin B\}$

**Complement of a set:**

If  $U$  is a universal set containing the set  $A$ , then  $U - A$  is called the complement of  $A$ . It is denoted by  $A^c$ . Thus  $A^c = (A^c) = \{x: x \notin A\}$

**Inclusion-Exclusion Principle:**

The inclusion–exclusion principle is a counting technique which generalizes the familiar method of obtaining the number of elements in the union of two finite sets; symbolically expressed as

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

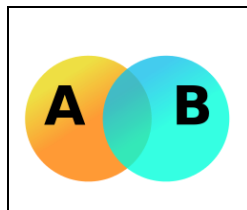


Fig.Venn diagram showing the union of sets A and B

where  $A$  and  $B$  are two finite sets and  $|S|$  indicates the cardinality of a set  $S$  (which may be considered as the number of elements of the set, if the set is finite). The formula expresses the fact that the sum of the sizes of the two sets may be too large since some elements may be counted twice. The double-counted elements are those in the intersection of the two sets and the count is corrected by subtracting the size of the intersection.

The principle is more clearly seen in the case of three sets, which for the sets  $A, B$  and  $C$  is given by

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |C \cap B| - |A \cap C| + |A \cap B \cap C|.$$

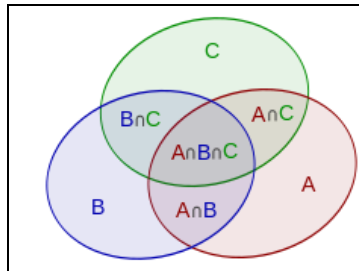


Fig. Inclusion–exclusion illustrated by a Venn diagram for three sets

This formula can be verified by counting how many times each region in the Venn diagram figure is included in the right-hand side of the formula. In this case, when removing the contributions of over-counted elements, the number of elements in the mutual intersection of the three sets has been subtracted too often, so must be added back in to get the correct total.

In general, Let  $A_1, \dots, A_p$  be finite subsets of a set  $U$ . Then,

$$|A_1 \cup A_2 \cup \dots \cup A_p| = \sum_{1 \leq i \leq p} |A_i| - \sum_{1 \leq i_1 < i_2 \leq p} |A_{i_1} \cap A_{i_2}| + \sum_{1 \leq i_1 < i_2 < i_3 \leq p} |A_{i_1} \cap A_{i_2} \cap A_{i_3}| - \dots + (-1)^{p-1} |A_1 \cap A_2 \cap \dots \cap A_p|,$$

Example: How many natural numbers  $n \leq 1000$  are not divisible by any of 2, 3?

Ans: Let  $A_2 = \{n \in \mathbb{N} \mid n \leq 1000, 2|n\}$  and  $A_3 = \{n \in \mathbb{N} \mid n \leq 1000, 3|n\}$ .

Then,  $|A_2 \cup A_3| = |A_2| + |A_3| - |A_2 \cap A_3| = 500 + 333 - 166 = 667$ .

So, the required answer is  $1000 - 667 = 333$ .

Example: How many integers between 1 and 10000 are divisible by none of 2, 3, 5, 7?

Ans: For  $i \in \{2, 3, 5, 7\}$ , let  $A_i = \{n \in \mathbb{N} \mid n \leq 10000, i|n\}$ .

Therefore, the required answer is  $10000 - |A_2 \cup A_3 \cup A_5 \cup A_7| = 2285$ .

## Relations

Let  $A$  and  $B$  be two sets, a binary relation from  $A$  to  $B$  is a subset of  $A \times B$ .

**Cartesian Product:** Let  $A$  and  $B$  be two sets, then the Cartesian product of  $A$  and  $B$  is defined as  $A \times B = \{ (a,b) | a \in A \text{ and } b \in B \}$ .

$A \times B \neq B \times A$  (Cartesian product is not commutative).

Cartesian product of  $A_1, A_2, A_3, \dots, A_n$  defined as  $A_1 \times A_2 \times A_3 \times \dots \times A_n = \{ (a_1, a_2, a_3, \dots, a_i), a_i \in A_i, i=1,2,3,\dots,n \}$ .

Suppose  $A$  has  $m$  elements,  $B$  has  $n$  elements then  $A \times B$  has  $mn$  elements.

$$A \times (B \cup C) = (A \times B) \cup (A \times C)$$

Let  $A$  and  $B$  be two sets, a binary relation from  $A$  to  $B$  is a subset of  $A \times B$ .

i.e., A binary relation from  $A$  to  $B$  is a set of  $R$  of order pairs comes first element from set  $A$  and second element from set  $B$ .

$$(a, b) \in R \text{ or } a R b$$

$$A = \{1,2,3,4\}, B = \{a,b,c\}$$

$$R = \{ (1,a), (1,b), (2,a), (2,c), (3,a), (3,b), (4,c) \}.$$

Definition: Any set of ordered pairs defines a *binary relation*.

We shall call a binary relation simply a relation. Binary relations represent relationships between elements of two sets. If  $R$  is a relation, a particular ordered pair, say  $(x, y) \in R$  can be written as  $xRy$  and can be read as “ $x$  is in relation  $R$  to  $y$ ”.

Example: Give an example of a relation.

Solution: The relation “greater than” for real numbers is denoted by ‘ $>$ ’. If  $x$  and  $y$  are any two real numbers such that  $x > y$ , then we say that  $(x, y) \in >$ .

Thus the relation  $R: >$  is  $\{ \} > = (x, y) : x \text{ and } y \text{ are real numbers and } x > y$

$$A = \{1,2,3,4,5\}$$

$$R: \{ (2,1), (3,1), (3,2), (4,1), (4,2), (4,3), (5,1), (5,2), (5,3), (5,4) \}$$

Example: Define a relation between two sets  $A = \{5, 6, 7\}$  and  $B = \{x, y\}$ .

Solution: If  $A = \{5, 6, 7\}$  and  $B = \{x, y\}$ , then the subset  $R = \{(5, x), (5, y), (6, x), (6, y)\}$  is a relation from  $A$  to  $B$ .

Definition: Let  $S$  be any relation. The *domain* of the relation  $S$  is defined as the set of all first elements of the ordered pairs that belong to  $S$  and is denoted by  $D(S)$ .

$$D(S) = \{ x : (x, y) \in S, \text{ for some } y \}$$

The *range* of the relation  $S$  is defined as the set of all second elements of the ordered pairs that belong to  $S$  and is denoted by  $R(S)$ .

$$R(S) = \{ y : (x, y) \in S, \text{ for some } x \}$$

Example:  $A = \{2, 3, 4\}$  and  $B = \{3, 4, 5, 6, 7\}$ .

Define a relation from  $A$  to  $B$  by  $(a, b) \in R$  if  $a$  divides  $b$ .

(  $a$  divides  $b$   $a|b = b = ka$  where  $k \in \mathbb{Z}$

Ex :  $2|4$  : 2 divides 4,  $(4 = 2 \cdot 2)$ ,  $2|6$ ,

$3|6$ : 3 divides 6,  $6 = 2 \cdot 3$

,  $4|8$ : 4 divides 8)

$$8 = 2 \cdot 4$$

$$2|4, 2|6$$

$$3|3, 3|6$$

$$4|4,$$

Solution: We obtain  $R = \{(2, 4), (2, 6), (3, 3), (3, 6), (4, 4)\}$ .

Domain of  $R = \{2, 3, 4\}$  and range of  $R = \{3, 4, 6\}$ .

Divisors of 5.  $R = \{(5,5), (5,10), (5,15), (5,20), (5,25)\}$

## Properties of Binary Relations in a Set

A relation  $R$  on a set  $X$  is said to be

- **Reflexive** relation if  $xRx$  or  $(x, x) \in R, \forall x \in X$

$A = \{1, 2, 3\}$

$R = \{(1,1), (2,2), (3,3)\}$  is reflexive.

- A relation  $R$  on a set  $X$  is said to be **Symmetric** relation if  $x R y$  then  $y R x$ ,  
*i.e., if  $(x, y) \in R$ , then  $(y, x) \in R$ .*

$A = \{1, 2, 3, 4\}$

$R = \{(1, 2), (2, 1), (3, 2), (2, 3), (4, 1), (1, 4)\}$  is symmetric.

- A relation  $R$  on a set  $X$  is said to be **Transitive** relation if  $x R y$  and  $y R z$  then  $x R z$ ,  
*i.e., if  $(x, y) \in R$  and  $(y, z) \in R$  then  $(x, z) \in R$ .*

$A = \{1, 2, 3, 4\}$

$R = \{(1, 2), (2, 3), (1, 3), (2, 3), (3, 4), (2, 4)\}$

. A relation  $R$  on a set  $X$  is said to be **Antisymmetric** relation if for every  $x$  and  $y$  in  $X$ ,  
whenever  $xRy$  and  $yRx$ , then  $x = y$ .

*i.e., if  $(x, y) \in R$  and  $(y, x) \in R$  then  $x=y$ .*

Examples: (i). If  $R_1 = \{(1, 1), (1, 2), (2, 2), (2, 3), (3, 3)\}$  be a relation on  $A = \{1, 2, 3\}$ , then  $R_1$  is a reflexive relation, since for every  $x \in A$ ,  $(x, x) \in R_1$ .

(ii). If  $R_2 = \{(1, 1), (1, 2), (2, 3), (3, 3)\}$  be a relation on  $A = \{1, 2, 3\}$ , then  $R_2$  is not a reflexive relation, since for every  $2 \in A$ ,  $(2, 2) \notin R_2$ .

(iii). If  $R_3 = \{(1, 1), (1, 2), (1, 3), (2, 2), (2, 1), (3, 1)\}$  be a relation on  $A = \{1, 2, 3\}$ , then  $R_3$  is a symmetric relation.

(iv). If  $R_4 = \{(1, 2), (2, 2), (2, 3)\}$  on  $A = \{1, 2, 3\}$  is an antisymmetric.

(v) If  $R_5 = \{(1,1), (2,2), (1,2), (2,1)\}$  is transitive.

Example: Given  $S = \{1, 2, \dots, 10\}$  and a relation  $R$  on  $S$ , where  $R = \{(x, y) / x + y = 10\}$ .

What are the properties of the relation  $R$ ? SYMMETRIC

Solution: Given that

$S = \{1, 2, \dots, 10\}$

$R = \{(x, y) / x + y = 10\}$

$R = \{(1, 9), (9, 1), (2, 8), (8, 2), (3, 7), (7, 3), (4, 6), (6, 4), (5, 5)\}$ .

(i). For any  $x \in S$  and  $(x, x) \notin R$ . Here,  $1 \in S$  but  $(1, 1) \notin R$ .

$\Rightarrow$  the relation  $R$  is not reflexive. It is also not irreflexive, since  $(5, 5) \in R$ .

(ii).  $(1, 9) \in R \Rightarrow (9, 1) \in R$

$$(2, 8) \in R \Rightarrow (8, 2) \in R \dots$$

$\Rightarrow$  the relation is symmetric, but it is not antisymmetric. (iii).  $(1, 9) \in R$  and  $(9, 1) \in R$

$\Rightarrow (1, 1) \notin R$

$\Rightarrow$  The relation  $R$  is not transitive. Hence,  $R$  is symmetric.

Ex: Consider the following relations on  $A = \{1, 2, 3, 4\}$ .

$$R1 = \{(1,1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)\}$$

$$R2 = \{(1, 1), (1, 2), (2, 1)\}$$

$$R3 = \{(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (3, 3), (4, 1), (4, 4)\}$$

$$R4 = \{(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)\}$$

$$R5 = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)\}$$

$$R6 = \{(3,4)\}$$

Which of the relations are (a) reflexive, (b) symmetric (c) anti symmetric and (d) Transitive?

(a) The reflexive relations are  $R3$  and  $R5$ .

Both the relations contains all the pairs of the form  $(a, a)$  namely  $(1,1), (2,2), (3,3)$  and  $(4,4)$ .

(b) The symmetric relations are  $R2$  and  $R3$ .

$R2$  contains  $(1, 2), (2, 1)$

$R3$  contains  $(1, 2), (2, 1), (1, 4), (4, 1)$ .

(c) The antisymmetric relations are  $R4, R5$  and  $R6$ .

For example  $(2,1)$  in  $R4$  but not  $(1,2)$ .....

(d) The transitive relations are  $R2, R3, R4$  and  $R5$ .

In  $R3$  Since we have  $(1,2), (2,1)$  then we have  $(1,1)$ ,

Since we have  $(1,4), (4,1)$  then we have  $(1,1)$ ,

Since we have  $(2,1), (1,2)$  then we have  $(2,2)$ ,

Since we have  $(4, 4), (4, 1)$  then we have  $(4, 1)$ .

In  $R4$  since we have  $(4, 2), (2, 1)$  then we have  $(4, 1)$ ,

Since we have  $(4, 3), (3,1)$  then we have  $(4,1)$ .

Ex: Consider the relations on set of integers

$$R1 = \{(a,b) / a \leq b\}$$

$$R2 = \{(a,b) / a > b\}$$

$$R3 = \{(a,b) / a = b \text{ or } a = -b\}$$

$$R4 = \{(a,b) / a = b\}$$

$$R5 = \{(a,b) / a = b + 1\}$$

$$R6 = \{(a,b) / a + b \leq 3\}$$

Which of the relations are (a) reflexive, (b) symmetric (c) anti symmetric and (d) Transitive?

(a) The reflexive relations are  $R1, R3$  and  $R4$ .

(b) The symmetric relations are  $R3, R4$  and  $R6$ .

$R3$  is symmetric for, if  $a = b$  or  $a = -b$  then  $b = a$  or  $b = -a$

$$(a,b) \in R3, (b,a) \in R3$$

$R4$  is symmetric because  $a = b$  implies  $b = a$ .

$R6$  is symmetric because  $a + b \leq 3$  implies that  $b + a \leq 3$ .

The relations  $R1, R2, R4$  and  $R5$  are antisymmetric.

R1 is antisymmetric because  $a \leq b$  and  $b \leq a$  implies  $a=b$

R2 is antisymmetric because it is impossible for  $a > b$  and  $b > a$ .

**R4** is antisymmetric because  $a=b$ ,  $b=a$  iff  $a=b$ .

(Because two elements are related with respect to R4 if and only if they are equal)

R5 is antisymmetric because it is impossible that  $a = b+1$  and  $b = a+1$ .

The relations R1, R2, R3 and R4 are Transitive.

R1 is transitive because  $a \leq b$  and  $b \leq c$  implies  $a \leq c$

R2 is transitive because  $a > b$  and  $b > c$  implies  $a > c$

R3 is transitive because  $a = +$  or  $- b$  and  $b = +$  or  $- c$  implies  $a = +$  or  $- c$

R4 is transitive because  $a = b$  and  $b = c$  implies  $a = c$ .

EX: Is the relation “divides” on the set of positive integers Reflexive?, symmetric?, antisymmetric? and Transitive?

Sol:  $a$  divides  $b$  ( $a|b$ ) is  $b = ka$  where  $k$  is a positive integer.

Examples :  $2|4$ ,  $2|8$ ,  $5|30$

$a|b$  means  $b$  is a multiple of  $a$ .

Since  $a|a$  ( $a$  divides  $a$ ), whenever  $a$  is a positive integer

Therefore the relation “divides” is reflexive.

If  $a|b$  then  $b \nmid a$  ( $b$  does not divide  $a$ ) ( $a \neq b$ )

$2|4$ , but not  $4|2$

Therefore the relation “divides” is not symmetric.

If  $a|b$  and  $b|a$  then  $a=b$ .

Therefore the relation “divides” is anti symmetric.

Suppose  $a|b$  and  $b|c$ , then we must have  $a|c$

$(a,b) \in R$ ,  $(b,c) \in R$  then  $(a,c) \in R$ .

$a|b$  :  $b = ka$  where  $k \in \mathbb{Z}$

$b|c$  :  $c = lb$  where  $l \in \mathbb{Z}$

now  $c = lb$

$$= l(ka)$$

$$= ma \text{ where } m = lk \in \mathbb{Z}$$

Therefore  $a|c$  and hence the relation divides ( $|$ ) is Transitive.

$2|6$ ,  $6|18$  and also  $2|18$

$2|6$ :  $6 = 3 \cdot 2$

$6|18$ :  $18 = 3 \cdot 6$

$2|18$ :  $18 = 9 \cdot 2$

## Matrix and the Graph Representation of a Relation

Relation Matrix: A relation  $R$  from a finite set  $X$  to a finite set  $Y$  can be represented using a zero-one matrix is called the *relation matrix* of  $R$ .

Let  $X = \{x_1, x_2, \dots, x_m\}$  and  $Y = \{y_1, y_2, \dots, y_n\}$  be finite sets containing  $m$  and  $n$  elements, respectively, and  $R$  be the relation from  $X$  to  $Y$ . Then  $R$  can be represented by an  $m \times n$  matrix  $M_R = [m_{ij}]_{m \times n}$ , which is defined as follows:

$$m_{ij} = \begin{cases} 1, & \text{if } (x_i, y_j) \in R \\ 0, & \text{if } (x_i, y_j) \notin R \end{cases}$$

In the otherwords, the zero-one matrix representing  $R$  has 1 as its  $(i, j)$  entry  $x_i$  is related to  $y_j$ , and a 0 in this position if  $x_i$  is not related to  $y_j$

*Example.* Let  $A = \{1, 2, 3, 4\}$  and  $B = \{b_1, b_2, b_3\}$ . Consider the relation  $R = \{(1, b_2), (1, b_3), (3, b_2), (4, b_1), (4, b_3)\}$ . Determine the matrix of the relation.

Solution:  $A = \{1, 2, 3, 4\}$ ,  $B = \{b_1, b_2, b_3\}$ .

Relation  $R = \{(1, b_2), (1, b_3), (3, b_2), (4, b_1), (4, b_3)\}$ .

Matrix of the relation  $R$  is written as

$$\text{That is } M_R = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

EX: Suppose that  $A = \{1, 2, 3\}$  and  $B = \{1, 2\}$ . Let  $R$  be the relation from  $A$  to  $B$

containing  $(a, b)$  if  $a \in A$ ,  $b \in B$  and  $a > b$ .

Sol :  $R = \{(2, 1), (3, 1), (3, 2)\}$

The Matrix Representation is  $M_R = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$

Example: Let  $A = \{1, 2, 3, 4\}$ . Find the relation  $R$  on  $A$  determined by the matrix

$$M_R = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}$$

Solution: The relation  $R = \{(1, 1), (1, 3), (2, 3), (3, 1), (4, 1), (4, 2), (4, 4)\}$ .

**Properties of a relation in a set:**

(i). If a relation is reflexive, then all the diagonal entries must be 1.

$$\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}$$

(ii). If a relation is symmetric, then the relation matrix is symmetric, i.e.,  $m_{ij} = m_{ji}$  for every  $i$  and  $j$ .  $a_{ij}$  it symmetric element is  $a_{ji}$

$$\begin{matrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{matrix}$$

(iii). If a relation is antisymmetric, then its matrix is such that if  $m_{ij} = 1$  then  $m_{ji} = 0$  for  $i \neq j$ .

$$\begin{matrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{matrix}$$

**Graph of a Relation:** A relation can also be represented pictorially by drawing its *graph*.

Let  $R$  be a relation in a set  $X = \{x_1, x_2, \dots, x_m\}$ . The elements of  $X$  are represented by points or circles called *nodes*. These nodes are called *vertices*. If  $(x_i, x_j) \in R$ , then we connect the nodes  $x_i$  and  $x_j$  by means of an arc and put an arrow on the arc in the direction from  $x_i$  to  $x_j$ . This is called an *edge*. If all the nodes corresponding to the ordered pairs in  $R$  are connected by arcs with proper arrows, then we get a graph of the relation  $R$ .

Note: (i). If  $x_i R x_j$  ( $x_i, x_j$ )  $\in R$  and  $x_j R x_i$  ( $x_j, x_i$ )  $\in R$ , then we draw two arcs between  $x_i$  and  $x_j$  **with arrows pointing in both directions**.

(ii). If  $x_i R x_i$ , ( $x_i, x_i$ )  $\in R$  then we get an arc which starts from node  $x_i$  and returns to node  $x_i$ . This arc is called a **loop**.

**Properties of relations:**

(i). If a relation is reflexive, then there must be a **loop at each node**.

On the other hand, if the relation is irreflexive, then there is **no loop** at any node.

(ii). If a relation is symmetric and if one node is connected to another, then there must be a return arc from the second node to the first.

(iii). For antisymmetric relations, no such **direct return path should exist**.

(iv). If a relation is transitive, the situation is **not so simple**.

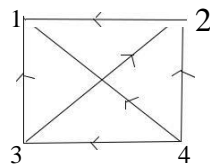
Example: Let  $X = \{1, 2, 3, 4\}$  and  $R = \{(x, y) / x > y\}$ .

Draw the graph of  $R$  and also give its matrix.

Solution:  $R = \{(4, 1), (4, 3), (4, 2), (3, 1), (3, 2), (2, 1)\}$ .

$R = \{(2,1), (3,1), (3,2), (4,1), (4,2), (4,3)\}$

The graph of  $R$  and the matrix of  $R$  are

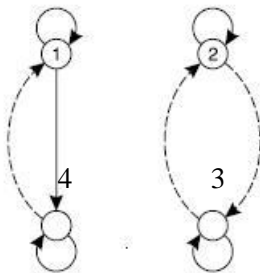


Graph of R

$$M_R = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

EX:2 Let  $X = \{1, 2, 3, 4\}$  and  $R = \{(1, 1), (1, 4), (4, 1), (4, 4), (2, 2), (2, 3), (3, 2), (3, 3)\}$ .  
Representation of the relation  $R$  in matrix and graph.

$$M_R = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$



### Combining Relations

Because relations from  $A$  to  $B$  are subsets of  $A \times B$ , two relations from  $A$  to  $B$  can be combined.

Ex:  $A = \{1, 2, 3\}$  and  $B = \{1, 2, 3, 4\}$  and the relations

$R_1 = \{(1, 1), (2, 2), (3, 3)\}$  and  $R_2 = \{(1, 1), (1, 2), (1, 3), (1, 4)\}$  can be combined.

$$R_1 \cup R_2 = \{(1, 1), (2, 2), (3, 3), (1, 2), (1, 3), (1, 4)\}$$

$$R_1 \cap R_2 = \{(1, 1)\}$$

$$R_1 - R_2 = \{(2, 2), (3, 3)\}$$

$$R_2 - R_1 = \{(1, 2), (1, 3), (1, 4)\}$$

Example : Let the relations  $R_1$  and  $R_2$  be  $R_1 = \{(x, y) / x < y\}$ ,  $R_2 = \{(x, y) / x > y\}$

$(x, y) \in R_1 \cup R_2 \leftrightarrow (x, y) \in R_1$  or  $(x, y) \in R_2$ .

$(x, y) \in R_1 \cup R_2$  if and only if  $x < y$  or  $x > y$ .

Because the condition  $x < y$  or  $x > y$  is the same as  $x \neq y$ .

i.e.,  $R_1 \cup R_2 = \{(x, y) / x \neq y\}$ .

$(x, y) \in R_1 \cap R_2$  if and only if  $x < y$  and  $x > y$ .

$$\begin{aligned} & x < y \wedge x > y \\ & = (x, y) \in R_1 \wedge (x, y) \in R_2 \\ & = \Phi. \end{aligned}$$

$(x, y) \in R_1 - R_2$  if and only if  $(x, y) \in R_1$  and  $(x, y) \notin R_2$

$$\begin{aligned} & x < y \text{ and } x > / y \\ & = R_1 \end{aligned}$$

$$R_1 - R_2 = R_1$$

$$\text{and } R_2 - R_1 = R_2$$

$$R_1 \oplus R_2 = (R_1 \cup R_2) - (R_1 \cap R_2).$$

$$\begin{aligned} & = x < y \text{ or } x > y - \Phi \\ & = x \neq y. \end{aligned}$$

The Boolean operations Join and meet of matrices representing the union and intersection of those relations are given by  $M_{R_1 \cup R_2} = M_{R_1} \vee M_{R_2}$

$$M_{R_1 \cap R_2} = M_{R_1} \wedge M_{R_2} .$$

EX: Suppose that the relations  $R_1$  and  $R_2$  on a set  $A$  are represented by the matrices

$$M_{R_1} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad M_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

Find the matrices representing  $R_1 \cup R_2$  and  $R_1 \cap R_2$ .

$$\text{Sol: } M_{R_1 \cup R_2} = M_{R_1} \vee M_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$M_{R_1 \cap R_2} = M_{R_1} \wedge M_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$1 \cup 1 = 1, 1 \cup 0 = 1, 1 \wedge 1 = 1, 1 \wedge 0 = 0, 0 \cup 0 = 0, 0 \wedge 0 = 0$

## Composition of Binary Relations

Let  $R$  be a relation from  $X$  to  $Y$  and

$S$  be a relation from  $Y$  to  $Z$ . Then a relation written as  $R \circ S$  is called a *composite relation* of  $R$  and  $S$  where

$$R \circ S = \{(x, z) \mid x \in X, z \in Z, \text{ and there exists } y \in Y \text{ with } (x, y) \in R \text{ and } (y, z) \in S \}.$$

Theorem: If  $R$  is relation from  $A$  to  $B$ ,  $S$  is a relation from  $B$  to  $C$  and  $T$  is a relation from  $C$  to  $D$  then  $T \circ (S \circ R) = (T \circ S) \circ R$

Example: Let  $R = \{(1, 2), (3, 4), (2, 2)\}$  and  $S = \{(4, 2), (2, 5), (3, 1), (1, 3)\}$ . Find  $R \circ S$ ,  $S \circ R$ ,  $R \circ (S \circ R)$ ,  $(R \circ S) \circ R$ ,  $R \circ R$ ,  $S \circ S$ , and  $(R \circ R) \circ R$ .

Solution: Given  $R = \{(1, 2), (3, 4), (2, 2)\}$  and  $S = \{(4, 2), (2, 5), (3, 1), (1, 3)\}$ .

$$R \circ S = \{(1, 5), (3, 2), (2, 5)\}$$

$$S \circ R = \{(4, 2), (3, 2), (1, 4)\} \neq R \circ S$$

$$(R \circ S) \circ R = \{(3, 2)\}$$

$$R \circ (S \circ R) = \{(3, 2)\} = (R \circ S) \circ R$$

$$R \circ R = \{(1, 2), (2, 2)\}$$

$$R \circ R \circ S = \{(4, 5), (3, 3), (1, 1)\}$$

Example: Let  $A = \{a, b, c\}$ , and  $R$  and  $S$  be relations on  $A$  whose matrices are as given below:

$$M_R = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \quad \text{and} \quad M_S = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

Find the composite relations  $R \circ S$ ,  $S \circ R$ ,  $R \circ R$ ,  $S \circ S$  and their matrices.

Solution:  $R = \{(a, a), (a, c), (b, a), (b, b), (b, c), (c, b)\}$

$S = \{(a, a), (b, b), (b, c), (c, a), (c, c)\}$ . From these, we find that

$R \circ S = \{(a, a), (a, c), (b, a), (b, b), (b, c), (c, b), (c, c)\}$

$S \circ R = \{(a, a), (a, c), (b, b), (b, a), (b, c), (c, a), (c, b), (c, c)\}$

$R \circ R = R^2 = \{(a, a), (a, c), (a, b), (b, a), (b, c), (b, b), (c, a), (c, b),$

$(c, c)\}$   $S \circ S = S^2 = \{(a, a), (b, b), (b, c), (b, a), (c, a), (c, c)\}$ .

The matrices of the above composite relations are as given below:  $M_{RO S} =$

$$; M_{SOR} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}; M_{RO R} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}; M_{SO S} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

## Transitive Closure

Let  $X$  be any finite set and  $R$  be a relation in  $X$ . The relation  $R^+ = R \cup R^2 \cup R^3 \cup \dots \cup R^n$  in  $X$  is called the *transitive closure* of  $R$  in  $X$ .

Example: Let the relation  $R = \{(1, 2), (2, 3), (3, 3)\}$  on the set  $\{1, 2, 3\}$ . What is the transitive closure of  $R$ ?

Solution: Given that  $R = \{(1, 2), (2, 3), (3, 3)\}$ .

The transitive closure of  $R$  is  $R^+ = R \cup R^2 \cup R^3 \cup \dots =$

$R = \{(1, 2), (2, 3), (3, 3)\}$

$R^2 = R \circ R = \{(1, 2), (2, 3), (3, 3)\} \circ \{(1, 2), (2, 3), (3, 3)\} =$   
 $\{(1, 3), (2, 3), (3, 3)\}$

$R^3 = R^2 \circ R = \{(1, 3), (2, 3), (3, 3)\}$

$R^4 = R^3 \circ R = \{(1, 3), (2, 3), (3, 3)\}$

$R^+ = R \cup R^2 \cup R^3 \cup R^4 \cup \dots$

$= \{(1, 2), (2, 3), (3, 3)\} \cup \{(1, 3), (2, 3), (3, 3)\} \cup \{(1, 3), (2, 3), (3, 3)\} \cup \dots$

$= \{(1, 2), (1, 3), (2, 3), (3, 3)\}$ .

Therefore  $R^+ = \{(1, 2), (1, 3), (2, 3), (3, 3)\}$ .

Example: Let  $X = \{1, 2, 3, 4\}$  and  $R = \{(1, 2), (2, 3), (3, 4)\}$  be a relation on  $X$ .

Find  $R^+$ .

Solution: Given  $R = \{(1, 2), (2, 3), (3, 4)\}$

$R^2 = R \circ R = \{(1, 2), (2, 3), (3, 4)\} \circ \{(1, 2), (2, 3), (3, 4)\}$

$R^2 = \{(1, 3), (2, 4)\}$

$R^3 = R^2 \circ R = \{(1, 3), (2, 4)\} \circ \{(1, 2), (2, 3), (3, 4)\}$

$= \{(1, 4)\}$

$R^4 = R^3 \circ R = \{(1, 4)\} \circ \{(1, 2), (2, 3), (3, 4)\}$

$\{(1, 4)\}$

.

$R^n = \{(1, 4)\}$

And  $R^+ = \{(1, 2), (2, 3), (3, 4), (1, 3), (2, 4), (1, 4)\}$ .

### ***Equivalence Relations***

A relation  $R$  in a set  $X$  is called an *equivalence relation* if it is **reflexive, symmetric and transitive.**

Reflexive: for every  $x \in R$ ,  $(x, x) \in R$

Symmetry: if  $(x, y) \in R$  then  $(y, x) \in R$

Transitive : if  $(x, y) \in R$ , and  $(y, z) \in R$  then  $(x, z) \in R$ .

The following are some examples of equivalence relations:

1. Equality of numbers on a set of real numbers.
2. Equality of subsets of a universal set.

**Example:** Let  $X = \{1, 2, 3, 4\}$  and

$R = \{(1, 1), (1, 4), (4, 1), (4, 4), (2, 2), (2, 3), (3, 2), (3, 3)\}$ .

Prove that  $R$  is an equivalence relation.

Reflexive :  $\{(1, 1), (4, 4), (2, 2), (3, 3)\}$ .

Symmetric :  $\{(1, 4), (4, 1), (2, 3), (3, 2)\}$ .

Transitive :  $\{(1, 1), (1, 4), (4, 1), (4, 4), (2, 2), (2, 3), (3, 2), (3, 3)\}$ .

$(1, 1), (1, 4)$  transitive is  $(1, 4)$

$(1, 4), (4, 1)$  transitive is  $(1, 1)$

$(4, 4), (4, 1)$  transitive is  $(4, 1)$

$(2, 2), (2, 3)$  transitive is  $(2, 3)$

$(2, 3), (3, 2)$ , transitive is  $(2, 2)$

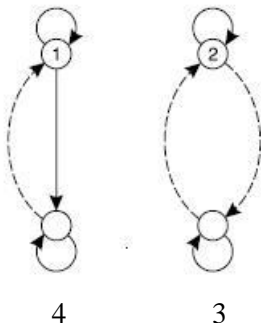
$(3, 2), (2, 3)$ , transitive is  $(3, 3)$

$(3, 3), (3, 2)$ , transitive is  $(3, 2)$

Therefore  $R$  is transitive.

Clearly, the relation  $R$  is reflexive, symmetric and transitive. Hence,  $R$  is an equivalence relation.

$$M_R = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}, M^T = M,$$



The corresponding graph of  $R$  is shown in figure:

Example: Let  $X = \{1, 2, 3, 4, 5, 6, 7\}$  and  $R = \{(x, y) \mid (x - y) \text{ is divisible by } 3\}$ . Show that  $R$  is an equivalence relation.

Solution: (i). For any  $x \in X$ ,  $x - x = 0$  is divisible by 3.

$$\therefore xRx \text{ i.e., } (x, x) \in R$$

$\Rightarrow R$  is reflexive.

(ii). For any  $x, y \in X$ , if  $xRy$  ( $(x, y) \in R$ ), then  $(x - y)$  is divisible by 3.

$$\Rightarrow -(x - y) \text{ is divisible by } 3.$$

$$\Rightarrow y - x \text{ is divisible by } 3.$$

$$\Rightarrow yRx$$

$$\text{i.e., } (y, x) \in R$$

Thus, the relation  $R$  is symmetric.

(iii). For any  $x, y, z \in X$ , let  $xRy$  ( $(x, y) \in R$ ) and  $yRz$  ( $(y, z) \in R$ )

$$\Rightarrow (x - y) + (y - z) \text{ is divisible by } 3$$

$$\Rightarrow x - z \text{ is divisible by } 3$$

$$\Rightarrow xRz$$

$$(x, z) \in R$$

Hence, the relation  $R$  is transitive.

Thus, the relation  $R$  is an equivalence relation.

**Congruence Relation:** Let  $I(\mathbb{Z}^+)$  denote the set of all positive integers, and let  $m$  be a positive integer. For  $x \in I$  and  $y \in I$ , define  $R$  as  $R = \{(x, y) \mid x - y \text{ is divisible by } m\}$

The statement “ $x - y$  is divisible by  $m$ ” is equivalent to the statement that both  $x$  and  $y$  have the same remainder when each is divided by  $m$ .

In this case, denote  $R$  by  $\equiv$  and to write  $xRy$  as

$x \equiv y \pmod{m}$ , which is read as “ $x$  is congruent to  $y$  modulo  $m$ ”.

The relation  $\equiv$  is called a *congruence relation*.

Example:  $83 \equiv 13 \pmod{5}$ , since  $83-13=70$  is divisible by 5.

$a \equiv b \pmod{m}$  that mean  $a-b$  is divisible by  $m$ .

Example: Prove that the relation “congruence modulo  $m$ ” over the set of positive integers is an equivalence relation.

Or

**Congruence Modulo  $m$ .** Let  $m$  be a positive integer with  $m > 1$ .

Show that the relation  $R = \{(a,b)/a \equiv b \pmod{m}\}$  is an equivalence relation on the set of integers.

Solution: Let  $\mathbb{Z}^+$  be the set of all positive integers and  $m$  be a positive integer.

**We define the relation “congruence modulo  $m$ ” on  $\mathbb{Z}^+$  as follows:**

**Let  $a, b \in \mathbb{Z}^+$ .  $a \equiv b \pmod{m}$  if and only if  $a - b$  is divisible by  $m$ . ( $m|a-b$ ,  $m$  divides  $a-b$ )**

Let  $a, b, c \in \mathbb{Z}^+$ .

(i). Note that  $a-a=0$  is divisible by  $m$

$$a - a = 0 = 0 \cdot m \Rightarrow a \equiv a \pmod{m} \text{ for all } a \in \mathbb{Z}^+$$

, so congruence modulo  $m$  is reflexive.

(ii). Now suppose that  $a \equiv b \pmod{m}$ . Then,  $a - b$  is divisible by  $m$ .

So  $a-b = k \cdot m$ , where  $k$  is an integer.

Now it follows that  $b-a = (-k)m$ , so  $b \equiv a \pmod{m}$

$\Rightarrow -(a - b) = b - a$  is divisible by  $m$ .

i.e.,  $b \equiv a \pmod{m}$

$\therefore$  the relation  $\equiv$  is symmetric.

Hence, congruence modulo  $m$  is symmetric.

(iii) Next suppose that  $a \equiv b \pmod{m}$ , and  $b \equiv c \pmod{m}$ .

Then  $a-b$  and  $b-c$  are divisible by  $m$ .

$m$  divides both  $a-b$  and  $b-c$ .

$\Rightarrow a - b$  and  $b - c$  are divisible by  $m$ .

Therefore there exists  $k$  and  $l$  such that  $a-b = km$ , and  $b-c = lm$ .

Adding these two

Now  $a-c = (a - b) + (b - c) = km + lm = (k+l)m$ .

$m$  divides  $a-c$

i.e.,  $a - c$  is divisible by  $m$ .

$\Rightarrow a \equiv c \pmod{m}$

$\therefore$  the relation  $\equiv$  is transitive.

Since the relation  $\equiv$  is reflexive, symmetric and transitive,

the relation *congruence modulo  $m$*  is an **equivalence relation**..

Ex: Show that the “divides” relation on the set of positive integers in **not an equivalence relation**.

Sol: We know that the relation “divides” is **reflexive and Transitive but not symmetric**.  
( for  $a \neq b$ , if  $a$  divides  $b$ , then  $b$  does not divides  $a$ , therefore divides is not symmetric).

And hence the relation “divides” is not an equivalence relation.

$2|6$ , but not  $6|2$

Ex: Let  $R$  be the relation on the set of real numbers such that  $x R y$  if and only if  $x$  and  $y$  are real numbers that differs by less than 1, i.e.,  $|x-y| < 1$ . Show that  $R$  is not an equivalence relation.

$R = \{ (x,y) / |x-y| < 1 \}$  is not an equivalence relation.

Sol :  $R$  is reflexive and symmetric but not transitive.

For any  $x$ ,  $|x-x| = 0 < 1$  therefore  $(x,x) \in R$ .

**$R$  is reflexive.**

For  $x, y \in R$ ,  $(x,y) \in R$  implies  $|x-y| < 1$

$$|y-x| < 1$$

Therefore  $(y,x) \in R$ .

And hence  **$R$  is symmetric.**

But  $R$  is not transitive:

For ex Take  $x = 2.7$ ,  $y = 1.8$  and  $z = 1.1$

$$|x-y| = |2.7-1.8| = 0.9 < 1$$

$$|y-z| = |1.8-1.1| = 0.7 < 1$$

$$|x-z| = |2.7-1.1| = 1.6 > 1.$$

**It is not transitive.**

$R$  is not an equivalence relation.

Example: Let  $R$  denote a relation on the set of ordered pairs of positive integers such that  $(x,y) R(u, v)$  iff  $xv = yu$ . Show that  $R$  is an equivalence relation.

Solution: Let  $R$  denote a relation on the set of ordered pairs of positive integers.

Let  $x, y, u$  and  $v$  be positive integers. Given  $(x, y) R(u, v)$  if and only if  $xv = yu$ .

(i). Since  $xy = yx$  is true for all positive integers

$$\Rightarrow (x, y) R(x, y), \text{ for all ordered pairs } (x, y) \text{ of positive integers.}$$

$\therefore$  The relation  $R$  is reflexive.

(ii). Let  $(x, y) R(u, v)$

$$\Rightarrow xv = yu$$

$$\Rightarrow yu = xv$$

$$\Rightarrow uy = vx$$

$$\Rightarrow (u, v) R(x, y)$$

$\therefore$  The relation  $R$  is symmetric.

(iii). Let  $x, y, u, v, m$  and  $n$  be positive integers

Let  $(x, y) R(u, v)$  and  $(u, v) R(m, n)$

$$\Rightarrow xv = yu \text{ and } un = vm$$

$$\Rightarrow xvun = yuvm$$

$$\Rightarrow xn = ym, \text{ by canceling } uv$$

$$\Rightarrow (x, y) R(m, n)$$

∴ The relation  $R$  is transitive.

Since  $R$  is reflexive, symmetric and transitive, hence the relation  $R$  is an equivalence relation.

### **Equivalence Classes:**

Def: Let  $R$  be an equivalence relation on a set  $A$ . The set **of all elements that are related to an element  $a$  of  $A$**  is called an equivalence class of  $a$ .

The equivalence class of  $a$  with respect to  $R$  is denoted by  $[a]_R$ .

$$[a]_R = \{ s \mid (a, s) \in R \}$$

$$[D]_R = \{$$

If  $b \in [a]_R$  then  $b$  is called a representative of this equivalence class.

Ex: What is the equivalence classes of 0 and 1 for congruence modulo 4?

Sol: The equivalence class of 0 contains all integers  $a$  such that  $a \equiv 0 \pmod{4}$ .

The integers in this class are those divisible by 4.

Hence, the equivalence class of 0 for this relation is

$$[0] = \{ \dots, -8, -4, 0, 4, 8, \dots \}.$$

The equivalence class of 1 contains all integers  $a$  such that  $a \equiv 1 \pmod{4}$ .

The integers in this class are those that have a remainder 1 when divided by 4.

Hence, the equivalence class of 1 for this relation is

$$[1]_4 = \{ \dots, -7, -3, 1, 5, 9, \dots \}.$$

$$-7-1=-8, -3-1=-4, 1-1=0, 5-1=4 \equiv 1 \pmod{4}.$$

$$9 \equiv 1 \pmod{4}.$$

$$13 \equiv 1 \pmod{4}.$$

### **Partition and Covering of a Set**

Let  $S$  be a given set and  $A = \{A_1, A_2, \dots, A_m\}$  where each  $A_i, i = 1, 2, \dots, m$  is a subset of  $S$  and

$$\bigcup_{i=1}^m A_i = S.$$

Then the set  $A$  is called a *covering* of  $S$ , and the sets  $A_1, A_2, \dots, A_m$  are said to *cover*  $S$ . If, in addition, the elements of  $A$ , which are subsets of  $S$ , are mutually disjoint, then  $A$  is called a *partition* of  $S$ , and the sets  $A_1, A_2, \dots, A_m$  are called the *blocks* of the partition.

Example: Let  $S = \{a, b, c\}$  and consider the following collections of subsets of  $S$ .  $A = \{\{a, b\}, \{b, c\}\}$ ,  $B = \{\{a\}, \{a, c\}\}$ ,  $C = \{\{a\}, \{b, c\}\}$ ,  $D = \{\{a, b, c\}\}$ ,  $E = \{\{a\}, \{b\}, \{c\}\}$ , and  $F = \{\{a\}, \{a, b\}, \{a, c\}\}$ . Which of the above sets are covering?

Solution: The sets  $A, C, D, E, F$  are covering of  $S$ . But, the set  $B$  is not covering of  $S$ , since their union is not  $S$ .

Example: Let  $S = \{a, b, c\}$  and consider the following collections of subsets of  $S$ .  $A = \{\{a, b\}, \{b, c\}\}$ ,  $B = \{\{a\}, \{b, c\}\}$ ,  $C = \{\{a, b, c\}\}$ ,  $D = \{\{a\}, \{b\}, \{c\}\}$ , and  $E = \{\{a\}, \{a, c\}\}$ .

Which of the above sets are covering?

Solution: The sets  $B$ ,  $C$  and  $D$  are partitions of  $S$  and also they are covering. Hence, every partition is a covering.

The set  $A$  is a covering, but it is not a partition of a set, since the sets  $\{a, b\}$  and  $\{b, c\}$  are not disjoint. Hence, every covering need not be a partition.

The set  $E$  is not partition, since the union of the subsets is not  $S$ . The partition  $C$  has one block and the partition  $D$  has three blocks.

Example: List of all ordered partitions  $S = \{a, b, c, d\}$  of type  $(1, 2, 2)$ .

Solution:

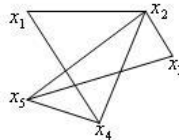
$(\{a\}, \{b\}, \{c, d\})$ ,	$(\{b\}, \{a\}, \{c, d\})$
$(\{a\}, \{c\}, \{b, d\})$ ,	$(\{c\}, \{a\}, \{b, d\})$
$(\{a\}, \{d\}, \{b, c\})$ ,	$(\{d\}, \{a\}, \{b, c\})$
$(\{b\}, \{c\}, \{a, d\})$ ,	$(\{c\}, \{b\}, \{a, d\})$
$(\{b\}, \{d\}, \{a, c\})$ ,	$(\{d\}, \{b\}, \{a, c\})$
$(\{c\}, \{d\}, \{a, b\})$ ,	$(\{d\}, \{c\}, \{a, b\})$ .

Example: Let  $X = \{\text{ball, bed, dog, let, egg}\}$ , and let the relation  $R$  be given by  $R = \{(x, y) \mid x, y \in X \wedge xRy \text{ if } x \text{ and } y \text{ contain some common letter}\}$ .

Then  $R$  is a compatibility relation, and  $x, y$  are called compatible if  $xRy$ .

Note:  $\text{ball} \approx \text{bed}$ ,  $\text{bed} \approx \text{egg}$ . But  $\text{ball} \not\approx \text{egg}$ . Thus  $\approx$  is not transitive.

Denoting "ball" by  $x_1$ , "bed" by  $x_2$ , "dog" by  $x_3$ , "let" by  $x_4$ , and "egg" by  $x_5$ , the graph of  $\approx$  is given as follows:



## Partial Ordering

Let  $S$  be a non empty set a binary relation  $R$  on a set  $S$  is called a **partial order relation** or a **partial ordering** in  $S$  if and only if  $R$  is reflexive, antisymmetric, and transitive.

i.e., POSet: **R, AS, T**

- **Reflexive:**  $aRa$  for all  $a \in S$  i.e.,  $(a, a) \in R$  for all  $a \in S$
- **Antisymmetric:**  $aRb$  and  $bRa \Rightarrow a = b$  i.e., If  $(a, b) \in R$  then  $(b, a) \in R$  then  $a=b$ .
- **Transitive:**  $aRb$  and  $bRc \Rightarrow aRc$  i.e., If  $(a, b) \in R$  and  $(b, c) \in R$  then  $(a, c) \in R$

A set  $P$  together with a partial ordering  $R$  is called a **partial ordered set or POset**.

The relation  $R$  is often denoted by the symbol  $\leq$  ( $\preceq$ ) which is different from the usual less than equal to symbol.

Thus, if  $\leq$  ( $\preceq$ ) is a partial order in  $P$ , then the ordered pair  $(P, \leq)$  or  $(P, \preceq)$  is called a POSET.

Example: Show that the relation “greater than or equal to” is a partial ordering on the set of integers.

Solution: Let  $Z$  be the set of all integers and the relation  $R = \geq$

(i). Since  $a \geq a$  for every integer  $a$ , the relation  $\geq$  is **reflexive**.

Ex :  $2 \geq 2, 6 \geq 6, 9 \geq 9$ .

(ii). Let  $a$  and  $b$  be any two integers.

Let  $a R b$  and  $b R a \Rightarrow a \geq b$  and  $b \geq a \Rightarrow a = b$   
 $6 \geq 2$  but not  $2 \geq 6$ .

$\therefore$  the relation  $\geq$  is **antisymmetric**.

i.e., Antisymmetric is possible only in the equal numbers,

(iii) Let  $a, b$  and  $c$  be any three integers.

Let  $a R b$  and  $b R c \Rightarrow a \geq b$  and  $b \geq c \Rightarrow a \geq c$

i.e., if  $(a,b) \in R, (b,c) \in R$  then  $(a,c) \in R$

$\therefore$  The relation  $\geq$  is **transitive**.

Ex.  $6 \geq 5, 5 \geq 4$  which implies  $6 \geq 4$ .

Since the relation  $\geq$  is **reflexive, antisymmetric and transitive,**

$\geq$  is partial ordering on the set of integers. Therefore,  $(Z, \geq)$  is a poset.

$(Z, \leq)$  POset?

Example: Show that the inclusion  $\subseteq$  is a partial ordering on the set power set of a set  $S$ .

Solution: Since (i).  $A \subseteq A$  for all  $A \subseteq S$ ,  $\subseteq$  is reflexive.

(ii).  $A \subseteq B$  and  $B \subseteq A \Rightarrow A = B$ ,  $\subseteq$  is antisymmetric.

(iii).  $A \subseteq B$  and  $B \subseteq C \Rightarrow A \subseteq C$ ,  $\subseteq$  is transitive.

Thus, the relation  $\subseteq$  is a partial ordering on the power set of  $S$ .

Example: Show that the divisibility relation  $\mid$  (divides) is a partial ordering on the set of positive integers.

Solution: Let  $Z^+$  be the set of positive integers.

$a \mid b$  (read as  $a$  divides  $b$ ) is  $b = ka$  where  $k$  belongs to  $Z^+$  ( set of positive integers)

$a$  divides  $b$  :  $b$  is divisible by  $a$

Since (i).  $a \mid a$  for all  $a \in Z^+$ ,  $\mid$  is reflexive.

i.e.,  $a$  divides  $a$  for all  $a$  in Positive integers.

EX:  $2 \mid 2, 6 \mid 6, 9 \mid 9, 29 \mid 29$ .... (every number divides itself) and  $\mid$  is reflexive.

(ii).  $a \mid b$  and  $b \mid a \Rightarrow a = b$ ,  $\mid$  is antisymmetric.

$2 \mid 6, (6 = 3 \cdot 2)$

$6 \nmid 2 (2 = K \cdot 6 \text{ } k = 1/3)$

i.e.,  $a$  divides  $b$  and  $b$  divides  $a$  then  $a = b$

$a$  divides  $b$  and  $b$  divides  $a$  possible in the equal case.

(iii).  $a/b$  and  $b/c \Rightarrow a/c$ ,  $/$  is transitive.

If  $a$  divides  $b$  and  $b$  divides  $c$  then we have  $a$  divides  $c$  and hence divides is transitive.

EX:  $2/4$  ( $4=2.2$ ), and  $4/12$  ( $12=3.4$ ), then we have  $2/12$  ( $12=6.2$ )

It follows that  $/$  is a partial ordering on  $Z^+$  and  $(Z^+, /)$  is a **poset**.

(Divides is **POSet** on Positive integers).

Note: On the set of all integers, the above relation is not a partial order as  $a$  and  $-a$  both divide each other, but  $a = -a$ . i.e., the relation is not antisymmetric.

### Definition:

The elements **a** and **b** of a Poset  $(S, \preceq)$  are called **comparable** if either  $a \preceq b$  or  $b \preceq a$ .

When  $a$  and  $b$  are elements of  $S$  such that neither  $a \preceq b$  nor  $b \preceq a$ ,  $a$  and  $b$  are called **incomparable**.

Example : In the Poset  $(Z^+, /)$  are the integers 3 and 9 are comparable?

Ans: Yes ,

The integers 3 and 9 are comparable, since  $3/9$  (3 divides 9)

i.e.,  $9=3.3$ .

Example : In the Poset  $(Z^+, /)$  are the integers 5 and 7 are comparable?

Ans: No , The integers 5 and 7 are incomparable, since  $5 \nmid 7$  (5 does not divides 7)

**Definition:** Let  $(P, \leq)$  be a partially ordered set . If for every  $x, y \in P$  we have either  $x \leq y$  or  $y \leq x$ , then  $\leq$  is called a *simple ordering* or *linear ordering* on  $P$  , and  $(P, \leq)$  is called a **totally ordered or simply ordered set or a chain**.

Note: It is not necessary to have  $x \leq y$  or  $y \leq x$  for every  $x$  and  $y$  in a poset  $P$  .

In fact,  $x$  may not be related to  $y$ , in which case we say that  $x$  and  $y$  are incomparable.

( $x$  is related to  $y$  then we say that  $x$  and  $y$  are comparable)

Examples:

(i). The poset  $(Z, \leq)$  is a totally ordered.

Since  $a \leq b$  or  $b \leq a$  whenever  $a$  and  $b$  are integers.

(ii). The divisibility relation  $/$  is a partial ordering on the set of positive integers.

Therefore  $(Z^+, /)$  is a poset and it is not a totally ordered, since it contain elements that are incomparable, such as 5 and 7, 3 and 5 (5 does not divides 7, and 3 does not divides 5) so

**Definition:** In a poset  $(P, \leq)$ , an element  $y \in P$  is said to *cover* an element  $x \in P$  if  $x < y$  and if there does not exist any element  $z \in P$  such that  $x \leq z$  and  $z \leq y$ ; that is,  $y$  covers  $x \Leftrightarrow (x < y \wedge (x \leq z \leq y \Rightarrow x = z \vee z = y))$ .

**Definition :** If  $(S, \preceq)$  is a Poset and every two elements of  $S$  are **comparable**,  $S$  is called a **totally ordered set or Linearly ordered set**, and  $\preceq$  is called total order or a linear order . A totally ordered set is also called a chain.

**Definition: Let**  $(S, \leq)$  be a partially ordered set. If for every  $x, y \in P$  we have either  $x \leq y$  or  $y \leq x$ , then  $\leq$  is called a **simple ordering or linear ordering** on  $P$  , and  $(P, \leq)$  is called a **totally ordered or simply ordered set or a chain**.

Note: It is not necessary to have  $x \leq y$  or  $y \leq x$  for every  $x$  and  $y$  in a poset  $P$ , In fact,  $x$  may not be related to  $y$ , in which case we say that  $x$  and  $y$  are incomparable. Examples:

(i). The poset  $(\mathbb{Z}, \leq)$  is a totally ordered.

Since  $a \leq b$  or  $b \leq a$  whenever  $a$  and  $b$  are integers.

(ii). The divisibility relation  $/$  is a partial ordering on the set of positive integers.

Therefore  $(\mathbb{Z}^+, /)$  is a poset and it is not a totally ordered, since it contain elements that are incomparable, such as 5 and 7, 3 and 5.

**Definition:**  $(S, \preceq)$  is a Well ordered set if it is a Poset such that  $\preceq$  is a totally ordering and every non empty subset of  $S$  has a least element.

EX: The set of ordered pairs of positive integers ,  $\mathbb{Z}^+ \times \mathbb{Z}^+$  , with  $(a_1, a_2) \preceq (b_1, b_2)$  if  $1 < b_1$  or if  $a_1 = b_1$  and  $a_2 \leq b_2$  is a well –ordered set

Lexicographic ordering is well ordered set.