



**Centurion**  
**UNIVERSITY**  
*Shaping Lives...  
Empowering Communities...*

# Centurion University of Technology And Management

.



**Centurion**  
**UNIVERSITY**  
*Shaping Lives...  
Empowering Communities...*

# Complex Analysis



Centurion  
UNIVERSITY

Shaping Lives...  
Empowering Communities...

## Definitions

**Definition 1.1** Complex numbers are defined as ordered pairs  $(x, y)$

Points on a complex plane. Real axis, imaginary axis, purely imaginary numbers. Real and imaginary parts of complex number. Equality of two complex numbers.

**Definition 1.2** The sum and product of two complex numbers are defined as follows:

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$$

$$(x_1, y_1) \cdot (x_2, y_2) = (x_1x_2 - y_1y_2, x_1y_2 + x_2y_1)$$

In the rest of the chapter use  $z, z_1, z_2, \dots$  for complex numbers and  $x, y$  for real numbers. introduce  $i$  and  $z = x + iy$  notation.

## Algebraic Properties

### 1. Commutativity

$$z_1 + z_2 = z_2 + z_1, \quad z_1 z_2 = z_2 z_1$$

### 2. Associativity

$$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3), \quad (z_1 z_2) z_3 = z_1 (z_2 z_3)$$

### 3. Distributive Law

$$z(z_1 + z_2) = zz_1 + zz_2$$

### 4. Additive and Multiplicative Identity

$$z + 0 = z, \quad z \cdot 1 = z$$

### 5. Additive and Multiplicative Inverse

$$-z = (-x, -y)$$

$$z^{-1} = \left( \frac{x}{x^2 + y^2}, \frac{-y}{x^2 + y^2} \right), \quad z \neq 0$$



**Centurion  
UNIVERSITY**

*Shaping Lives...  
Empowering Communities...*

#### 6. Subtraction and Division

$$z_1 - z_2 = z_1 + (-z_2), \quad \frac{z_1}{z_2} = z_1 z_2^{-1}$$

#### 7. Modulus or Absolute Value

$$|z| = \sqrt{x^2 + y^2}$$

#### 8. Conjugates and properties

$$\begin{aligned} \bar{z} &= x - iy = (x, -y) \\ \overline{z_1 \pm z_2} &= \bar{z}_1 \pm \bar{z}_2 \\ \overline{z_1 z_2} &= \bar{z}_1 \bar{z}_2 \\ \overline{\left(\frac{z_1}{z_2}\right)} &= \frac{\bar{z}_1}{\bar{z}_2} \end{aligned}$$

#### 9.

$$\begin{aligned} |z|^2 &= z\bar{z} \\ \operatorname{Re} z &= \frac{z + \bar{z}}{2}, \operatorname{Im} z = \frac{z - \bar{z}}{2i} \end{aligned}$$

#### 10. Triangle Inequality

$$|z_1 + z_2| \leq |z_1| + |z_2|$$

## Polar Coordinates and Euler Formula

#### 1. Polar Form: for $z \neq 0$ ,

$$z = r(\cos \theta + i \sin \theta)$$

where  $r = |z|$  and  $\tan \theta = y/x$ .  $\theta$  is called the argument of  $z$ . Since  $\theta + 2n\pi$  is also an argument of  $z$ , the principle value of argument of  $z$  is taken such that  $-\pi < \theta \leq \pi$ . For  $z = 0$  the arg  $z$  is undefined.

#### 2. Euler formula: Symbolically,

$$e^{i\theta} = (\cos \theta + i \sin \theta)$$

#### 3.

$$\begin{aligned} z_1 z_2 &= r_1 r_2 e^{i(\theta_1 + \theta_2)} \\ \frac{z_1}{z_2} &= \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)} \\ z^n &= r^n e^{in\theta} \end{aligned}$$

#### 4. de Moivre's Formula

$$(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta$$



Centurion  
UNIVERSITY

Shaping Lives...  
Empowering Communities...

## Roots of Complex Numbers

Let  $z = re^{i\theta}$  then

$$z^{1/n} = r^{1/n} \exp \left( i \left( \frac{\theta}{n} + \frac{2k\pi}{n} \right) \right)$$

There are only  $n$  distinct roots which can be given by  $k = 0, 1, \dots, n - 1$ . If  $\theta$  is a principle value of  $\arg z$  then  $\theta/n$  is called the principle root.

**Example 1.1** The three possible roots of  $\left(\frac{1+i}{\sqrt{2}}\right)^{1/3} = (e^{i\pi/4})^{1/3}$  are  $e^{i\pi/12}$ ,  $e^{i\pi/12+i2\pi/3}$ ,  $e^{i\pi/12+i4\pi/3}$ .

## Regions in Complex Plane

1.  $\epsilon$  - nbd of  $z_0$  is defined as a set of all points  $z$  which satisfy

$$|z - z_0| < \epsilon$$

2. Deleted nbd of  $z_0$  is a nbd of  $z_0$  excluding point  $z_0$ .
3. Interior Point, Exterior Point, Boundary Point, Open set and closed set.
4. Domain, Region, Bounded sets, Limit Points.



Centurion  
UNIVERSITY

Shaping Lives...  
Empowering Communities...

## Limits

A function  $f$  is defined in a deleted nbd of  $z_0$ .

**Definition 3.1** The limit of the function  $f(z)$  as  $z \rightarrow z_0$  is a number  $w_0$  if, for any given  $\epsilon > 0$  there exists a  $\delta > 0$  such that

$$|z - z_0| < \delta \Rightarrow |f(z) - w_0| < \epsilon.$$

**Example 3.1**  $f(z) = 5z$ . Show that  $\lim_{z \rightarrow z_0} f(z) = 5z_0$ .

**Example 3.2**  $f(z) = z^2$ . Show that  $\lim_{z \rightarrow z_0} f(z) = z_0^2$ .

**Example 3.3**  $f(z) = z/\bar{z}$ . Show that the limit of  $f$  does not exist as  $z \rightarrow 0$ .

**Theorem 3.1** Let  $f(z) = u(x, y) + iv(x, y)$  and  $w_0 = u_0 + iv_0$ .  $\lim_{z \rightarrow z_0} f(z) = w_0$  if and only if  $\lim_{(x,y) \rightarrow (x_0,y_0)} u = u_0$  and  $\lim_{(x,y) \rightarrow (x_0,y_0)} v = v_0$ .

**Example 3.4**  $f(z) = \sin z$ . Show that  $\lim_{z \rightarrow z_0} f(z) = \sin z_0$

**Example 3.5**  $f(z) = 2x + iy^2$ . Show that  $\lim_{z \rightarrow 2i} f(z) = 4i$ .

**Theorem 3.2** If  $\lim_{z \rightarrow z_0} f(z) = w_0$  and  $\lim_{z \rightarrow z_0} F(z) = W_0$ ,

$$\lim_{z \rightarrow z_0} [f(z) + F(z)] = w_0 + W_0;$$

$$\lim_{z \rightarrow z_0} f(z) F(z) = w_0 W_0;$$

$$\lim_{z \rightarrow z_0} f(z) / F(z) = \frac{w_0}{W_0} \quad W_0 \neq 0.$$

This theorem immediately makes available the entire machinery and tools used for real analysis to be applied to complex analysis. The rules for finding limits then can be listed as follows:



**Centurion  
UNIVERSITY**

*Shaping Lives...  
Empowering Communities...*

1.  $\lim_{z \rightarrow z_0} c = c.$
2.  $\lim_{z \rightarrow z_0} z^n = z_0^n.$
3.  $\lim_{z \rightarrow z_0} P(z) = P(z_0)$  if  $P$  is a polynomial in  $z$ .
4.  $\lim_{z \rightarrow z_0} \exp(z) = \exp(z_0).$
5.  $\lim_{z \rightarrow z_0} \sin(z) = \sin z_0.$

## Continuity

**Definition 3.2** A function  $f$ , defined in some nbd of  $z_0$  is continuous at  $z_0$  if

$$\lim_{z \rightarrow z_0} f(z) = f(z_0).$$

This definition clearly assumes that the function is defined at  $z_0$  and the limit on the LHS exists. The function  $f$  is continuous in a region if it is continuous at all points in that region.

If functions  $f$  and  $g$  are continuous at  $z_0$  then  $f + g$ ,  $fg$  and  $f/g$  ( $g(z_0) \neq 0$ ) are also continuous at  $z_0$ .

If a function  $f(z) = u(x, y) + iv(x, y)$  is continuous at  $z_0$  then the component functions  $u$  and  $v$  are also continuous at  $(x_0, y_0)$ .

## Derivative

**Definition 3.3** A function  $f$ , defined in some nbd of  $z_0$  is differentiable at  $z_0$  if

$$\lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists. The limit is called the derivative of  $f$  at  $z_0$  and is denoted by  $f'(z_0)$  or  $\frac{df}{dz}(z_0)$ .

**Example 3.6**  $f(z) = z^2$ . Show that  $f'(z) = 2z$ .

**Example 3.7**  $f(z) = |z|^2$ . Show that this function is differentiable only at  $z = 0$ . In real analysis  $|x|$  is not differentiable but  $|x|^2$  is.

If a function is differentiable at  $z$ , then it is continuous at  $z$ .





Centurion  
UNIVERSITY

Shaping Lives...  
Empowering Communities

## Analytic Functions

**Definition 3.4** A function is analytic in an open set if it has a derivative at each point in that set.

**Definition 3.5** A function is analytic at a point  $z_0$  if it is analytic in some nbd of  $z_0$ .

**Definition 3.6** A function is an entire function if it is analytic at all points of  $C$ .

**Example 3.13**  $f(z) = 1/z$  is analytic at all nonzero points.

**Example 3.14**  $f(z) = |z|^2$  is not analytic anywhere.

A function is not analytic at a point  $z_0$ , but is analytic at some point in each nbd of  $z_0$  then  $z_0$  is called the singular point of the function  $f$ .

## Harmonic Functions

**Definition 3.7** A real valued function  $H(x, y)$  is said to be harmonic in a domain of  $xy$  plane if it has continuous partial derivatives of the first and second order and satisfies Laplace equation:

$$H_{xx}(x, y) + H_{yy}(x, y) = 0.$$



# Cauchy-Riemann Equations



Centurion  
UNIVERSITY

Shaping Lives...  
Empowering Communities...

**Theorem 3.3** If  $f'(z_0)$  exists, then all the first order partial derivatives of component function  $u(x, y)$  and  $v(x, y)$  exist and satisfy Cauchy-Riemann Conditions:

$$\begin{aligned}u_x &= v_y \\ u_y &= -v_x.\end{aligned}$$

**Example 3.8**  $f(z) = z^2 = x^2 - y^2 + i2xy$ . Show that Cauchy-Riemann Conditions are satisfied.

**Example 3.9**  $f(z) = |z|^2 = x^2 + y^2$ . Show that the Cauchy-Riemann Conditions are satisfied only at  $z = 0$ .

**Theorem 3.4** Let  $f(z) = u(x, y) + iv(x, y)$  be defined in some nbd of the point  $z_0$ . If the first partial derivatives of  $u$  and  $v$  exist and are continuous at  $z_0$  and satisfy Cauchy-Riemann equations at  $z_0$ , then  $f$  is differentiable at  $z_0$  and

$$f'(z) = u_x + iv_x = v_y - iu_y.$$

**Example 3.10**  $f(z) = \exp(z)$ . Show that  $f'(z) = \exp(z)$ .

**Example 3.11**  $f(z) = \sin(z)$ . Show that  $f'(z) = \cos(z)$ .

**Example 3.12**  $f(z) = \frac{z^2}{z}$ . Show that the CR conditions are satisfied at  $z = 0$  but the function  $f$  is not differentiable at 0.



1. The *Cauchy-Riemann equations* for two functions  $u(x, y)$  and  $v(x, y)$  are

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

- (a) Show that  $u(x, y) = x^2 - y^2$  and  $v(x, y) = 2xy$  satisfy the Cauchy-Riemann equations.
- (b) The *Laplace equation* for a function  $f(x, y)$  is

$$f_{xx} + f_{yy} = 0.$$

Suppose that  $a$  and  $b$  satisfy the Cauchy-Riemann equations and have equal mixed partial derivatives (i.e.  $a_{xy} = a_{yx}$ , etc.).

Show that  $a + b$  satisfies the Laplace equation.

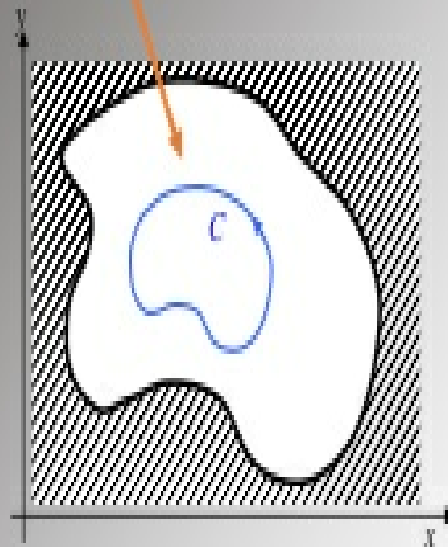


**Centurion  
UNIVERSITY**

*Shaping Lives...  
Empowering Communities...*

# Cauchy Integral Formula

**R** - a simply connected region

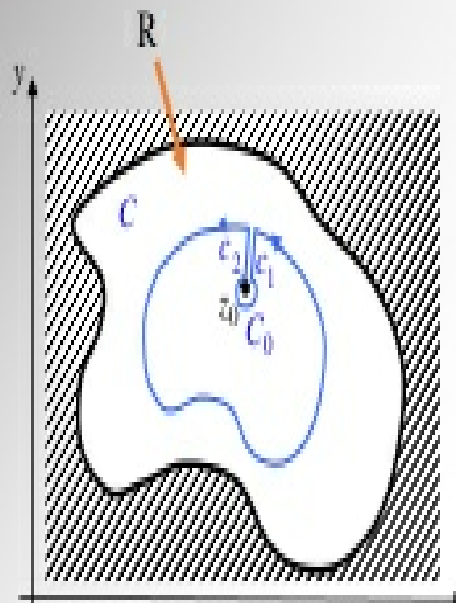


#.  $f(z)$  is assumed analytic in  $R$  but we multiply by a factor  $\frac{1}{(z - z_0)}$  that is analytic except at  $z_0$  and consider the integral around  $C$

$$\int_C \frac{f(z)}{(z - z_0)} dz$$

#. To evaluate, consider the path  $C + c_1 + c_2 + C_0$  shown that encloses a simply connected region for which the integrand is analytic on and inside the path:

$$\int_{C + c_1 + c_2 + C_0} \frac{f(z)}{(z - z_0)} dz = 0 \Rightarrow \int_C \frac{f(z)}{(z - z_0)} dz = - \int_{C_0} \frac{f(z)}{(z - z_0)} dz$$



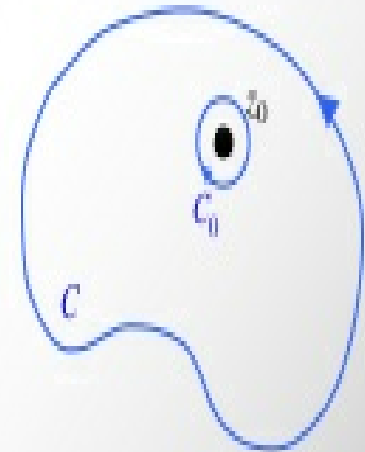


**Centurion  
UNIVERSITY**

*Shaping Lives...  
Empowering Communities...*

# Cauchy Integral Formula, cont'd

$$\int_C \frac{f(z)}{z - z_0} dz = - \int_{C_0} \frac{f(z)}{z - z_0} dz$$



Evaluate the  $C_0$  integral on a circular path,  $z - z_0 = re^{i\theta}$ ,  $dz = rie^{i\theta} d\theta$

$$\int_{C_0} \frac{f(z)}{(z - z_0)} dz \stackrel{r \rightarrow 0}{=} f(z_0) \int_{2\pi}^0 \frac{\cancel{f} i e^{i\theta} d\theta}{\cancel{f} e^{i\theta}} = -2\pi i f(z_0)$$

for  $r \rightarrow 0$

$$\Rightarrow \int_C \frac{f(z)}{z - z_0} dz = 2\pi i f(z_0)$$

$$\Rightarrow f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0} dz$$

Cauchy Integral  
Formula

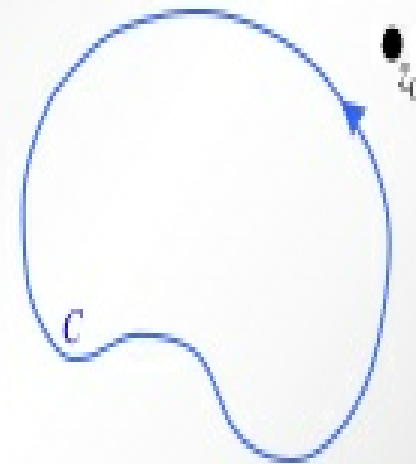
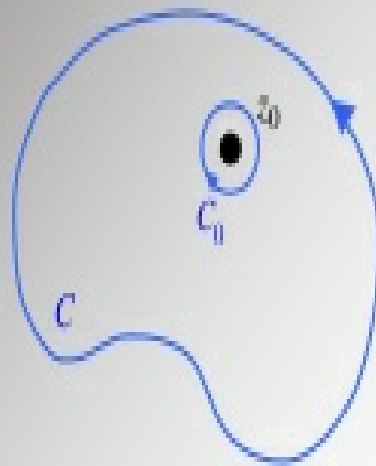
The value of  $f(z)$  at  $z_0$  is completely determined by its values on  $C$ !



Centurion  
UNIVERSITY

Shaping Lives...  
Empowering Communities...

# Cauchy Integral Formula, cont'd



Note that if  $z_0$  is outside  $C$ , the integrand is analytic inside  $C$ , hence by the Cauchy Integral Theorem,

$$\frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0} dz = 0$$

In summary,

$$\oint_C \frac{f(z)}{z - z_0} dz = \begin{cases} 2\pi i f(z_0), & z_0 \text{ inside } C \\ 0, & z_0 \text{ outside } C \end{cases}$$



Centurion  
UNIVERSITY

Shaping Lives...  
Empowering Communities...

## Contour Integral

If  $C$  is a contour in complex plane defined by  $z(t) = x(t) + iy(t)$  and a function  $f(z) = u(x, y) + iv(x, y)$  is defined on it. The integral of  $f(z)$  along the contour  $C$  is denoted and defined as follows:

$$\begin{aligned}\int_C f(z) dz &= \int_a^b f(z) z'(t) dt \\ &= \int_a^b (ux' - vy') dt + i \int_a^b (uy' + vx') dt \\ &= \int (udx - vdy) + i \int (udy + vdx)\end{aligned}$$

The component integrals are usual real integrals and are well defined. In the last form appropriate limits must be placed in the integrals.

Some very straightforward rules of integration are given below:

1.  $\int_C w f(z) dz = w \int_C f(z) dz$  where  $w$  is a complex constant.
2.  $\int_C (f(z) + g(z)) dz = \int_C f(z) dz + \int_C g(z) dz.$
3.  $\int_{C_1+C_2} f(z) dz = \int_{C_1} f(z) dz + \int_{C_2} f(z) dz.$
4.  $|\int_C f(z) dz| \leq \int_C |f(z(t)) z'(t)| dt.$
5. If  $|f(z)| \leq M$  for all  $z \in C$  then  $|\int_C f(z) dz| \leq ML$ , where  $L$  is length of the contour  $C$ .

**Example 4.4**  $f(z) = z^2$ . Find integral of  $f$  from  $(0, 0)$  to  $(2, 1)$  along a straight line and also along st line path from  $(0, 0)$  to  $(2, 0)$  and from  $(2, 0)$  to  $(2, 1)$ .

**Example 4.5**  $f(z) = 1/z$ . Find the integral from  $(2, 0)$  to  $(-2, 0)$  along a semicircular path in upper plane given by  $|z| = 2$ .

**Example 4.6** Show that  $|\int_C f(z) dz| \leq \frac{\pi}{3}$  for  $f(z) = 1/(z^2 - 1)$  and  $C : |z| = 2$  from  $2$  to  $2i$ .



Centurion  
UNIVERSITY

Shaping Lives...  
Empowering Communities...

## Cauchy-Goursat Theorem

**Theorem 4.1 (Jordan Curve Theorem)** Every simple and closed contour in complex plane splits the entire plane into two domains one of which is bounded. The bounded domain is called the interior of the contour and the other one is called the exterior of the contour.

Define a sense direction for a contour.

**Theorem 4.2** Let  $C$  be a simple closed contour with positive orientation and let  $D$  be the interior of  $C$ . If  $P$  and  $Q$  are continuous and have continuous partial derivatives  $P_x, P_y, Q_x$  and  $Q_y$  at all points on  $C$  and  $D$ , then

$$\int_C (P(x, y)dx + Q(x, y)dy) = \iint_D [Q_x(x, y) - P_y(x, y)] dx dy$$

**Theorem 4.3 (Cauchy-Goursat Theorem)** Let  $f$  be analytic in a simply connected domain  $D$ . If  $C$  is any simple closed contour in  $D$ , then

$$\int_C f(z) dz = 0.$$

**Example 4.7**  $f(z) = z^2, \exp(z), \cos(z)$  etc are entire functions so integral about any loop is zero.

**Theorem 4.4** Let  $C_1$  and  $C_2$  be two simple closed positively oriented contours such that  $C_2$  lies entirely in the interior of  $C_1$ . If  $f$  is an analytic function in a domain  $D$  that contains  $C_1$  and  $C_2$  both and the region between them, then

$$\int_{C_1} f(z) dz = \int_{C_2} f(z) dz.$$

**Example 4.8**  $f(z) = 1/z$ . Find  $\int_C f(z) dz$  if  $C$  is any contour containing origin. Choose a circular contour inside  $C$ .

**Example 4.9**  $\int_C \frac{1}{z-z_0} dz = 2\pi i$  if  $C$  contains  $z_0$ .

**Example 4.10** Find  $\int_C \frac{2z dz}{z^2+2}$  where  $C : |z| = 2$ . Extend the Cauchy Goursat theorem to multiply connected domains.





Centurion  
UNIVERSITY

Shaping Lives...  
Empowering Communities...

## Singularities

**Definition 6.1** If a function  $f$  fails to be analytic at  $z_0$  but is analytic at some point in each neighbourhood of  $z_0$ , then  $z_0$  is a singular point of  $f$ .

**Definition 6.2** If a function  $f$  fails to be analytic at  $z_0$  but is analytic at each  $z$  in  $0 < |z - z_0| < \delta$  for some  $\delta$ , then  $f$  is said to be an isolated singular point of  $f$ .

**Example 6.1**  $f(z) = 1/z$  has an isolated singularity at 0.

**Example 6.2**  $f(z) = 1/\sin(\pi z)$  has isolated singularities at  $z = 0, \pm 1, \dots$

**Example 6.3**  $f(z) = 1/\sin(\pi/z)$  has isolated singularities at  $z = 1/n$  for integral  $n$ , also has a singularity at  $z = 0$ .

**Example 6.4**  $f(z) = \log z$  all points of negative  $x$ -axis are singular.

## Types of singularities

If a function  $f$  has an isolated singularity at  $z_0$  then  $\exists$  a  $\delta$  such that  $f$  is analytic at all points in  $0 < |z - z_0| < \delta$ . Then  $f$  must have a Laurent series expansion about  $z_0$ . The part  $\sum_{n=1}^{\infty} b_n (z - z_0)^{-n}$  is called the principal part of  $f$ .

1. If there are infinite nonzero  $b_i$  in the principal part then  $z_0$  is called an essential singularity of  $f$ .
2. If for some integer  $m$ ,  $b_m \neq 0$  but  $b_i = 0$  for all  $i > m$  then  $z_0$  is called a pole of order  $m$  of  $f$ . If  $m = 1$  then it is called a simple pole.
3. If all  $b$ s are zero then  $z_0$  is called a removable singularity.



**Centurion  
UNIVERSITY**

*Shaping Lives...  
Empowering Communities...*

## Residues

Suppose a function  $f$  has an isolated singularity at  $z_0$ , then there exists a  $\delta > 0$  such that  $f$  is analytic for all  $z$  in deleted nbd  $0 < |z - z_0| < \delta$ . Then  $f$  has a Laurent series representation

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}.$$

The coefficient

$$b_1 = \frac{1}{2\pi i} \int_C f(z) dz$$

where  $C$  is any contour in the deleted nbd, is called the residue of  $f$  at  $z_0$ .

**Example 6.6**  $f(z) = \frac{1}{z - z_0}$ . Then  $\int_C f(z) dz = 2\pi i b_1 = 2\pi i$  if  $C$  contains  $z_0$ , otherwise 0.

**Example 6.7**  $f(z) = \frac{1}{z(z - z_0)^2}$ . Show  $\int_C f(z) dz = -\frac{\pi i}{8}$  if  $C : |z - 2| = 1$ .

**Example 6.8**  $f(z) = z \exp\left(\frac{1}{z}\right)$ . Show  $\int_C f(z) dz = \pi i$  if  $C : |z| = 1$ .

**Example 6.9**  $f(z) = \exp\left(\frac{1}{z^2}\right)$ . Show  $\int_C f(z) dz = 0$  if  $C : |z| = 1$  even though it has a singularity at  $z = 0$ .

**Theorem 6.1** If a function  $f$  is analytic on and inside a positively oriented contour  $C$ , except for a finite number of points  $z_1, z_2, \dots, z_k$  inside  $C$ , then

$$\int_C f(z) dz = 2\pi i \sum_{i=1}^k \text{Res } f(z_i).$$

**Example 6.10** Show that  $\int \frac{5z-2}{z(z-1)} dz = 10\pi i$

## Residues of Poles

**Theorem 6.2** If a function  $f$  has a pole of order  $m$  at  $z_0$  then

$$\text{Res } f(z_0) = \lim_{z \rightarrow z_0} \left[ \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} ((z - z_0)^m f(z)) \right].$$

**Example 6.11**  $f(z) = \frac{1}{z - z_0}$ . Simple pole  $\text{Res } f(z_0) = 1$ .



Centurion  
UNIVERSITY

Shaping Lives...  
Empowering Communities...

**Example 6.12**  $f(z) = \frac{1}{z(z-z_0)^4}$ . Simple pole at  $z = 0$ .  $\text{Res } f(0) = 1/16$ . Pole of order 4 at  $z = 2$ .  $\text{Res } f(2) = -1/16$ .

**Example 6.13**  $f(z) = \frac{\cos z}{z^2(z-\pi)^3}$ .  $\text{Res } f(0) = -3/\pi^4$ .  $\text{Res } f(\pi) = -(6 - \pi^2)/2\pi^4$ .

## Quotients of Analytic Functions

**Theorem 6.3** If a function  $f(z) = \frac{P(z)}{Q(z)}$ , where  $P$  and  $Q$  are analytic at  $z_0$ , then

1.  $f$  is singular at  $z_0$  iff  $Q(z_0) = 0$ .
2.  $f$  has a simple pole at  $z_0$  if  $Q'(z_0) \neq 0$ . Then residue of  $f$  at  $z_0$  is  $P(z_0)/Q'(z_0)$ .



# Singularities

## Isolated singularities

Removable singularities

$$\frac{\sin(z)}{z}, \frac{1 - \cos(z)}{z^2}$$

Poles of finite order

$$\frac{1}{z}, \frac{1}{z^2}, \frac{1}{(z-1)^m},$$

$$\frac{2z+3}{(z-1)^2(z+2)}$$

Isolated essential singularities  
(pole of infinite order)

$$\sin\left(\frac{1}{z}\right), e^{1/z}$$

These are each discussed in more detail next.



Essential singularities can be distinguished from poles by the fact that they cannot be removed by multiplying by a factor of finite value.

Example:

$$w = e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2! z^2} + \dots + \frac{1}{n! z^n} + \dots \leftarrow \text{infinite at the origin}$$

We try to remove the singularity of the function at the origin by multiplying  $z^p$

$$z^p w = z^p + z^{p-1} + \frac{z^{p-2}}{2!} + \dots + \frac{z^{p-n}}{n!} + \dots$$

It consists of a finite number of positive powers of  $z$ , followed by an infinite number of negative powers of  $z$ .

All terms are positive

$$\text{As } z \rightarrow 0, \quad z^p w \rightarrow \infty$$

It is impossible to find a finite value of  $p$  which will remove the singularity in  $e^{1/z}$  at the origin.

The singularity is “essential”.



# Singularities

## Examples of singularities:

(These will be discussed in more detail later.)

*If expanded about  
the singularity, we  
can have:*

T = Taylor

L = Laurent

N = Neither

T  $\frac{\sin(z)}{z}$  removable singularity at  $z = 0$

L  $\frac{1}{(z - z_s)^p}$  pole of order  $p$  at  $z = z_s$  ( if  $p = 1$ , pole is a *simple pole*)

L  $e^{1/z}$  isolated essential singularity at  $z = z_0$  (pole of infinite order)

N  $\frac{1}{\sin\left(\frac{1}{z}\right)}$  non-isolated essential singularity  $z = 0$

N  $z^{1/2}$  branch point (*not* an isolated singularity)



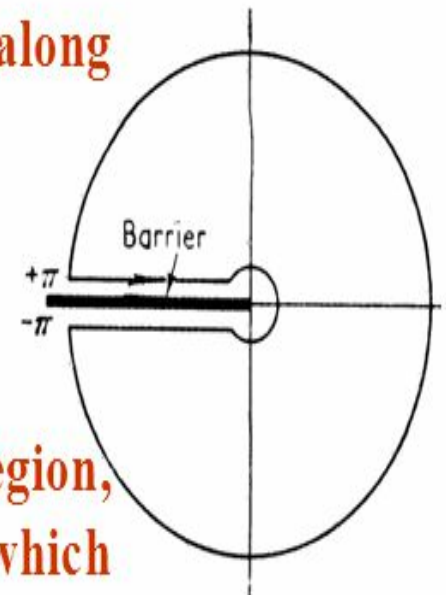


# Barrier - Branch Cut

The barrier must start from the branch point but it can go to infinity in any direction in the  $z$  plane, and may be either curved or straight.

In most normal applications, the barrier is drawn along the negative real axis.

- The branch is termed the “principle branch”.
- The barrier is termed the “branch cut”.
- For the example given in the previous slide, the region, the barrier confines the function to the region in which the argument of  $z$  is within the range  $-\pi < \theta < \pi$ .



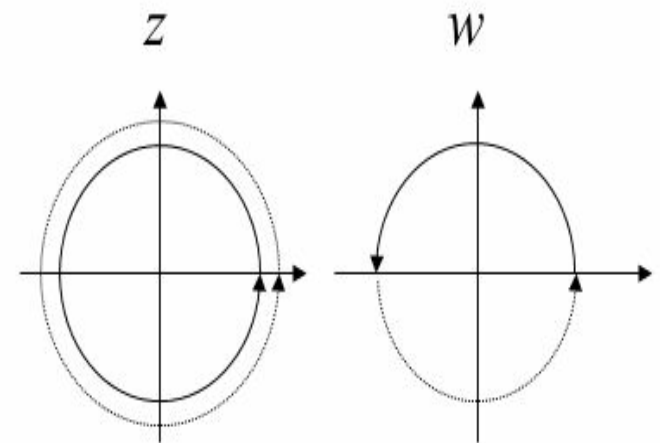




# Branch points

- The singularities described above arise from the non-analytic behaviour of single-valued functions.
- However, multi-valued functions frequently arise in the solution of engineering problems.
- For example:

$$\boxed{w = z^{\frac{1}{2}}} \xrightarrow{z = re^{i\theta}} \boxed{w = r^{\frac{1}{2}} e^{\frac{1}{2}i\theta}}$$



For any value of  $z$  represented by a point on the circumference of the circle in the  $z$  plane, there will be two corresponding values of  $w$  represented by points in the  $w$  plane.



# Laurent's Theorem

➔ **Laurent series** of a complex function  $f(z)$  is a representation of that function as a power series which includes terms of negative degree. It may be used to express complex functions in cases where a Taylor series expansion cannot be applied.

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - c)^n$$

$$C = z_0$$

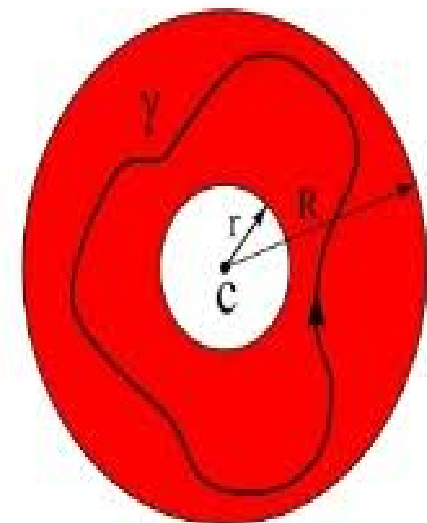
$$K = n$$

$$A = z_0$$

Also say that ,

$$f(z) = \sum_{k=0}^{\infty} a_k (z - a)^k + \sum_{k=1}^{\infty} b_k (z - a)^{-k},$$

$$a_n = \frac{1}{2\pi i} \oint_{\gamma} \frac{f(z) dz}{(z - c)^{n+1}}.$$





# Taylor's series

- **Taylor series** is a representation of a function as an infinite sum of terms that are calculated from the values of the function's derivatives at a single point.

$$f(x) =$$

$$f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \dots$$

Here,  $a = z_0$ , center  
 $x = z$  inside circle  $C$

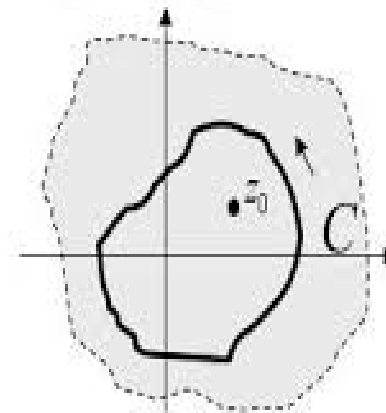


# The Residue Theorem

We saw that the integral of an analytic function  $f(z)$  over a closed curve  $C$  when  $f(z)$  has one singular point inside  $C$  is

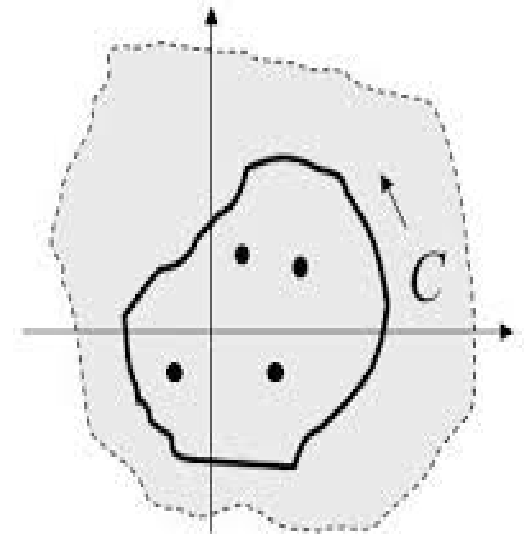
$$\oint_C f(z) dz = 2\pi j b_1$$

$b_1$  is the residue of  $f(z)$  at  $z_0$



Residue Theorem: Let  $f(z)$  be an analytic function inside and on a closed path  $C$  except for at  $k$  singular points inside  $C$ . Then

$$\oint_C f(z) dz = 2\pi j \sum_{i=1}^k \text{Res } f(z)_{z=z_i}$$





**Centurion**  
**UNIVERSITY**

*Shaping Lives...  
Empowering Communities...*

Numericals from various sources