

TOPIC:

Scalar and Vector Spherical Harmonics

Mahmoud Solomon
Department of Surveying and Geoinformatics
UNIVERSITY OF LAGOS

1.0 INTRODUCTION AND BACKGROUND

1.1 INTRODUCTION

- This presentation is about Vector and Scalar Spherical Harmonics
- Emphasis
 - Introduction and background
 - Mathematical Representation
 - Spherical Coordinates
 - Scalar Spherical Harmonics
 - Vector Spherical Harmonics
 - Properties
 - Similarities and Differences
- Applications

1.0 INTRODUCTION AND BACKGROUND Cont'd

1.2 BACKGROUND

- Origin

- Firstly connected with Newton's Law of Universal Gravitation in 3D
- In 1782, Pierre-Simon de Laplace in his publication "***Mecanique Celeste***" show the relationship between gravitational potential and a point is given by:

$$V(x) = \sum_i \frac{m_i}{|x_i - x|} \quad 1.1$$

- Prior to Laplace's determination, Andrien-Marie Legendre had investigated the expansion of the Newtonian Potential in powers of $r = |x|$ and $r_1 = |x_1|$. He discovered that

$$\frac{1}{|x_1 - x|} = P_0(\cos \gamma) \frac{1}{r_1} + P_1(\cos \gamma) \frac{r}{r_1^2} + P_2(\cos \gamma) \frac{r^3}{r_1^3} + \dots \quad 1.2$$

Where γ is the angle between the vectors x and x_1 . The functions P_i are the Legendre Polynomials, and they are a special case of spherical harmonics (Muller, 1966).

1.0 INTRODUCTION AND BACKGROUND Cont'd

1.2 BACKGROUND

- In 1867 William Thomson (Lord Kelvin) and Peter Guthrie Tait introduced the Solid Spherical harmonics in their *Treatise on Natural Philosophy*, and also first introduced the name of “*Spherical Harmonics*” for these functions.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

1.3

- Since its involvement, Spherical harmonics are closely associated with the basic theory of gravitational and magnetic fields such as those of the Earth and planets; for this reason, they are important both in geodesy and in Earth and Planetary Physics (Colombo, 1981).

2.0 FUNDAMENTALS

2.1 SO WHAT THEN ARE SPHERICAL HARMONICS???

- These are functions which are solutions to the Laplace's Differential Equation SUCH THAT $\nabla^2 F = 0$ 2.1

Where ∇^2 = divergence and F is a function. Laplace's equation imposes that the divergence of the gradient of a scalar field f is zero.

- This means that every point on a sphere has a numeric value.
- Conversely, not all functions that have values for every point on a sphere represent a sphere.
- Examples →



Marbles



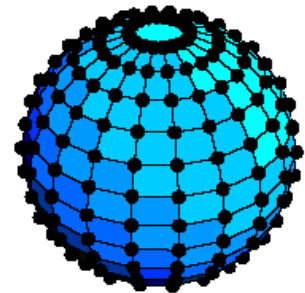
Eye



Bubble



Balls



Planet

2.0 FUNDAMENTALS Cont'd

2.2 LAPLACE EQUATION IN SPHERICAL HARMONICS

- The most important harmonic functions are the so-called Spherical Harmonics (Heiskanen and Moritz, 1993).

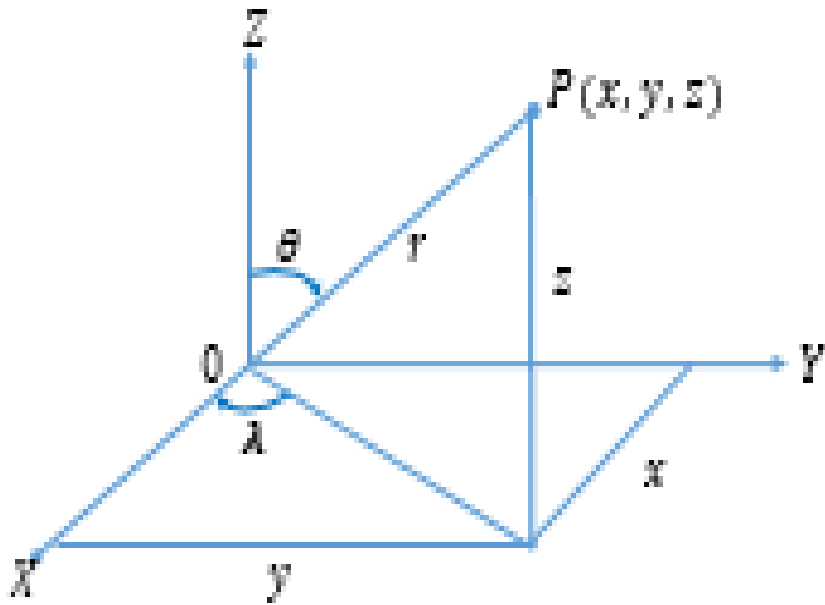


Figure 1

Where r (Radius Vector), θ = polar distance, and λ is the geocentric latitude

$$x = r \sin \theta \cos \lambda$$

$$y = r \sin \theta \sin \lambda$$

$$z = r \cos \theta$$

$$r = \sqrt{x^2 + y^2 + z^2}$$

$$\tan \theta = \frac{\sqrt{x^2 + y^2}}{z}$$

$$\tan \lambda = \frac{y}{x}$$

2.2

2.0 FUNDAMENTALS Cont'd

2.3 SPHERICAL COORDINATES

- Spherical Coordinates or spherical polar coordinates form a coordinate system for the three-dimensional real space \mathbb{R}^3 .
- Three numbers, two angles and a length specify any point in \mathbb{R}^3 .
- Spherical coordinates are natural for describing positions on a sphere or spheroid.

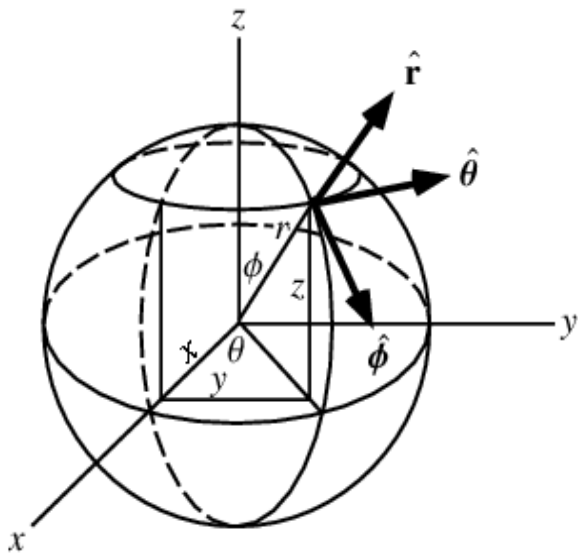


Figure 2

θ is the azimuthal angle in the $x y$ – plane from the x-axis with $0 \leq \theta < 2\pi$ (denoted λ when referred to as the longitude), ϕ to be the polar angle (also known as the zenith angle and colatitude, with $\phi = 90^\circ - \delta$ where δ is the latitude) from the positive z-axis with $0 \leq \phi \leq \pi$, and r to be the distance (radius) from a point to the origin. This is the convention commonly used in mathematics.

3.0 SPHERICAL HARMONICS

3.1 MATHEMATICS

$$Y_l^m(\theta, \phi) = N e^{im\phi} P_l^m(\cos\theta) \quad \text{where } N = \sqrt{\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!}} \quad 3.1$$

- Here, Y_l^m is called spherical harmonic function of degree l and order m
- P_l^m is an associated Legendre polynomial, N is a normalization constant, and θ and ϕ represent colatitude and longitude, respectively
- θ , or polar angle, ranges from 0 at the North Pole to π at the South Pole, assuming the value of $\pi/2$ at the Equator, and ϕ , or azimuth, may assume all values with $0 \leq \phi < 2\pi$

Example of first few harmonics	$Y_0^0(\theta, \phi) = \frac{1}{2} \frac{1}{\sqrt{\pi}}$	$Y_2^2(\theta, \phi) = \frac{1}{4} \sqrt{\frac{15}{2\pi}} \sin^2 \theta e^{2i\phi}$	$ Y_0^0(\theta, \phi) ^2$		
	$Y_1^0(\theta, \phi) = \frac{1}{2} \sqrt{\frac{3}{\pi}} \cos \theta$	$Y_1^1(\theta, \phi) = -\frac{1}{2} \sqrt{\frac{3}{2\pi}} \sin \theta e^{i\phi}$	$ Y_1^0(\theta, \phi) ^2$	$ Y_1^1(\theta, \phi) ^2$	
	$Y_2^0(\theta, \phi) = \frac{1}{4} \sqrt{\frac{5}{\pi}} (3\cos^2 \theta - 1)$	$Y_2^1(\theta, \phi) = -\frac{1}{2} \sqrt{\frac{15}{2\pi}} \sin \theta \cos \theta e^{i\phi}$	$ Y_2^0(\theta, \phi) ^2$	$ Y_2^1(\theta, \phi) ^2$	$ Y_2^2(\theta, \phi) ^2$

3.0 SCALAR AND VECTOR SPHERICAL HARMONICS

3.1 SCALAR SPHERICAL HARMONICS

- Important in the modeling of geomagnetic field
- Useful in the analysis of the secular variation, diurnal variation of the earth

$$Y_l^m(\theta, \phi) = (-)^m \sqrt{\frac{(2n+1)(n-m)!}{(n+m)!}} P_{n,m}(\cos\theta) e^{im\phi} \quad 3.1.1$$

3.2 VECTOR SPHERICAL HARMONICS

- The definition of vector spherical harmonics used herein are defined by (James, 1974)

$$Y_l^m(\theta, \phi) = (-)^{n-m} \sqrt{2n+1} \sum_{m_1, \mu} \begin{pmatrix} n & n_1 & 1 \\ m & -m_1 & -\mu \end{pmatrix} Y_{n_1}^{m_1} \mathbf{e}_\mu \quad 3.2.1$$

Where the complex basis vectors \mathbf{e}_μ are defined in terms of the unit vectors $\mathbf{1}_x, \mathbf{1}_y, \mathbf{1}_z$ of the Cartesian coordinate system $(x, y, z) = (r \sin\theta \cos\phi, r \sin\theta \sin\phi, r \cos\theta)$ by $\mathbf{e}_0 := \mathbf{1}_z$ and $\mathbf{e}_{\pm 1} := \mp(\mathbf{1}_x \pm i\mathbf{1}_y)/\sqrt{2}$.

3.0 SCALAR AND VECTOR SPHERICAL HARMONICS

3.3 REVIEW OF SCALAR AND VECTOR SPHERICAL HARMONICS

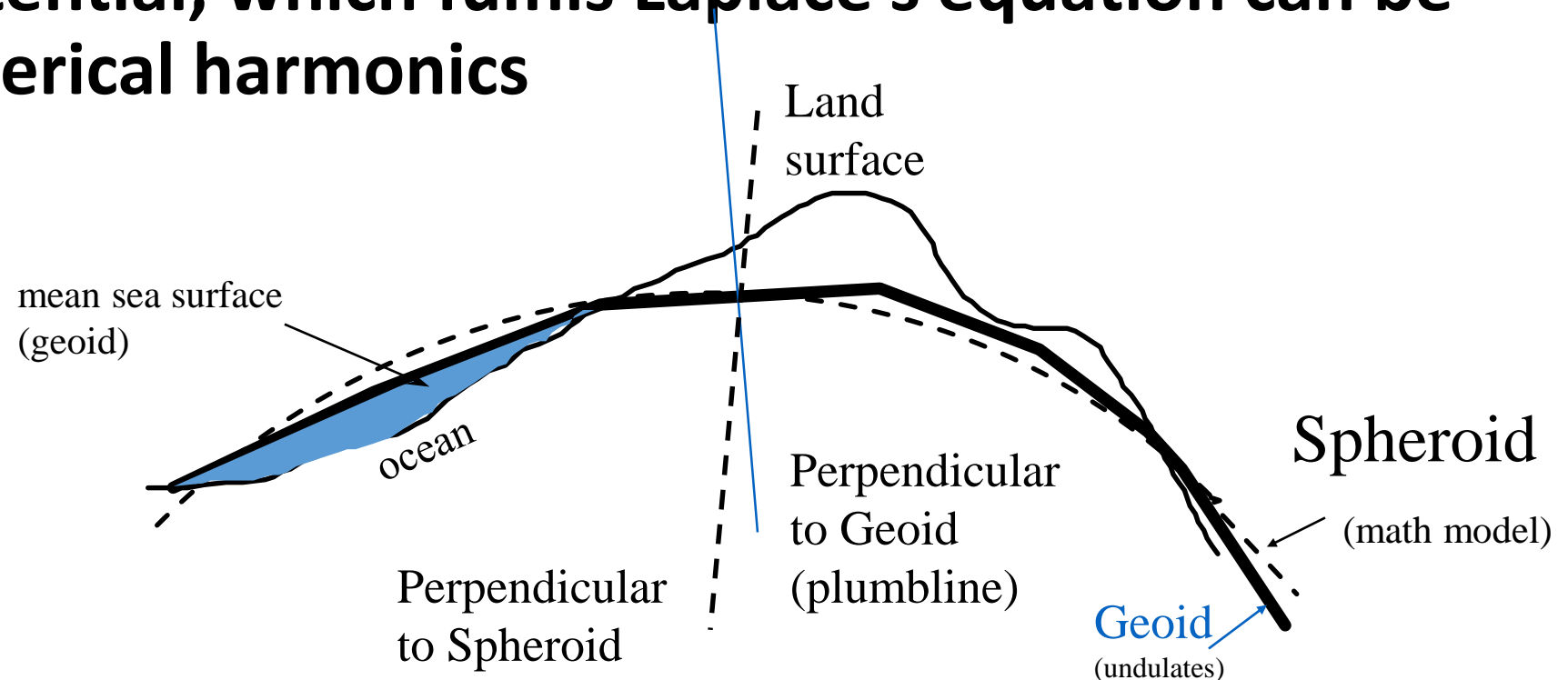
SCALAR	VECTOR
SIMILARITIES	
Solutions to the Laplace equation of the sphere	
Double Product Integral	
Triple Product Integral	
Double Product Projection	
DIFFERENCES	
Used in spectral method approaches for numerical solution of scalar PDEs or data analysis of scalar data on the sphere (Kostelec <i>et al.</i> , 2001).	May be used effectively in cases where the data consists of vector fields on the sphere (Kostelec <i>et al.</i> , 2001).
Scalar fields such as pressure, temperature are expanded in term of scalar harmonies (Swarztrauber, 1993).	Vector fields such as wind, velocity are expanded in terms of vector harmonics (Swarztrauber, 1993).
Provides suitable basis for scalar functions such as divergence and vorticity which are smooth at the poles (Swarztrauber, 1993)	Vector functions such as the velocity are multivalued and, hence, discontinuous at the poles (Swarztrauber, 1993)

3.0 SCALAR AND VECTOR SPHERICAL HARMONICS

3.3 APPLICATIONS

3.3.1 Geoid Determination

- Like stated, SH are orthogonal set of solution to Laplace Equation represented in a system of spherical coordinates.
- **Each harmonic potential, which fulfils Laplace's equation can be expanded into spherical harmonics**



3.0 SCALAR AND VECTOR SPHERICAL HARMONICS

3.3 APPLICATIONS

3.3.1 Geoid Determination

- For this reason the stationary part of the Earth's gravitational potential W_a at any point (r, λ, φ) on and above the Earth's surface is expressed on a global scale conveniently by summing up over degree and order of a spherical harmonic expansion.

$$W_a(r, \lambda, \varphi) = \frac{GM}{r} \sum_{l=0}^{l_{max}} \sum_{m=0}^l \left(\frac{R}{r}\right)^l P_{lm}(\sin\varphi) (C_{lm}^W \cos m\lambda + S_{lm}^W \sin m\lambda) \quad 3.3.1.1$$

- (r, λ, φ) - Spherical geocentric coordinates of computational point (radius, longitude, latitude). R - Reference radius, GM - Product of gravitational constant and mass of the Earth. l, m - degree, order of spherical harmonics, P_{lm} Fully normalized Legendre functions. C_{lm}^W, S_{lm}^W - Stokes' coefficient (fully normalized).

3.0 SCALAR AND VECTOR SPHERICAL HARMONICS

3.3 APPLICATIONS

3.3.2 Tide Prediction

- According to US CGS, tides can be expressed mathematically by the sum of a series of harmonic terms having certain relations to astronomical conditions.
- The general equation for the height (h) of the tide at any time (t) may be written

$$h = H_o + A\cos(at + \alpha) + B\cos(bt + \beta) + C\cos(ct + \gamma) + \text{etc} \dots \quad 3.2.2.1$$

- H – height of the tide, H_o is the height of the mean water level above the datum used, α, β, γ etc. refer to the initial phases of the constituent angles, t - time
- Scientists have been able to make reasonable and accurate tide predictions by the harmonic method (Schureman P. , 1941).

3.0 SCALAR AND VECTOR SPHERICAL HARMONICS

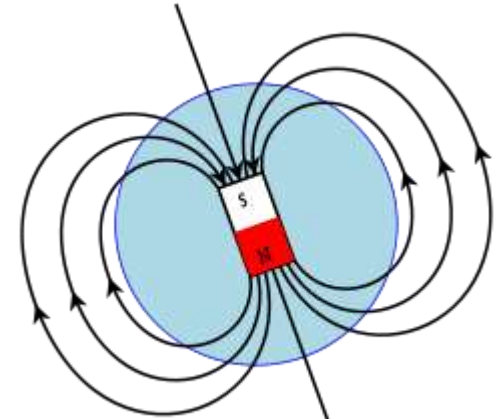
3.3 APPLICATIONS

3.3.3 Magnetic Field of Planetary Bodies

- Magnetic Field extends from the interior of a planetary body out into out space.
- Campbell (2008) stated that Carl Friedrich Gauss was the first to analyze the global variations in the Earth's magnetic field using a set of spherical harmonics.
- This is because Spherical Harmonics are functions that oscillate over the surface of a sphere.

$$V_m(r, \theta, \varphi) = a \left\{ I_l^m \left(\frac{a}{r} \right)^{l+1} + E_l^m \left(\frac{r}{a} \right)^l \right\} P_l^m(\cos\theta) \quad 3.3.3.1$$

- Where I_l^m and E_l^m are the amplitude factors of the contributions of the internal and external sources, respectively.



A sketch of Earth's magnetic field

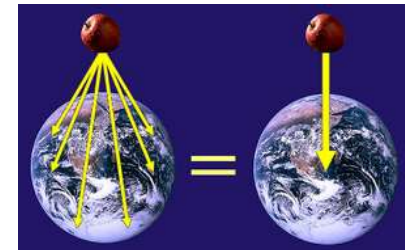
3.0 SCALAR AND VECTOR SPHERICAL HARMONICS

3.3 APPLICATIONS

3.3.4 Gravitational Field

- This is a model used to explain the influence that a massive body extends into the space around itself, producing a force on another massive body.
- Gravitational models explain gravitational phenomenon.
- Since it is a force between two masses, SHA is appropriate.

$$u = -\frac{\mu}{r} + \sum_{n=2}^{N_z} \frac{J_n P_n^0(\sin\theta)}{r^{n+1}} + \sum_{n=2}^{N_t} \sum_{m=1}^n \frac{P_n^m(\sin\theta)(C_n^m \cos m\varphi + S_n^m \sin m\varphi)}{r^{n+1}} \quad 3.3.4.1$$



- where $\mu = GM$, C_n^m , S_n^m Stokes' coefficients (fully normalized), P_n^0 = fully normalized Legendre function, r is the distance of body separation.

3.0 SCALAR AND VECTOR SPHERICAL HARMONICS

3.3 APPLICATIONS

3.3.5 Climate Change

- According to UNFCC, “Climate” is the statistics of weather.
- Spherical Harmonics being a function that decomposes components on a sphere combines components of various wavelengths to generate functions values on a sphere.
- According to Hwang Kao (2005), the temporal variation of the Earth’s gravity field is closely related to global climate change.
- The mass variation within the Earth system may be induced by changes in ocean, atmosphere, precipitation (snow and rainfall), water table, glacier, ice sheet, etc which are all important variables in the climate change process.
- The introduction of GRACE can be used to derive gravity variations.
- SHA and synthesis are important tools for investigating these gravity variations

3.0 SCALAR AND VECTOR SPHERICAL HARMONICS

3.3 APPLICATIONS

3.3.5 Additional Areas of application

- Atomic orbital electron configurations
- Cosmic microwave background radiations
- Ocean dynamic topography
- 3D Computer Graphics

3.0 CONCLUSION

- Spherical Harmonics

- Great tool for solving Laplace equation of the sphere.
- Second order partial differential equation named after Pierre-Simon de Laplace.
- Have series of applications as mentioned earlier.

- A MATLAB PROGRAM TO COMPUTE VARIOUS SPHERICAL HARMONICS ON THE SPHERE.