

Navier-Stokes Equation



- Newtonian Fluid
 - Constant Density, Viscosity
 - Cartesian, Cylindrical, spherical coordinates

Cartesian Coordinates

$$\rho \frac{DV}{Dt} = -\nabla P + \mu \nabla^2 V + \rho g$$

$$\rho \left(\frac{\partial V_x}{\partial t} + V_x \frac{\partial V_x}{\partial x} + V_y \frac{\partial V_x}{\partial y} + V_z \frac{\partial V_x}{\partial z} \right) = -\frac{\partial P}{\partial x} + \mu \left[\frac{\partial^2 V_x}{\partial x^2} + \frac{\partial^2 V_x}{\partial y^2} + \frac{\partial^2 V_x}{\partial z^2} \right] + \rho g_x$$

$$\rho \left(\frac{\partial V_y}{\partial t} + V_x \frac{\partial V_y}{\partial x} + V_y \frac{\partial V_y}{\partial y} + V_z \frac{\partial V_y}{\partial z} \right) = -\frac{\partial P}{\partial y} + \mu \left[\frac{\partial^2 V_y}{\partial x^2} + \frac{\partial^2 V_y}{\partial y^2} + \frac{\partial^2 V_y}{\partial z^2} \right] + \rho g_y$$

$$\rho \left(\frac{\partial V_z}{\partial t} + V_x \frac{\partial V_z}{\partial x} + V_y \frac{\partial V_z}{\partial y} + V_z \frac{\partial V_z}{\partial z} \right) = -\frac{\partial P}{\partial z} + \mu \left[\frac{\partial^2 V_z}{\partial x^2} + \frac{\partial^2 V_z}{\partial y^2} + \frac{\partial^2 V_z}{\partial z^2} \right] + \rho g_z$$

Continuity

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho V) = 0$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} (\rho V_x) + \frac{\partial}{\partial y} (\rho V_y) + \frac{\partial}{\partial z} (\rho V_z) = 0$$

$$\frac{\partial \rho}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r} (\rho r V_r) + \frac{1}{r} \frac{\partial}{\partial \theta} (\rho V_\theta) + \frac{\partial}{\partial z} (\rho V_z) = 0$$

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2} \frac{\partial}{\partial r} (\rho r^2 V_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\rho V_\theta \sin \theta) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} (\rho V_\phi) = 0$$

Newton's law of viscosity

$$\tau = -\mu(\nabla V + (\nabla V)^+) + \left(\frac{2}{3}\mu - \kappa\right)(\nabla \cdot V)\delta$$

Constant Density, zero dilatational viscosity

$$\tau_{yx} = \tau_{xy} = -\mu \left[\frac{\partial V_y}{\partial x} + \frac{\partial V_x}{\partial y} \right]$$

Cylindrical coordinates

$$\tau_{r\theta} = \tau_{\theta r} = -\mu \left[r \frac{\partial}{\partial r} \left(\frac{V_\theta}{r} \right) + \frac{1}{r} \frac{\partial V_r}{\partial \theta} \right]$$

$$\tau_{z\theta} = \tau_{\theta z} = -\mu \left[r \frac{\partial V_z}{\partial \theta} + \frac{\partial V_\theta}{\partial z} \right]$$

$$\tau_{rz} = \tau_{zr} = -\mu \left[\frac{\partial V_r}{\partial z} + \frac{\partial V_z}{\partial r} \right]$$

N-S Equation: Examples



- ODE vs PDE
- Spherical and cylindrical coordinates
- Eqn for pipe flow (Hagen Poiseulle)
- Flow between rotating cylinders (not solved in class)
- Thin film flow with temp variation (not solved in class, steps were discussed briefly. BSL 'worked out' example)
- Radial flow between circular plates (BSL 3B.10)

Example problems



1. Pressure driven steady state flow of fluid
 between two infinite parallel plates
 inside a circular tube
2. Steady state Couvette flow of a fluid
 between two infinite parallel plates with top plate moving
 at a known velocity
 between two circular plates of finite radius, with the top
 plate rotating at a known angular velocity
 between two circular cylinders with outer cylinder
 rotating at a known angular velocity (end effects are negligible)
 between a cone and plate (stationary plate and cone is
 rotating at a known angular velocity). Angle of cone is very small
 (almost a parallel plate with almost zero gap)
3. Couette Poiseuille flow
 between two parallel plates

N-S Equation: Other examples

- To determine the velocity profile in a rectangular channel, where the top plate is moving at a constant velocity of V_0 , under steady state conditions
 - Try out a solution of the form “ V_0 -parallel-plate + V -correction”
 - Use separation of variable techniques, to determine V -correction
 - What happens if you try separation of variable in the first place?
- To determine the unsteady state solution for a flow in a cylindrical pipe, caused by sudden application of pressure
 - Try a solution of the form ‘Steady state + Transient’, just like the one we saw for flow between parallel plates
 - You will get Bessel Equations. Just like we represented functions in rectangular coordinates by sine and cosine functions, we can represent functions in cylindrical coordinates by Bessel functions, because they are orthogonal.