

Scalar And Vector Quantities

Physical Quantity

Observables, which can be measured and represented by numbers, are called physical quantities.

Some Examples

- ▶ mass of a body,
- ▶ volume of a body,
- ▶ temperature at a point etc.

Some More Examples

- ▶ position of a particle,
- ▶ electric field at a point
- ▶ size of a cuboid,
- ▶ thermodynamic state (ρ, P, T)

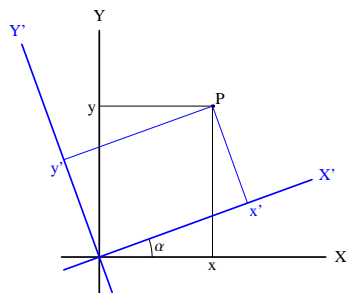
Scalar quantities have *magnitude* and vector quantities have *magnitude* as well as *direction*.

Vectors are represented by three real numbers.

Is a physical quantity represented by three numbers, a vector?

Alternate Definition: Vector Quantities

Two observers with different coordinate systems are shown in the figure.



The coordinates of a point P

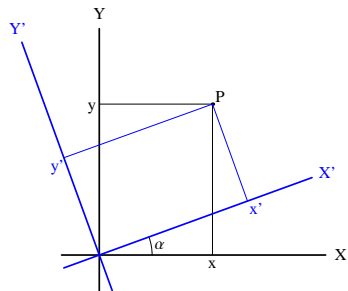
$$x' = x \cos \alpha + y \sin \alpha$$

$$y' = -x \sin \alpha + y \cos \alpha$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

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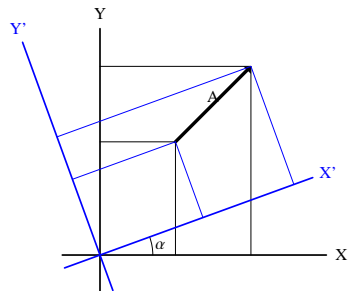


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Components of \mathbf{A} in two frames are related by

$$\begin{bmatrix} A'_x \\ A'_y \end{bmatrix} = \begin{bmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{bmatrix} \begin{bmatrix} A_x \\ A_y \end{bmatrix}$$

Alternate Definition: Vector Quantities

Definition

Suppose a physical quantity \mathbf{A} is represented by n components in n -dimensional space. If, under a coordinate transformation, the components of \mathbf{A} transform in the same way as coordinates, then the physical quantity \mathbf{A} is said to be a *vector quantity*.

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Scalar quantities do not depend on the coordinate transformations.

Size of cuboid, a state of a thermodynamic system are not vector quantities but are collections of three scalar quantities.

Scalar And Vector Functions of One Variable

Typically, in Physics, scalar and vector quantities are functions of time.
Examples are:

- ▶ Temperature of a body
- ▶ Amount of Radioactive material etc.
- ▶ Position of a particle
- ▶ Forces on a particle

We are quite familiar with scalar functions and calculus of such functions.
Calculus of vector valued functions is also familiar to us.

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Example

Suppose $\mathbf{r} : \mathbb{R} \rightarrow \mathbb{R}^3$ is position of a particle as a function of time. Let $\mathbf{r}(t) = x(t)\hat{\mathbf{x}} + y(t)\hat{\mathbf{y}} + z(t)\hat{\mathbf{z}}$. Then, velocity of the particle is given by

$$\mathbf{v}(t) = \frac{d}{dt}\mathbf{r}(t) = \frac{dx}{dt}(t)\hat{\mathbf{x}} + \frac{dy}{dt}(t)\hat{\mathbf{y}} + \frac{dz}{dt}(t)\hat{\mathbf{z}}.$$

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- ▶ Height of a location from sea level,
- ▶ Density of a fluid,
- ▶ Force Fields (Gravitational, Electrostatic),
- ▶ Potential energy of a force field etc.

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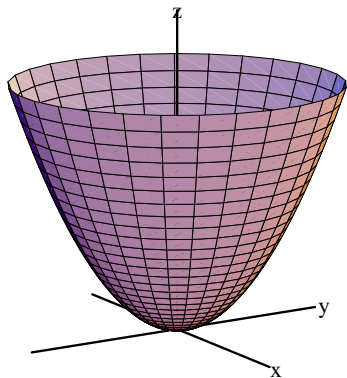
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Mathematically, these are functions from \mathbb{R}^m to \mathbb{R}^n , with $m > 1$. If $n = 1$, these functions are called scalar fields otherwise are called vector fields.

Scalar Fields (2D)



$$f(x, y) = x^2 + y^2$$

Scalar Fields (2D)

Definition

Let $f : \mathbb{R}^m \rightarrow \mathbb{R}$. Then, $S = \{a \in \mathbb{R}^m | f(a) = c\}$ is called a level set.

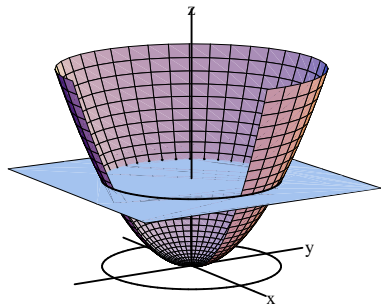
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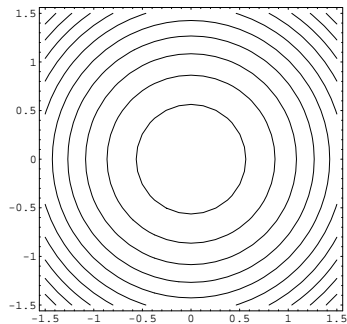
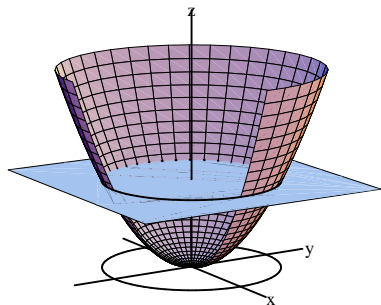


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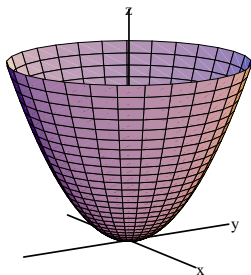
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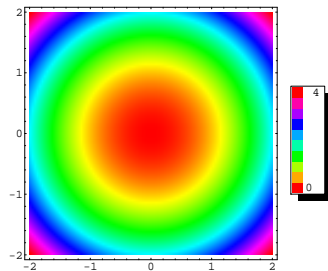
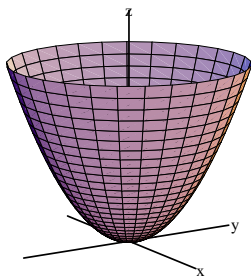


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Scalar Fields (3D)

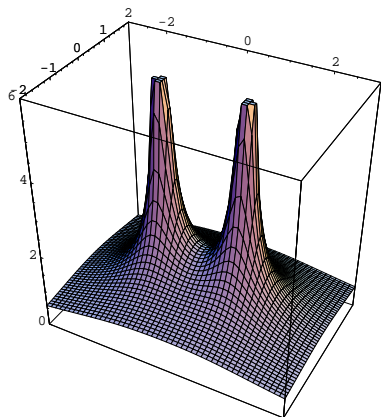
Two identical point charges at $(1, 0, 0)$ and $(-1, 0, 0)$. The potential in xy -plane

$$V(x, y, z = 0) = \frac{Q}{4\pi\epsilon_0} \left(\frac{1}{\sqrt{(x-1)^2 + y^2}} + \frac{1}{\sqrt{(x+1)^2 + y^2}} \right)$$

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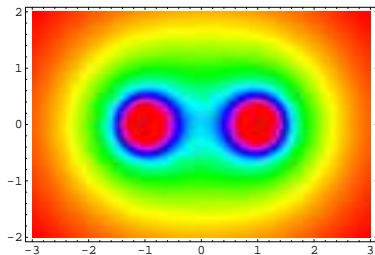
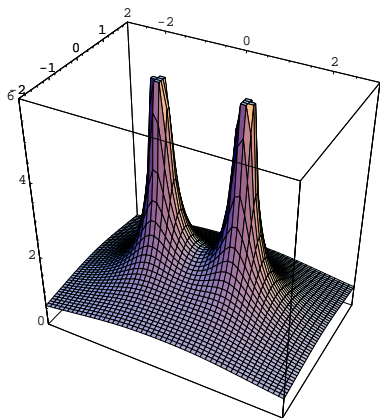
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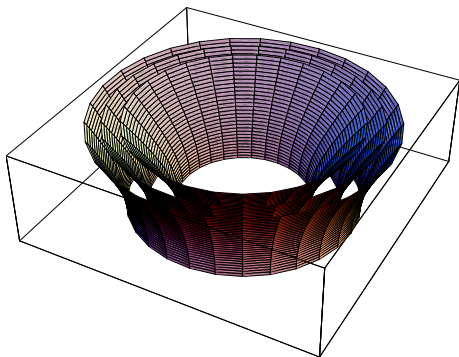
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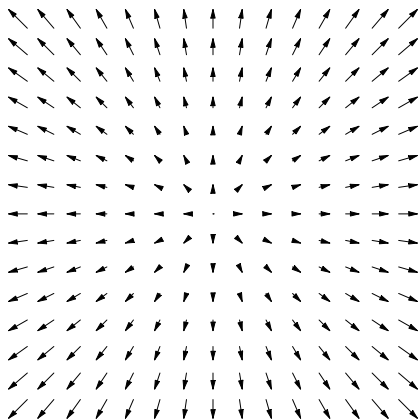
Scalar Fields (3D)



Level surfaces

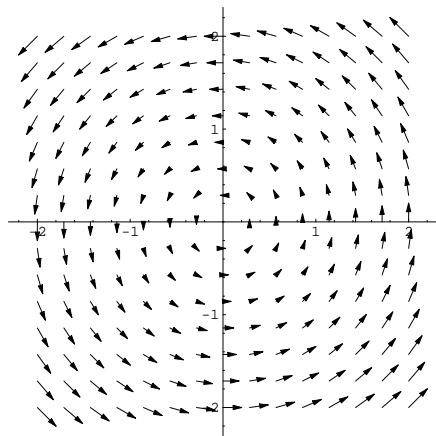
$$f(x, y, z) = x^2 + y^2 - z^2 = c$$

Vector Fields (2D)



$$f(x, y) = x\hat{x} + y\hat{y}$$

Vector Fields (2D)

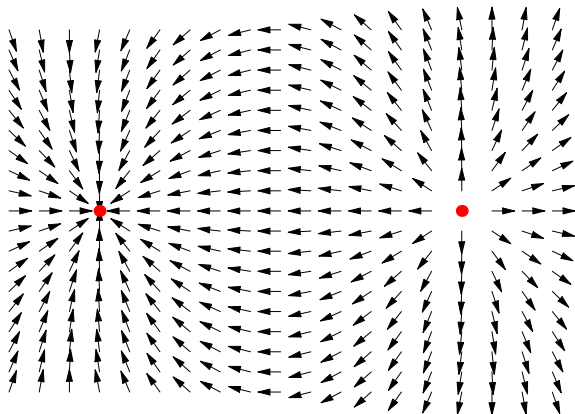


$$f(x, y) = -y\hat{x} + x\hat{y}$$

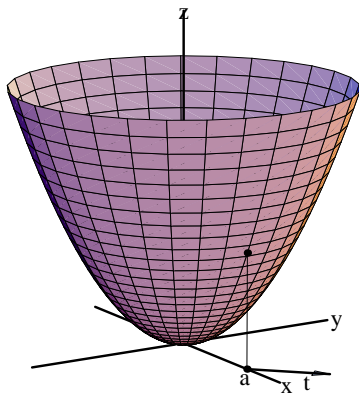
Vector Fields (2D)

Two opposite point charges at $(1, 0, 0)$ and $(-1, 0, 0)$. The electric field in xy -plane

$$\mathbf{E}(x, y) = \frac{Q}{4\pi\epsilon_0} \left(\frac{(x-1)\hat{x} + y\hat{y}}{((x-1)^2 + y^2)^{3/2}} - \frac{(x+1)\hat{x} + y\hat{y}}{((x+1)^2 + y^2)^{3/2}} \right)$$



Directional Derivative and Partial Derivatives



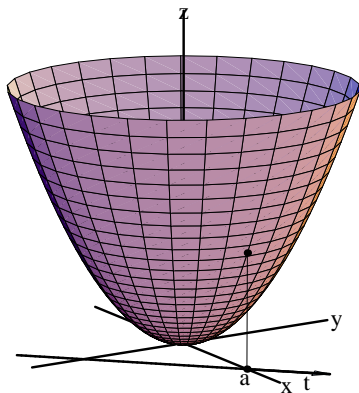
$$f(x, y) = x^2 + y^2$$

$$\mathbf{a} = (1, 0)$$

$$\mathbf{t} = (1, 1)/\sqrt{2}$$

Consider function $f(x, y) = x^2 + y^2$

Directional Derivative and Partial Derivatives



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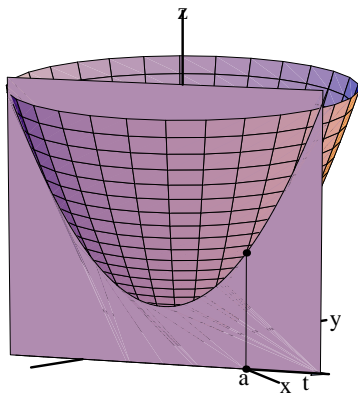
$$\mathbf{t} = (1, 1)/\sqrt{2}$$

Equation of line: $y = x - 1$.

Or parametric equation: $x(h) = 1 + \frac{1}{\sqrt{2}}h$ and $y(h) = \frac{1}{\sqrt{2}}h$.

Or $(x, y) = \mathbf{a} + h\mathbf{t}$.

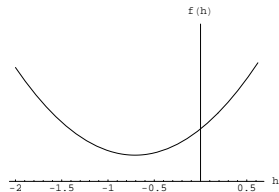
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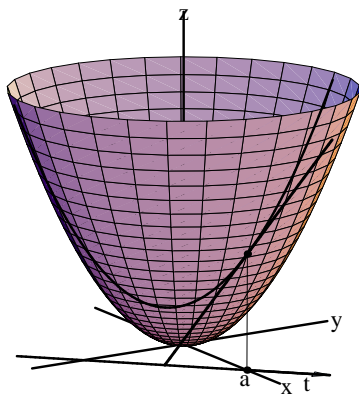
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Write: $f(h) = f(x(h), y(h)) = f(\mathbf{a} + h\mathbf{t}) = 1 + \sqrt{2}h + h^2$

$$f'(\mathbf{a}; \mathbf{t}) = \frac{df}{dh}(h=0) = \sqrt{2}$$

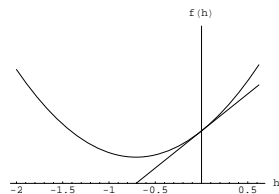
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Equation of the tangent to $f(h)$: $1 + \sqrt{2}h$

Equation of the tangent to $f(x, y)$: $x = 1 + h/\sqrt{2}$, $y = h/\sqrt{2}$, $z = \sqrt{2}h$

Directional Derivative and Partial Derivatives

- ▶ Let $f(x, y) = x^2 + y^2$. Let $\mathbf{a} = (1, 0)$ and $\mathbf{t} = (1, 1)/\sqrt{2}$.
- ▶ Line parallel to \mathbf{t} and passing through \mathbf{a} :

$$x = 1 + \frac{1}{\sqrt{2}}h \quad \text{and} \quad y = \frac{1}{\sqrt{2}}h.$$

with $(x, y) = \mathbf{a}$ at $h = 0$.

- ▶ Restriction of function f along this line:

$$f(h) = f(x(h), y(h)) = 1 + \sqrt{2}h + h^2$$

- ▶ Directional derivative

$$f'(\mathbf{a}; \mathbf{t}) = \frac{df}{dh}(h=0) = \sqrt{2}$$

- ▶ Equation of the tangent to $f(h)$: $y = 1 + \sqrt{2}h$.
- ▶ Equation of the tangent to $f(x, y)$: $x = 1 + t/\sqrt{2}$, $y = t/\sqrt{2}$, $z = \sqrt{2}t$

Directional Derivative and Partial Derivatives

Definition

Let $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ be a scalar field. Let $\mathbf{a} \in \mathbb{R}^3$ be a point in space. Suppose $\mathbf{t} \in \mathbb{R}^3$ be a unit vector. Then

$$\lim_{h \rightarrow 0} \frac{f(\mathbf{a} + h\mathbf{t}) - f(\mathbf{a})}{h}$$

is called *directional derivative* of f at \mathbf{a} in the direction of \mathbf{t} and is denoted by $f'(\mathbf{a}; \mathbf{t})$. (If the limit exists.)

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The directional derivative in any one of the coordinate unit vectors, that is, either $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$ or $\hat{\mathbf{z}}$, is also called *partial derivative*. Partial derivatives are usually denoted by

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Example

In previous example: $f(x, y) = x^2 + y^2$ and $\mathbf{a} = (1, 0)$ then

$$\frac{\partial}{\partial x} f(\mathbf{a}) = 2, \quad \frac{\partial}{\partial y} f(\mathbf{a}) = 0.$$

Differentiability

Notion of differentiability

If a function $f : \mathbb{R} \rightarrow \mathbb{R}$ is differentiable at a then

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In other words, we can approximate the graph of f by a straight line near a .

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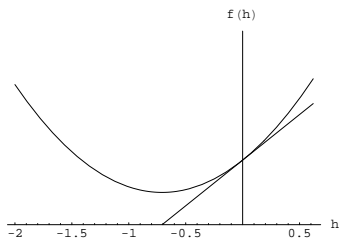
Example

In earlier example,

$f(x) = 1 + \sqrt{2}x + x^2$ and $a = 0$.

Thus $f(0) = 1$ and $f'(0) = \sqrt{2}$ Thus

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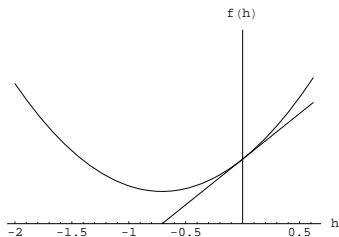
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A function is differentiable only if the first order Taylor expansion is valid.

Total Derivative

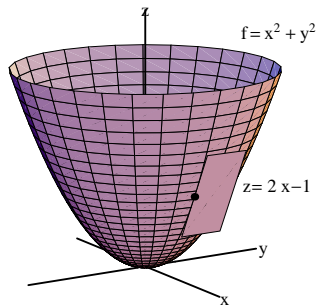
Differentiability for functions of two variables

If a function of two variables is to be differentiable, one must be able to approximate its graph by a plane rather than a line.

Example

Consider $f(x, y) = x^2 + y^2$ and $\mathbf{a} = (1, 0)$. Figure shows a plane given by a function $z(x, y) = 1 + 2(x - 1)$. Here is comparison

$$\begin{array}{ll} f(1.1, 0) = 1.21 & z(1.1, 0) = 1.20, \\ f(1, 0.1) = 1.01 & z(1, 0.1) = 1.00. \end{array}$$



Total Derivative

Definition

Let $f : \mathbb{R}^m \rightarrow \mathbb{R}$ be a scalar field. Let $\mathbf{a} \in \mathbb{R}^m$ be a point in space. f is said to be differentiable at \mathbf{a} if there exists a linear function $T_{\mathbf{a}} : \mathbb{R}^m \rightarrow \mathbb{R}$ such that

$$f(\mathbf{a} + \mathbf{v}) = f(\mathbf{a}) + T_{\mathbf{a}}(\mathbf{v}) + |\mathbf{v}| R_{\mathbf{a}}(\mathbf{v}).$$

where $R_{\mathbf{a}}(\mathbf{v}) \rightarrow 0$ as $\mathbf{v} \rightarrow 0$. The linear transformation $T_{\mathbf{a}}$ is called the derivative of f at \mathbf{a} .

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Linear Transformation $m = 2$

Let $\mathbf{v} = (x, y) \in \mathbb{R}^2$. A linear transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}$ can always be written as

$$T(\mathbf{v}) = t_x x + t_y y$$

where $t_x, t_y \in \mathbb{R}$. That is, one can always find a vector $\mathbf{t} = (t_x, t_y)$ such that

$$T(\mathbf{v}) = \mathbf{t} \cdot \mathbf{v}$$

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where $\mathbf{t} \in \mathbb{R}^m$ is some constant vector.

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Example

$f(x, y) = x^2 + y^2$, $\mathbf{a} = (1, 0)$, $\mathbf{v} = (x, y)$. Then

$$\begin{aligned} f(\mathbf{a} + \mathbf{v}) &= f(x + 1, y) = (x + 1)^2 + y^2 = 1 + 2x + 0y + x^2 + y^2 \\ &= 1 + (2, 0) \cdot \mathbf{v} + |\mathbf{v}|^2 \end{aligned}$$

Thus $T_{\mathbf{a}}(\mathbf{v}) = (2, 0) \cdot \mathbf{v}$. And remember $\frac{\partial}{\partial x} f(\mathbf{a}) = 2$, $\frac{\partial}{\partial y} f(\mathbf{a}) = 0$.

Gradient

Definition

Let $f : \mathbb{R}^m \rightarrow \mathbb{R}$ be a scalar field. Let $\mathbf{a} \in \mathbb{R}^m$ be a point in space. f is said to be differentiable at \mathbf{a} if there exists a linear function $T_{\mathbf{a}} : \mathbb{R}^m \rightarrow \mathbb{R}$ such that

$$f(\mathbf{a} + \mathbf{v}) = f(\mathbf{a}) + T_{\mathbf{a}}(\mathbf{v}) + |\mathbf{v}| R_{\mathbf{a}}(\mathbf{v}).$$

where $R_{\mathbf{a}}(\mathbf{v}) \rightarrow 0$ as $\mathbf{v} \rightarrow 0$. The linear transformation $T_{\mathbf{a}}$ is called the derivative of f at \mathbf{a} .

Let $T_{\mathbf{a}} = \mathbf{t} \cdot \mathbf{v}$ where $\mathbf{t} \in \mathbb{R}^m$, the vector \mathbf{t} is called the **gradient of f at \mathbf{a}** and is denoted by $\nabla f(\mathbf{a})$.

Properties of Gradient

Theorem

If $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ is a differentiable at \mathbf{a} with total derivative $T_{\mathbf{a}}$, then all partial derivatives exist and

$$T_{\mathbf{a}}(\mathbf{v}) = \left(\frac{\partial}{\partial x} f(\mathbf{a}), \frac{\partial}{\partial y} f(\mathbf{a}), \frac{\partial}{\partial z} f(\mathbf{a}) \right) \cdot \mathbf{v}$$

that is

$$\nabla f(\mathbf{a}) = \left(\frac{\partial}{\partial x} f(\mathbf{a}), \frac{\partial}{\partial y} f(\mathbf{a}), \frac{\partial}{\partial z} f(\mathbf{a}) \right).$$

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Example

$f(x, y) = x^2 + y^2$, $\mathbf{a} = (1, 0)$, $\mathbf{v} = (x, y)$. Then $T_{\mathbf{a}}(\mathbf{v}) = (2, 0) \cdot \mathbf{v}$. And $\frac{\partial}{\partial x} f(\mathbf{a}) = 2$, $\frac{\partial}{\partial y} f(\mathbf{a}) = 0$.

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The converse of the previous theorem does hold.

Example

$f(x, y) = \frac{xy^2}{x^2+y^2}$. Show that all partial derivatives of f exist at $(0, 0)$ but total derivative does not exist.

Properties of Gradient

Theorem

If $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ is a differentiable at \mathbf{a} , then directional derivatives $f'(\mathbf{a}; \mathbf{t})$ exist in all directions \mathbf{t} and

$$f'(\mathbf{a}; \mathbf{t}) = \nabla f(\mathbf{a}) \cdot \mathbf{t}$$

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If $\mathbf{t} = (1, 1)/\sqrt{2}$, then

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Also note:

$$|f'(\mathbf{a}; \mathbf{t})| = |\nabla f(\mathbf{a})| |\mathbf{t}| \cos \alpha \leq |\nabla f(\mathbf{a})|$$

- ▶ Change in f is maximum in the direction of gradient vector.

Properties of Gradient

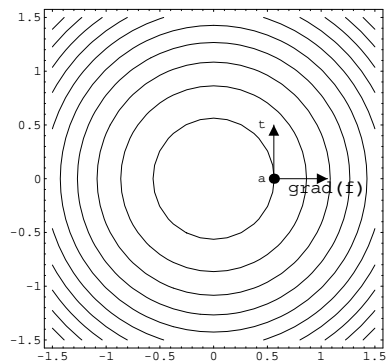
Geometric Interpretation

Gradient Vector is always perpendicular to level curves and level surfaces.

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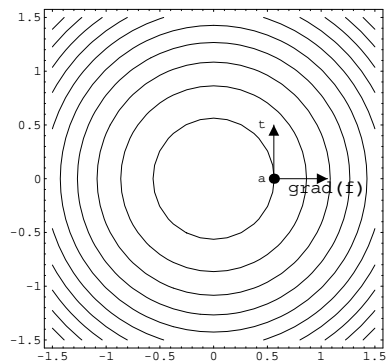
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Along the tangent direction to a level curve, directional derivative must be 0! Function value does not change in that direction. But then

$$f'(a; \mathbf{t}) = \nabla f(\mathbf{a}) \cdot \mathbf{t} = 0$$

Properties of Gradient

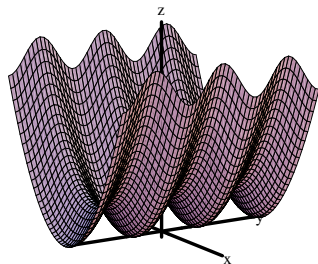
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$\nabla f(\mathbf{a}) = 0 \implies$ maximum or minimum or a saddle point.

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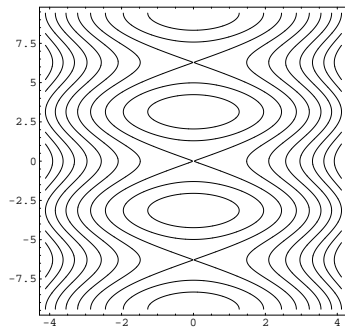
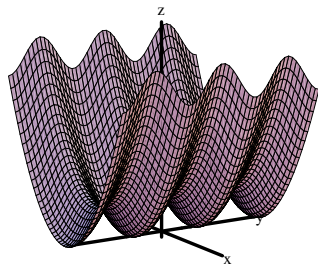
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Properties of Gradient

f and g are scalar fields and c is a real constant

- ▶ $\nabla(f + g)(a) = \nabla f(a) + \nabla g(a)$
- ▶ $\nabla(cf) = c\nabla f$
- ▶ $\nabla(fg) = f\nabla g + g\nabla f$
- ▶ If $\mathbf{r} = (x, y, z)$ and $r = \sqrt{x^2 + y^2 + z^2}$, then $\nabla(r^n) = nr^{n-2}\mathbf{r} = nr^{n-1}\hat{\mathbf{r}}$
- ▶ Chain Rule: $f : \mathbb{R} \rightarrow \mathbb{R}^n$ and $g : \mathbb{R}^n \rightarrow \mathbb{R}$ and $h = g \circ f$, then $h'(a) = \nabla g \cdot f'(a)$

Derivative of Vector Fields

Definition

Let $\mathbf{F} : \mathbb{R}^m \rightarrow \mathbb{R}^n$ be a vector field. Let $\mathbf{a} \in \mathbb{R}^m$ be a point in space. \mathbf{F} is said to be differentiable at \mathbf{a} if there exists a linear function $T_{\mathbf{a}} : \mathbb{R}^m \rightarrow \mathbb{R}^n$ such that

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Linear Transformation $m = n = 2$

Let $\mathbf{v} = (x, y) \in \mathbb{R}^2$. A linear transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ can always be written as

$$T(\mathbf{v}) = \begin{bmatrix} t_{11}x + t_{12}y \\ t_{21}x + t_{22}y \end{bmatrix} = \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

where $t_{11}, t_{12}, t_{21}, t_{22} \in \mathbb{R}$. That is, one can always find a matrix

$$\mathbf{t} = \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix} \text{ such that}$$

$$T(\mathbf{v}) = \mathbf{t} \cdot \mathbf{v}$$

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$$T_{\mathbf{a}}(\mathbf{v}) = \begin{bmatrix} \frac{\partial F_x}{\partial x} & \frac{\partial F_x}{\partial y} & \frac{\partial F_x}{\partial z} \\ \frac{\partial F_y}{\partial x} & \frac{\partial F_y}{\partial y} & \frac{\partial F_y}{\partial z} \\ \frac{\partial F_z}{\partial x} & \frac{\partial F_z}{\partial y} & \frac{\partial F_z}{\partial z} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Divergence and Curl of Vector Fields

Total derivative (which is like gradient) of a vector field is not very useful when it comes to application in science and engineering.

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If $\mathbf{A} = (A_x, A_y, A_z)$ is a differentiable vector field, then divergence of \mathbf{A} (denoted by $\nabla \cdot \mathbf{A}$) is defined as

$$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

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$$\nabla \times \mathbf{A} = \hat{\mathbf{x}} \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) + \hat{\mathbf{y}} \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) + \hat{\mathbf{z}} \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right)$$

Gradient as an operator

It is useful to treat the symbol ∇ as a vector operator, that is when it operates on a scalar field and produces a vector field. It is conveniently written as

$$\nabla = \hat{\mathbf{x}} \frac{\partial}{\partial x} + \hat{\mathbf{y}} \frac{\partial}{\partial y} + \hat{\mathbf{z}} \frac{\partial}{\partial z}$$