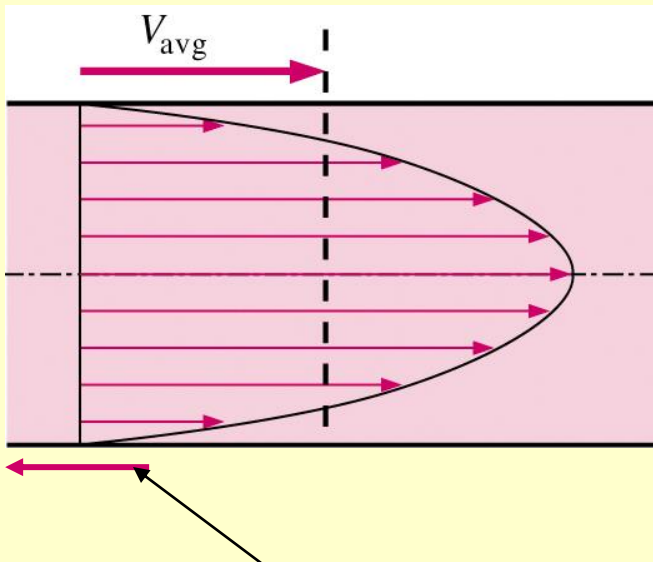


Flow of Fluids in Pipes

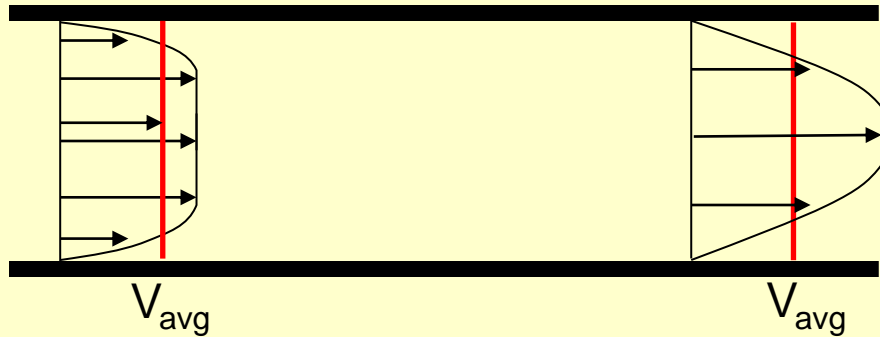
Introduction

- Average velocity in a pipe
 - Recall - because of the no-slip condition, the velocity at the walls of a pipe or duct flow is zero
 - We are often interested only in V_{avg} , which we usually call just V (drop the subscript for convenience)
 - Keep in mind that the no-slip condition causes shear stress and friction along the pipe walls



Friction force of wall on fluid

Introduction



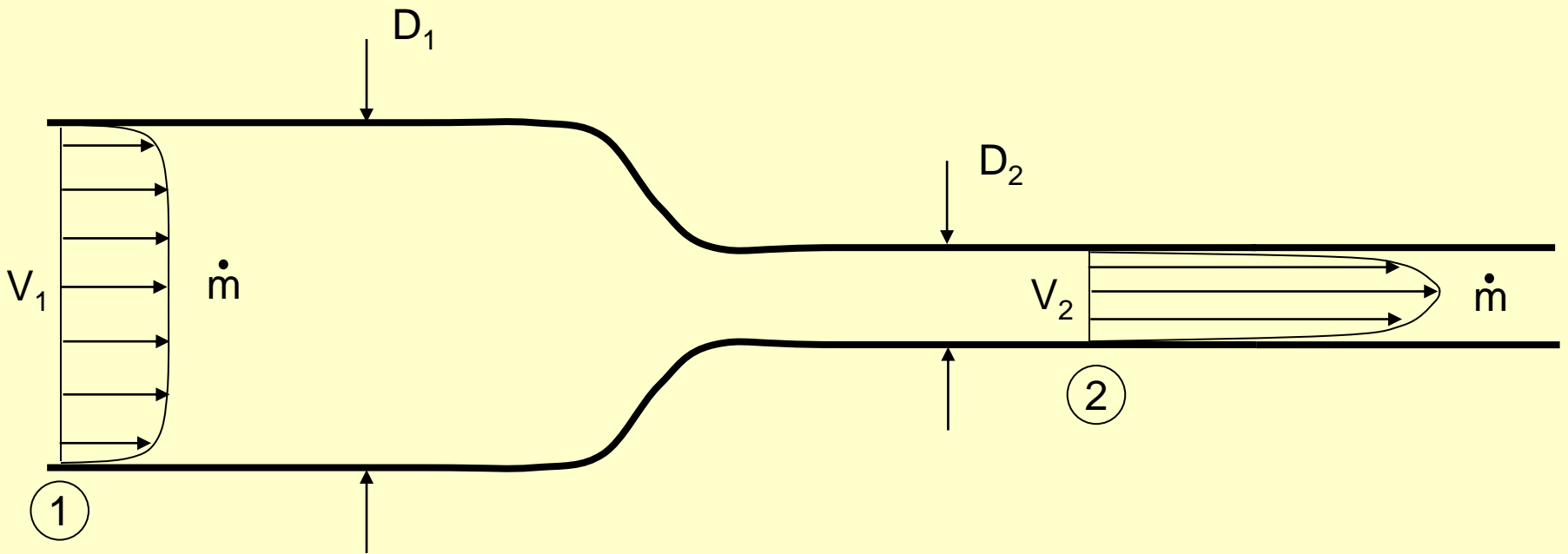
- For pipes of constant diameter and incompressible flow
 - V_{avg} stays the same down the pipe, even if the velocity profile changes
 - Why? Conservation of Mass

$$\dot{m} = \rho V_{avg} A = \text{constant}$$

same same same

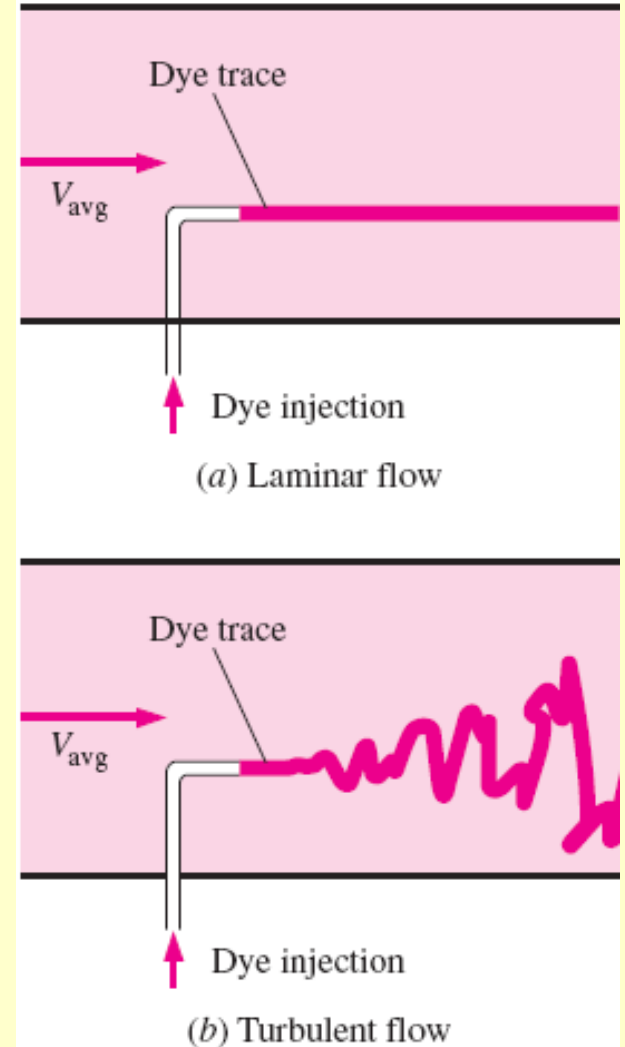
Introduction

- For pipes with variable diameter, \dot{m} is still the same due to conservation of mass, but $V_1 \neq V_2$



LAMINAR AND TURBULENT FLOWS

- **Laminar flow:** characterized by *smooth streamlines* and *highly ordered motion*.
- **Turbulent flow:** characterized by *velocity fluctuations* and *highly disordered motion*.
- The **transition** from laminar to turbulent flow does not occur suddenly; rather, it occurs over some region in which the flow fluctuates between laminar and turbulent flows before it becomes fully turbulent.



Reynolds Number

- The transition from laminar to turbulent flow depends on the *geometry, surface roughness, flow velocity, surface temperature, and type of fluid*, among other things.
- British engineer Osborne Reynolds (1842–1912) discovered that the flow regime depends mainly on the ratio of *inertial forces* to *viscous forces* in the fluid.
- The ratio is called the **Reynolds number** and is expressed for internal flow in a circular pipe as

$$\text{Re} = \frac{\text{Inertial forces}}{\text{Viscous forces}} = \frac{V_{\text{avg}}D}{\nu} = \frac{\rho V_{\text{avg}}D}{\mu}$$

Reynolds Number

- At large Reynolds numbers, the inertial forces are large relative to the viscous forces \Rightarrow Turbulent Flow
- At *small* or *moderate* Reynolds numbers, the viscous forces are large enough to suppress these fluctuations \Rightarrow Laminar Flow
- The Reynolds number at which the flow becomes turbulent is called the **critical Reynolds number**, Re_{cr} .
- The value of the critical Reynolds number is different for different geometries and flow conditions. For example, $Re_{cr} = 2300$ for internal flow in a circular pipe.

Reynolds Number

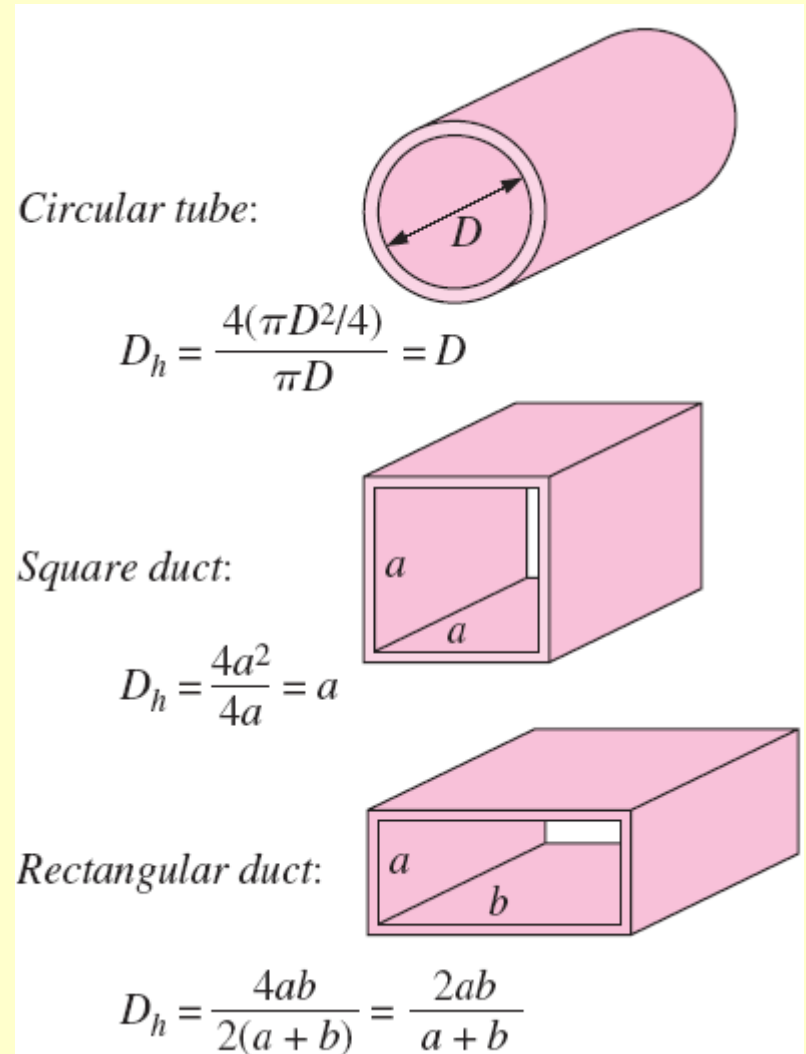
- For flow through noncircular pipes, the Reynolds number is based on the **hydraulic diameter** D_h defined as

$$D_h = \frac{4A_c}{p}$$

A_c = cross-section area

P = wetted perimeter

- The transition from laminar to turbulent flow also depends on the degree of disturbance of the flow by *surface roughness*, *pipe vibrations*, and *fluctuations in the flow*.

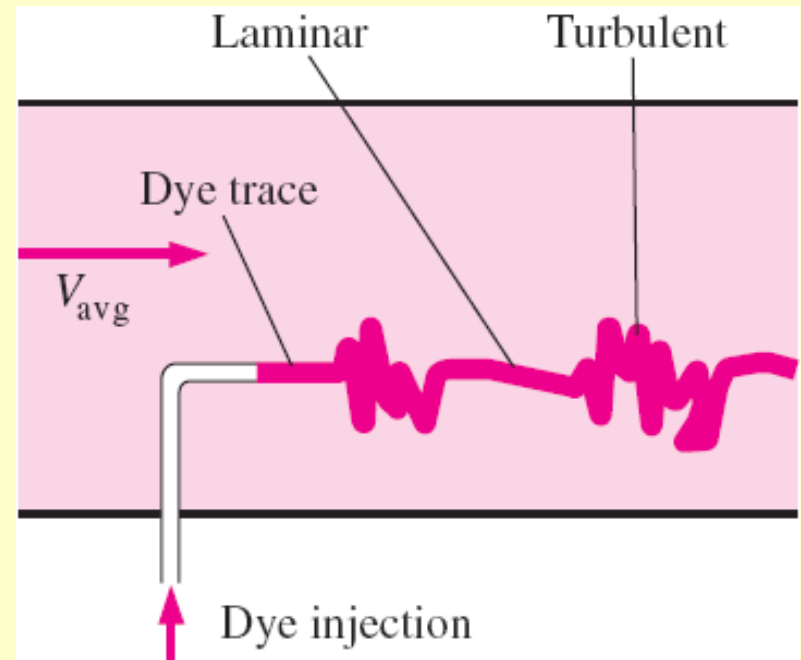


Reynolds Number

- Under most practical conditions, the flow in a circular pipe is

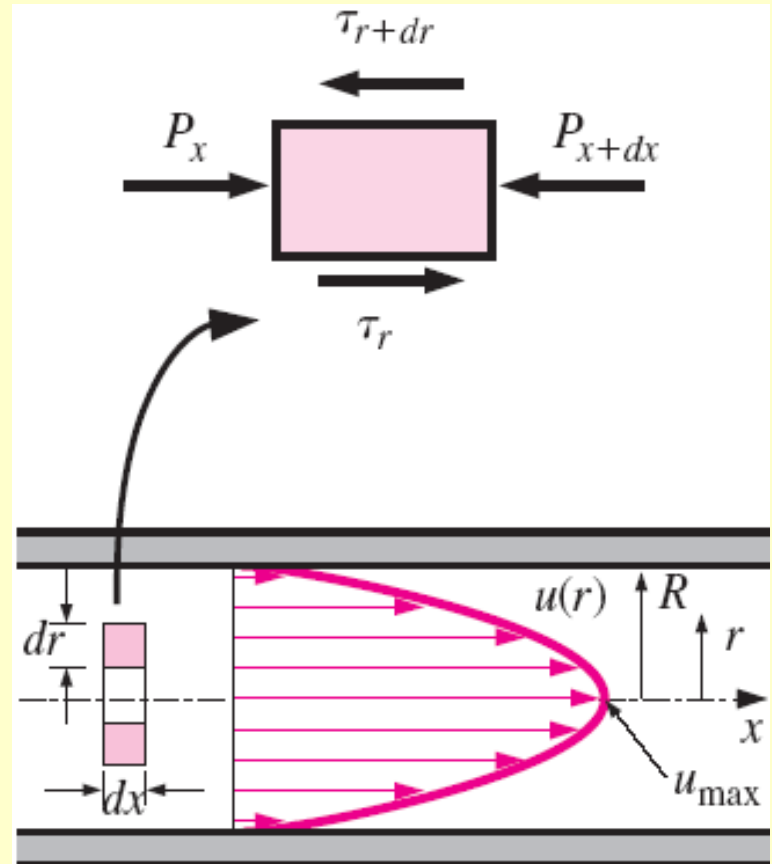
| | |
|----------------------------------|-------------------|
| $Re \lesssim 2300$ | laminar flow |
| $2300 \lesssim Re \lesssim 4000$ | transitional flow |
| $Re \gtrsim 4000$ | turbulent flow |

- In transitional flow, the flow switches between laminar and turbulent randomly.



LAMINAR FLOW IN PIPES

- In this section we consider the steady laminar flow of an incompressible fluid with constant properties in the fully developed region of a straight circular pipe.
- In fully developed laminar flow, each fluid particle moves at a constant axial velocity along a streamline and no motion in the radial direction such that no acceleration (since flow is steady and fully-developed).



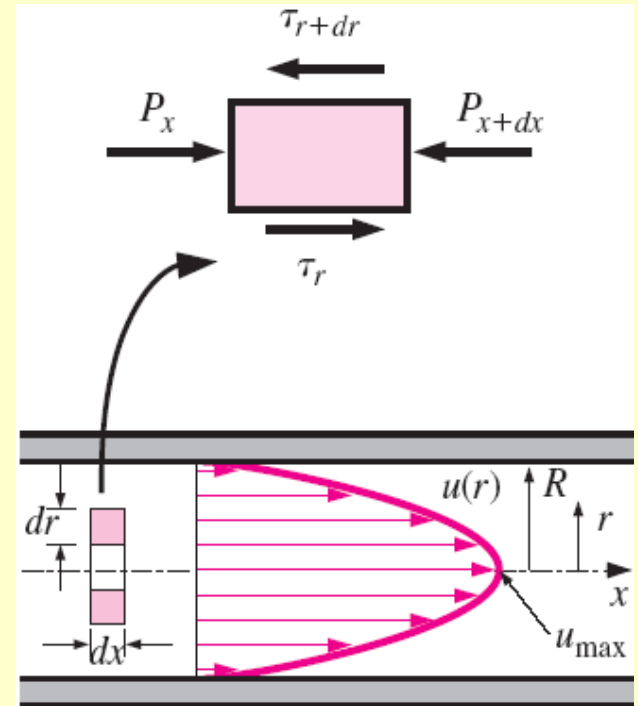
LAMINAR FLOW IN PIPES

- Now consider a ring-shaped differential volume element of radius r , thickness dr , and length dx oriented coaxially with the pipe. A force balance on the volume element in the flow direction gives

$$(2\pi r dr P)_x - (2\pi r dr P)_{x+dx} + (2\pi r dx \tau)_r - (2\pi r dx \tau)_{r+dr} = 0$$

- Dividing by $2\pi dr dx$ and rearranging,

$$r \frac{P_{x+dx} - P_x}{dx} + \frac{(r\tau)_{r+dr} - (r\tau)_r}{dr} = 0$$



LAMINAR FLOW IN PIPES

- Taking the limit as $dr, dx \rightarrow 0$ gives

$$r \frac{dP}{dx} + \frac{d(r\tau)}{dr} = 0$$

- Substituting $\tau = -\mu(du/dr)$ gives the desired equation,

$$\frac{\mu}{r} \frac{d}{dr} \left(r \frac{du}{dr} \right) = \frac{dP}{dx}$$

- The left side of the equation is a function of r , and the right side is a function of x . The equality must hold for any value of r and x ; therefore, $f(r) = g(x) = \text{constant}$.

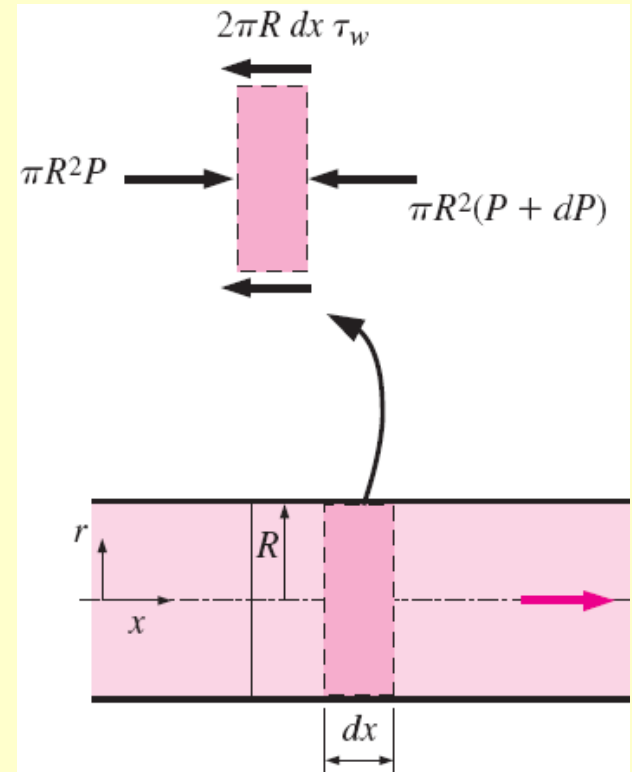
LAMINAR FLOW IN PIPES

- Thus we conclude that $dP/dx =$ constant and we can verify that

$$\frac{dP}{dx} = -\frac{2\tau_w}{R}$$

- Here τ_w is constant since the viscosity and the velocity profile are constants in the fully developed region. Then we solve the $u(r)$ eq. by rearranging and integrating it twice to give

$$u(r) = \frac{r^2}{4\mu} \left(\frac{dP}{dx} \right) + C_1 \ln r + C_2$$



Force balance:

$$\pi R^2 P - \pi R^2 (P + dP) - 2\pi R dx \tau_w = 0$$

Simplifying:

$$\frac{dP}{dx} = -\frac{2\tau_w}{R}$$

LAMINAR FLOW IN PIPES

- Since $\partial u / \partial r = 0$ at $r = 0$ (because of symmetry about the centerline) and $u = 0$ at $r = R$, then we can get $u(r)$

$$u(r) = -\frac{R^2}{4\mu} \left(\frac{dP}{dx} \right) \left(1 - \frac{r^2}{R^2} \right)$$

- Therefore, the velocity profile in fully developed laminar flow in a pipe is *parabolic*. Since u is positive for any r , and thus the dP/dx must be negative (i.e., pressure must decrease in the flow direction because of viscous effects).
- The average velocity is determined from

$$V_{\text{avg}} = \frac{2}{R^2} \int_0^R u(r)r \, dr = \frac{-2}{R^2} \int_0^R \frac{R^2}{4\mu} \left(\frac{dP}{dx} \right) \left(1 - \frac{r^2}{R^2} \right) r \, dr = -\frac{R^2}{8\mu} \left(\frac{dP}{dx} \right)$$

LAMINAR FLOW IN PIPES

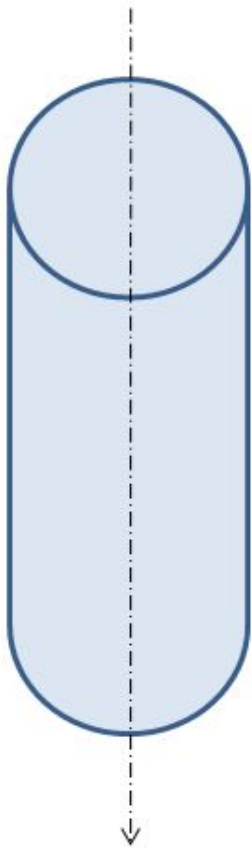
- The velocity profile is rewritten as

$$u(r) = 2V_{\text{avg}} \left(1 - \frac{r^2}{R^2} \right)$$

- Thus we can get

$$u_{\text{max}} = 2V_{\text{avg}}$$

- Therefore, *the average velocity in fully developed laminar pipe flow is one half of the maximum velocity.*



Poiseuille flow in a cylinder (Hagen-Poiseuille):

Assume: flow along O_z + rotational invariance: $\underline{v}(r, \theta, z) = v_z(r, z)\underline{e}_z$

Continuity equation: $\frac{\partial v_z}{\partial z} = 0 \Rightarrow v_z(r)$

Boundary conditions: $\underline{v}(r = R, \theta) = \underline{0}$

$P(z = 0) - P(z = L) \equiv \Delta P$ Pressure Gradient

Navier-Stokes equation: $\left\{ \begin{array}{l} \frac{\partial P}{\partial r} = \frac{\partial P}{r \partial \theta} = 0 \Rightarrow P(z) \\ \eta \left(\frac{\partial^2 v_z}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 v_z}{\partial \theta^2} + \frac{1}{r} \frac{\partial v_z}{\partial r} \right) = \frac{\partial P}{\partial z} - \rho g \end{array} \right.$

$$v_{z(r)} = \frac{\Delta P + \rho g L}{4\eta L} \cdot (R^2 - r^2)$$

Flow rate: $Q = \frac{\Delta P + \rho g L}{8\eta L} \cdot \pi \cdot R^4$

Friction force: $F_z = (\Delta P + \rho g L) \pi R^2$

Total pressure force: $F_z = \langle P \rangle 2\pi R L$

Pressure Drop and Head Loss

- The *pressure drop* ΔP of pipe flow is related to the power requirements of the fan or pump to maintain flow. Since $dP/dx = \text{constant}$, and integrating from $x = x_1$ where the pressure is P_1 to $x = x_1 + L$ where the pressure is P_2 gives

$$\frac{dP}{dx} = \frac{P_2 - P_1}{L}$$

- The pressure drop for laminar flow can be expressed as

$$\Delta P = P_1 - P_2 = \frac{8\mu L V_{\text{avg}}}{R^2} = \frac{32\mu L V_{\text{avg}}}{D^2}$$

- ΔP due to viscous effects represents an irreversible pressure loss, and it is called **pressure loss** ΔP_L to emphasize that it is a *loss*.

Pressure Drop and Head Loss

- In the analysis of piping systems, pressure losses are commonly expressed in terms of the *equivalent fluid column height*, called the **head loss** h_L .

$$h_L = \frac{\Delta P_L}{\rho g} = f \frac{L}{D} \frac{V_{\text{avg}}^2}{2g}$$

(Frictional losses due to viscosity)