

## FUNCTIONS CONTINUOUS ON METRIC SPACE

**Definition-:** let  $\langle M, \rho \rangle$  be metric space. If  $a \in M$  and  $r > 0$  then  $B[a, r]$  is defined as the set of all points in  $M$  whose distance to  $M$  is less than  $r$  or  $\rho(a, x) < r$ .  $B[a, r] = \{x \in M \mid \rho(a, x) < r\}$  Then  $B[a, r]$  is said to be an open ball radius  $r$  about  $a$ .

**Definition -:** The function  $f$  is said to be continuous at  $a \in M_1$  if 
$$\lim_{x \rightarrow a} f(x) = f(a)$$

**Theorem -:** The function  $f$  is continuous at  $a \in M_1$  if and only if any one of the following condition holds.

- Given  $\epsilon > 0$ ,  $\exists \delta > 0$  such that  $\rho_2(f(x), f(a)) < \epsilon$ , when  $\rho_1(x, a) < \delta$
- The inverse image under  $f$  of any  $B[f(a), \epsilon]$  about  $f(a)$  contains an open ball  $B[a, \delta]$  about  $a$ .
- Whenever the sequence  $\{x_n\}_{n=1}^{\infty}$  of points in  $M_1$  converging to  $a$  then the sequence  $f\{x_n\}_{n=1}^{\infty}$  in  $M_2$  convergent to  $f(a)$ .

**Theorem -:** let  $\langle M_1, \rho_1 \rangle, \langle M_2, \rho_2 \rangle$  and  $\langle M_3, \rho_3 \rangle$  be metric space and let  $f: M_1 \rightarrow M_2$ ,  $g: M_2 \rightarrow M_3$  if  $f$  is continuous at  $a \in M_1$  and  $g$  is continuous at  $f(a) \in M_2$  then  $g \circ f$  is continuous at  $a$ .

Proof-: Given  $f$  is continuous at  $a \in M$

$$\lim_{x \rightarrow a} f(x) = f(a)$$

Let  $\{x_n\}_{n=1}^{\infty}$  be a sequence of points in  $M_1$

For  $n > 0$ ,  $\exists \delta > 0$  such that

$$\rho_1(x_n, a) < \delta \text{ when } |x_n - a| < \delta$$

Or  $\rho_2(f(x_n), f(a)) < \epsilon$  when  $\rho_1(x_n, a)$

Let  $f\{x_n\}_{n=1}^{\infty} = y$  when  $|x_n - a| < \delta$

Again,  $g$  is continuous at  $f(a) = b \in M_2$

$$|g(y_n) - g(b)| < \epsilon \text{ when } |y_n - b| < \epsilon$$

Or  $|g(f(x_n)) - g(f(a))| < \epsilon$  when  $|y_n - b| < \epsilon$

$$\lim_{y_n \rightarrow f(a)} g(f(x_n)) = g(f(a))$$

Or  $\lim_{x \rightarrow a} g(f(x_n)) = g(f(a))$

Hence  $g \circ f$  is continuous at.

**Definition** -: let  $M_1$  and  $M_2$  be metric space and let  $f$  is a function  $f: M_1 \rightarrow M_2$  then  $f$  is said to be continuous from  $M_1$  into  $M_2$  if  $f$  is continuous at each point in  $M_1$ .

**OPEN SET** -: let  $M$  be metric space .consider  $G$  be the open subset that  $\forall x \in G, \exists a$  is  $r > 0$  such that the entire open ball  $B[x, r]$  contained in  $G$ .

**Q-: show that  $B[a, 1] = \{a\}$  is in  $R_d$**

Proof -:  $B[a, 1] = \{x \in R_d \mid d(a, x) < 1\}$

$$x \in R_d, d(a, x) = \begin{cases} 1 & \text{if } a \neq x \\ 0 & \text{if } a = x \end{cases}$$

$0 < 1$ , Hence  $a$  is the only one element in  $B[a, 1]$

**Q-: Single term set  $\{a\}$  is open in the metric space  $R_d$  whereas it is not open in  $R$ .**

1. Proof -: we know  $B[a, 1] = \{a\}$  in  $R_d$

$$\text{or } B[a, 1] \subset \{a\} \text{ in } R_d$$

Hence  $\{a\}$  is open set in  $R_d$

2.  $B[a, 1] = [1, -1]$  or the elements in the open interval  $[1 - 1]$

Hence  $B[a, 1]$  is not subset of  $\{a\}$  in  $R$

$$B[a, 1] \not\subset \{a\}$$

Hence  $\{a\}$  is not open set in  $R$

**Theorem -: for any metric space  $M$  both  $M$  and  $\emptyset$  are open.**

Proof -: for any  $x \in M$  we can construct an open ball  $B[x, r]$ , which is subset of  $M$

Hence  $M$  is an open set

We know  $\emptyset$  does not contain any element.

Hence it automatically satisfies the condition of open ball .

$\therefore \emptyset$  is open

**Theorem -; let  $F$  be any nonempty family of open subsets of a metric space  $M$ . Then  $\bigcup_{G \in F} G$  is also open subset of  $M$ .**

Proof -: let  $H = \bigcup_{G \in F} G$

**Case-1** -: let  $= \emptyset$  , as  $\emptyset$  has no element then automatically it satisfies all the condition of open set.

$\therefore$  It is open

**Case-2-:**  $H = \emptyset$

At least one element belongs to  $H$

i.e.  $x \in H$

$\Rightarrow x \in G$  (any)

We know  $G \in F$  that  $G$  is open

$\Rightarrow$  for  $r > 0$ ,  $\exists$  an open ball  $B[x, r] \subseteq G$

$\Rightarrow B[x, r] \subseteq G \subseteq H$

$\therefore B[x, r] \subseteq H$  hence  $H$  is open

**Theorem -:** Every subset of  $R_d$  is open

Proof -: let any  $G \subseteq R_d$

Case-1 if  $G$  is empty that  $G = \emptyset$

then automatically it satisfies all the condition of open set .

Hence  $G = \emptyset$  is open

Case-2 if  $\neq \emptyset$ , let  $x \in G$

We know that the single term set in  $R_d$  is open

Therefore any non-empty set in  $G$  is also open