

Introduction

- In previous chapters, \mathbf{E} was determined by Coulomb's law or Gauss law when charge distribution is known, or $\mathbf{E} = -\nabla V$ when potential V is known throughout the region.
- In most practical applications, however, neither the charge distribution nor the potential distribution is known.
- In this chapter, practical electrostatic problems are considered where only electrostatic conditions (charge & potential) at some boundaries are known, and \mathbf{E} and \mathbf{V} are to be found.
- Such problems are tackled using **Poisson's** or **Laplace's** equation or the method of images.

6.2 Poisson's and Laplace's Equations

- They are easily derived from Gauss's law.

$$\nabla \cdot \mathbf{D} = \nabla \cdot \epsilon \mathbf{E} = \rho_V$$

$$\mathbf{E} = -\nabla V$$

$$\nabla \cdot (-\epsilon \nabla V) = \rho_V \quad \text{for an inhomogeneous medium}$$

$$\boxed{\nabla^2 V = \frac{-\rho_V}{\epsilon}} \quad \text{for homogeneous medium (Poisson's equation)}$$

when $\rho_V = 0$, (for charge free region):

$$\boxed{\nabla^2 V = 0} \quad \text{(Laplace's equation)}$$

Laplace's Equation

Cartesian

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

$$\nabla^2 V = 0$$

Cylindrical

$$\frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial V}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

Spherical

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial V}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial V}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 V}{\partial \phi^2} = 0$$

Poisson's Equation

Cartesian

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = -\frac{\rho_V}{\epsilon}$$

$$\nabla^2 V = \frac{-\rho_V}{\epsilon}$$

Cylindrical

$$\frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial V}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} = -\frac{\rho_V}{\epsilon}$$

Spherical

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial V}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial V}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 V}{\partial \phi^2} = -\frac{\rho_V}{\epsilon}$$

Uniqueness Theorem

- There are several methods of solving a given problem (analytical, graphical, numerical, experimental, etc).
- The question is: If a solution of Laplace's equation satisfies a given set of boundary conditions, is this the only possible solution?
- The answer is **Yes**, there is only one solution, and it is unique.
- This is **uniqueness theorem**: If a solution of Laplace's equation can be found that satisfies the boundary conditions, then the solution is *unique*.

General Procedure for Solving Poisson's or Laplace's Equation

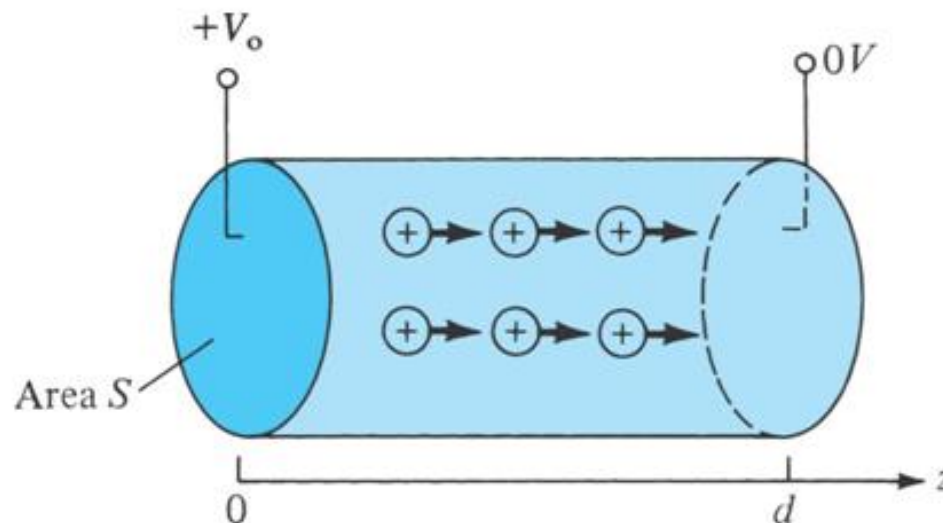
1. Solve **Laplace's** (if $\rho_V = 0$) or **Poisson's** (if $\rho_V \neq 0$) equation using:
 - (a) direct integration when **V** is a function of one variable, or
 - (b) separation of variables if **V** is a function of more than one variable.

The solution at this point is not unique but expressed in terms of unknown *integration constants* to be determined.

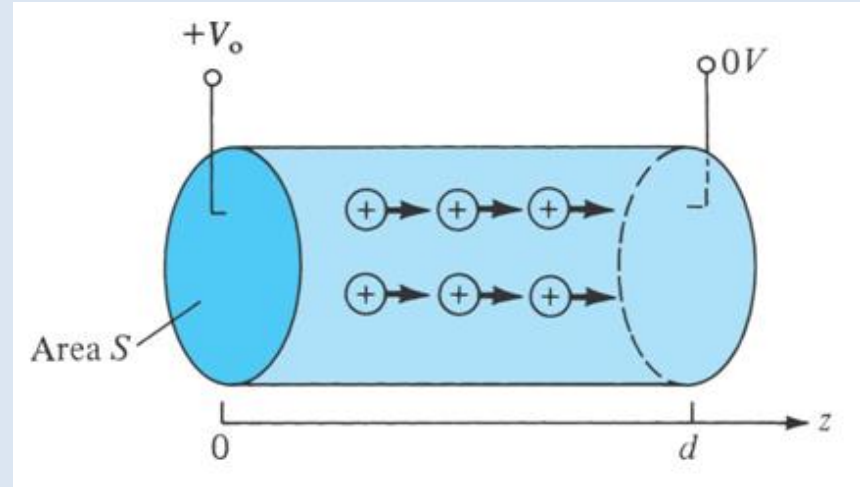
2. Apply the boundary conditions to determine a unique solution for **V**.
Imposing the given boundary conditions makes the solution unique.
3. Having obtained **V**, find **E** using $\mathbf{E} = -\nabla V$ and **D** using $\mathbf{D} = \epsilon \mathbf{E}$.

Example 6.1

Current-carrying components in high-voltage power equipment must be cooled to carry away the heat caused by ohmic losses. A means of pumping is based on the force transmitted to the cooling fluid by charges in an electric field. The electrohydrodynamic (EHD) pumping is modeled in the Figure. The region between the electrodes contains a uniform charge ρ_0 which is generated at the left electrode and collected at the right electrode. Calculate the pressure of the pump if $\rho_0 = 25 \text{ mC/m}^3$ and $V_0 = 22 \text{ kV}$.

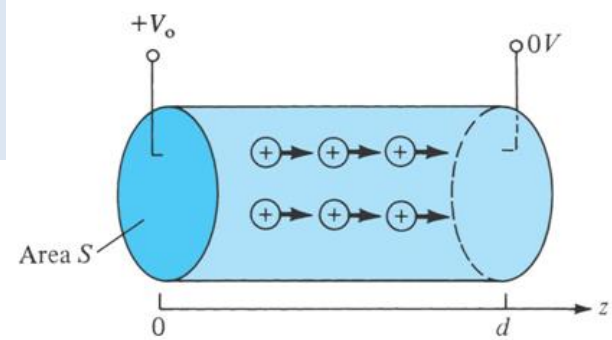


EHD pumps



- The technology promises to make it easier and more efficient to remove heat from small spaces, vastly expanding the capabilities of advanced instruments and microprocessors.
- Unlike traditional thermal-control technologies that rely on mechanical pumps and other moving parts, EHD cooling uses electric fields to pump coolant through tube.
- Electrodes apply the voltage that pushes the coolant through tube.
- EHD-based systems are lightweight and consume little power.

Example 6.1 - Solution



Since $\rho_V \neq 0$, we apply Poisson's equation

$$\nabla^2 V = \frac{-\rho_V}{\epsilon}$$

the boundary conditions $V(z=0)=V_0$ and $V(z=d)=0$ show that V depends only on z ,

$$\text{Hence } \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial V}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} = -\frac{\rho_V}{\epsilon} \quad \text{becomes: } \frac{\partial^2 V}{\partial z^2} = -\frac{\rho_0}{\epsilon}$$

$$\text{Integrating once gives } \frac{dV}{dz} = \frac{-\rho_0 z}{\epsilon} + A$$

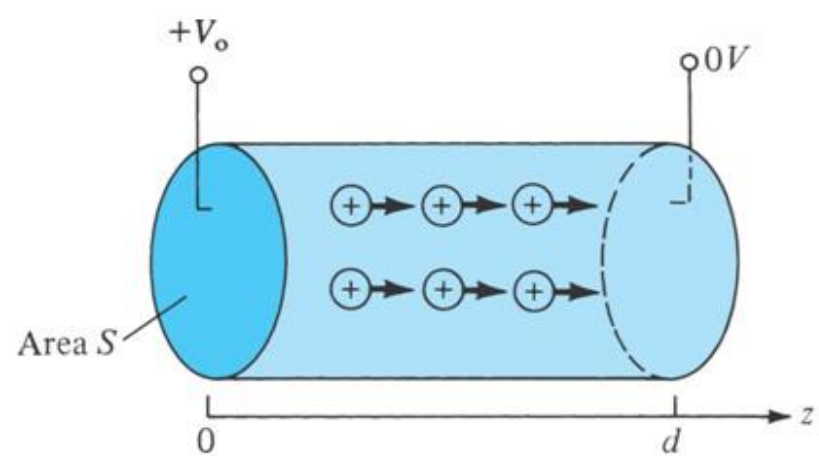
$$\text{Integrating again gives } V = \frac{-\rho_0 z^2}{2\epsilon} + Az + B$$

where A and B are integration constants to be determined by applying the boundary conditions.

$$\text{when } z = 0, V = V_0 \rightarrow V_0 = 0 + 0 + B \rightarrow B = V_0$$

$$\text{when } z = d, V = 0 \rightarrow 0 = \frac{-\rho_0 d^2}{2\epsilon} + Ad + V_0 \quad \text{or} \quad A = \frac{\rho_0 d}{2\epsilon} - \frac{V_0}{d}$$

Example 6.1 - Continued



The electric field is given by

$$\mathbf{E} = -\nabla V = -\left(\frac{\partial V}{\partial \rho} \mathbf{a}_\rho + \frac{1}{\rho} \frac{\partial V}{\partial \phi} \mathbf{a}_\phi + \frac{\partial V}{\partial z} \mathbf{a}_z \right)$$

$$\mathbf{E} = -\frac{\partial V}{\partial z} \mathbf{a}_z = \left(\frac{\rho_0 z}{\epsilon} - A \right) \mathbf{a}_z = \left[\frac{V_0}{d} + \frac{\rho_0}{\epsilon} \left(z - \frac{d}{2} \right) \right] \mathbf{a}_z$$

The net force is $\mathbf{F} = Q_T \mathbf{E}$, $Q_T = \int_v \rho_v dv$

$$\rightarrow \mathbf{F} = \int \rho_v \mathbf{E} dv = \rho_0 \int dS \int_{z=0}^d \mathbf{E} dz$$

$$= \rho_0 S \left[\frac{V_0 z}{d} + \frac{\rho_0}{2\epsilon} (z^2 - dz) \right] \Big|_0^d \mathbf{a}_z \rightarrow \mathbf{F} = \rho_0 S V_0 \mathbf{a}_z$$

Force per unit area (Pressure) is $\frac{F}{S} = \rho_0 S V_0 = 25 \times 10^{-3} \times 22 \times 10^3 = 550 \text{ N/m}^2$

$$V = \frac{-\rho_0 z^2}{2\epsilon} + Az + B$$

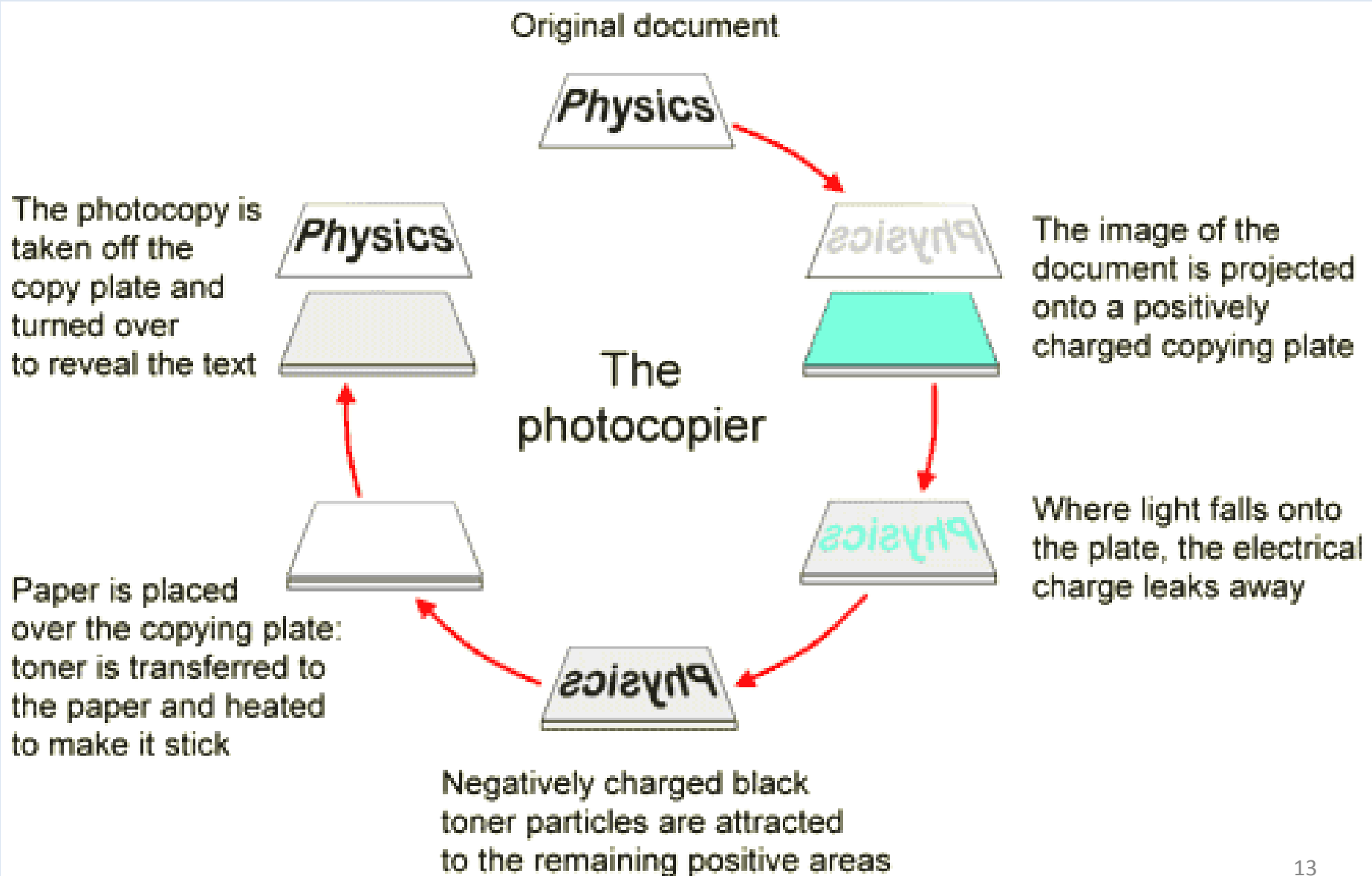
$$B = V_0$$

$$A = \frac{\rho_0 d}{2\epsilon} - \frac{V_0}{d}$$

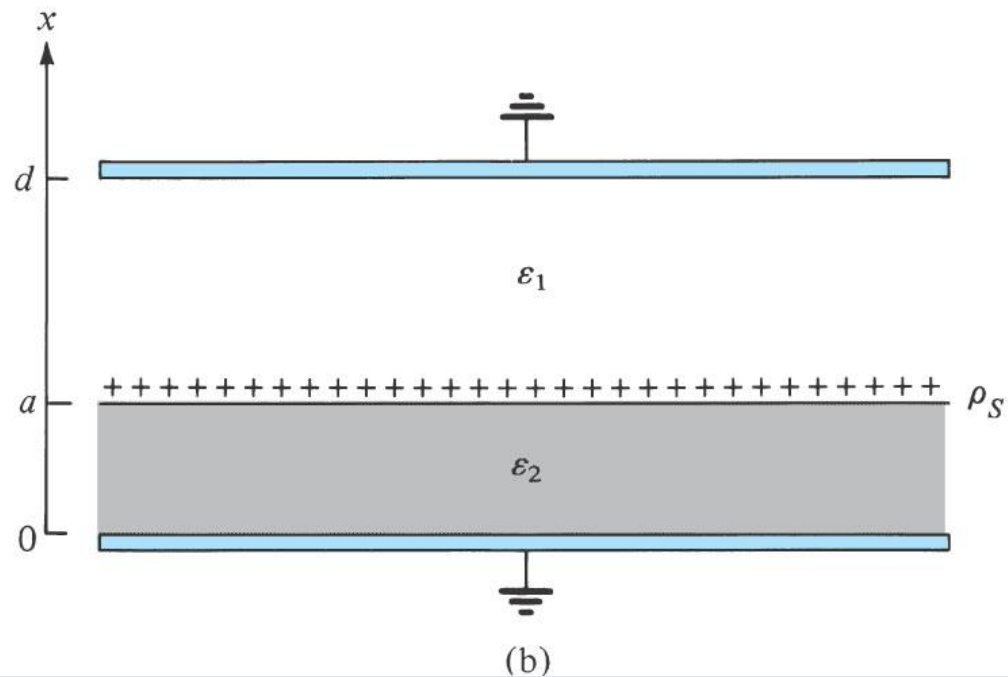
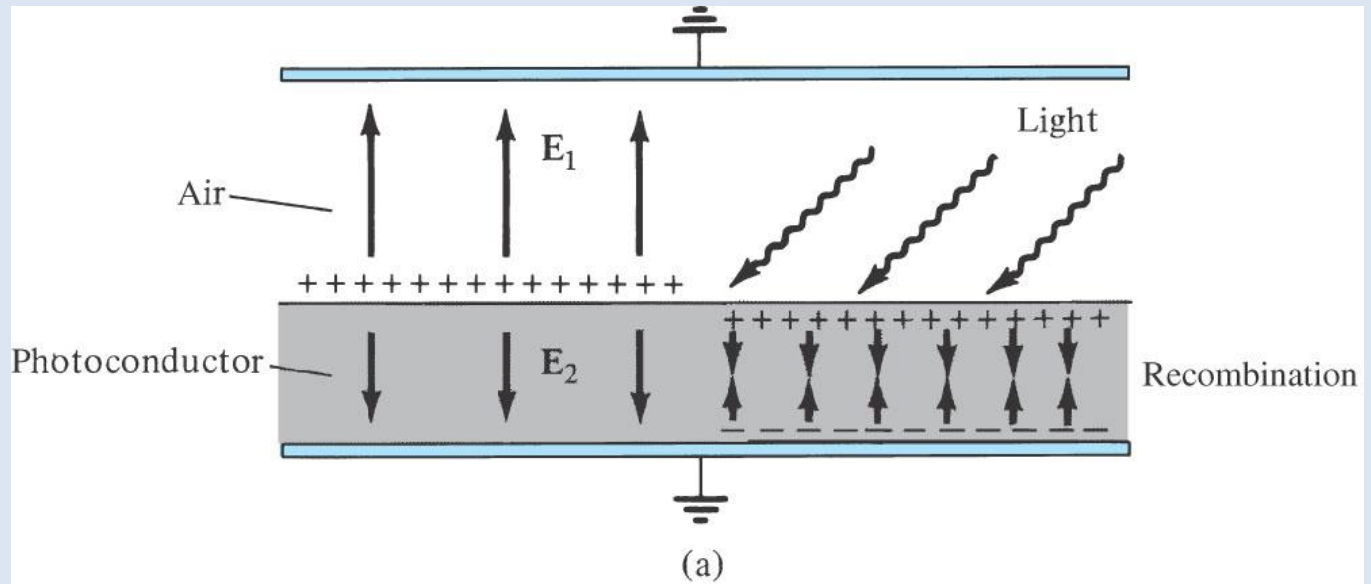
Example 6.2

The xerographic copying machine is an important application of electrostatics. The surface of the photoconductor is initially charged uniformly as in Figure 6.2(a). When light from the document to be copied is focused on the photoconductor, the charges on the lower surface combine with those on the upper surface to neutralize each other. The image is developed by pouring a charged black powder over the surface of the photoconductor. The electric field attracts the charged powder, which is later transferred to paper and melted to form a permanent image. We want to determine the electric field below and above the surface of the photoconductor.

Example 6.2



Example 6.2



Example 6.2 - Solution

Since $\rho_V = 0$, we apply Laplace's equation.

Also potential depends only on x .

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} = 0$$

Integrate twice gives

$$V = Ax + B$$

Let the potentials above $x=a$ be V_1 , and below $x=a$ be V_2 .

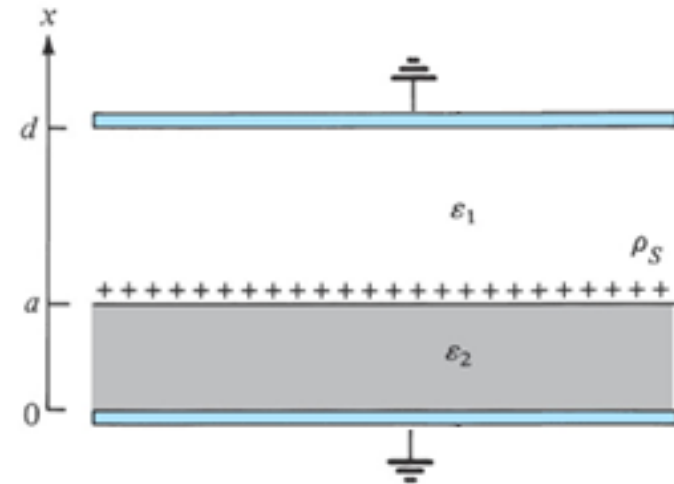
$$V_1 = A_1 x + B_1 \quad x > a$$

$$V_2 = A_2 x + B_2 \quad x < a$$

The boundary conditions at the grounded electrodes are,

$$V_1(x=d)=0 \rightarrow 0 = A_1 d + B_1 \rightarrow \boxed{B_1 = -A_1 d} \dots (1)$$

$$V_2(x=0)=0 \rightarrow 0 = 0 + B_2 \rightarrow \boxed{B_2 = 0} \dots (2)$$



Example 6.2 - Continued

At the surface of the photoconductor,

$$V_1(x=a) = V_2(x=a) \rightarrow \boxed{A_1 a + B_1 = A_2 a} \dots (3)$$

$$D_{1n} - D_{2n} = \rho_S \Big|_{x=a} \rightarrow \varepsilon_1 E_{1n} - \varepsilon_2 E_{2n} = \rho_S$$

$$\text{Since } \mathbf{E} = -\nabla V = -\left(\frac{dV}{dx} \mathbf{a}_x + \frac{dV}{dy} \mathbf{a}_y + \frac{dV}{dz} \mathbf{a}_z \right) \rightarrow E_n = -\frac{dV}{dx}$$

$$\text{Therefore } -\varepsilon_1 \frac{dV_1}{dx} + \varepsilon_2 \frac{dV_2}{dx} = \rho_S \rightarrow \boxed{\rho_S = -\varepsilon_1 A_1 + \varepsilon_2 A_2} \dots (4)$$

Solve equations (1) to (4) to find the constants.

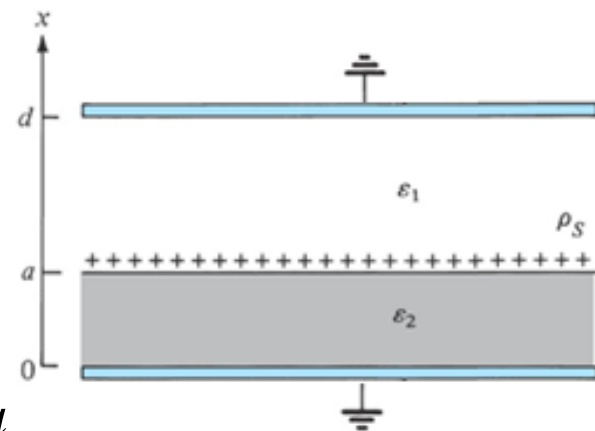
$$\mathbf{E}_1 = -\frac{dV_1}{dx} \mathbf{a}_x = -A_1 \mathbf{a}_x = \frac{\rho_S \mathbf{a}_x}{\varepsilon_1 \left[1 + \frac{\varepsilon_2 d}{\varepsilon_1 a} - \frac{\varepsilon_2}{\varepsilon_1} \right]}, \quad a \leq x \leq d$$

$$\mathbf{E}_2 = -\frac{dV_2}{dx} \mathbf{a}_x = -A_2 \mathbf{a}_x = \frac{-\rho_S \left(\frac{d}{a} - 1 \right) \mathbf{a}_x}{\varepsilon_1 \left[1 + \frac{\varepsilon_2 d}{\varepsilon_1 a} - \frac{\varepsilon_2}{\varepsilon_1} \right]}, \quad 0 \leq x \leq a$$

$$V_1 = A_1 x + B_1 \quad x > a$$

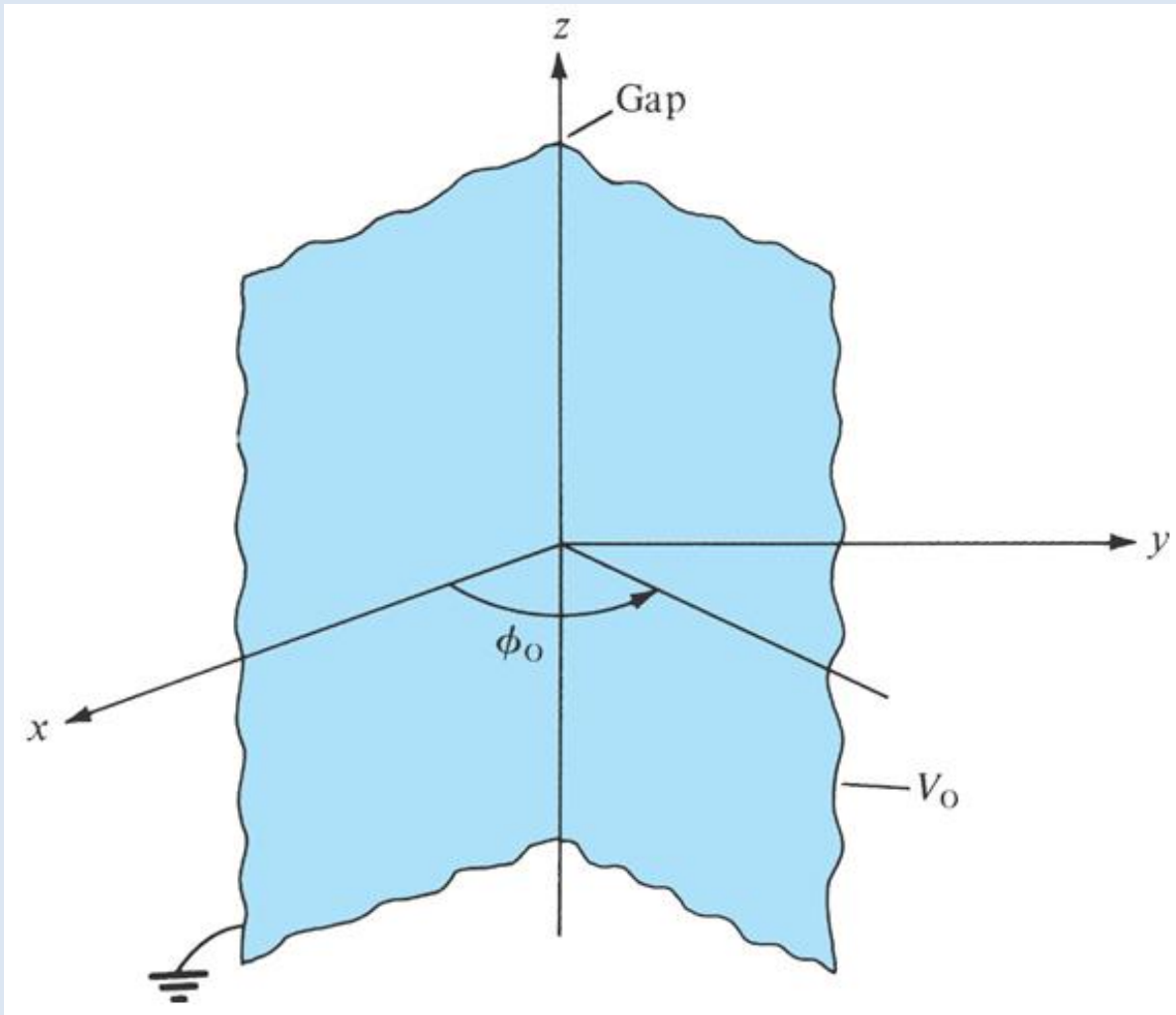
$$V_2 = A_2 x + B_2 \quad x < a$$

$$B_1 = -A_1 d, \quad B_2 = 0$$



Example 6.3

Semi-infinite conducting planes at $\Phi=0$ and $\Phi=\pi/6$ are separated by an infinitesimal insulating gap as in Figure 6.3. If $V(\Phi=0)=0$ and $V(\Phi=\pi/6)=100$ V, Calculate \mathbf{V} and \mathbf{E} in the region between the planes.



Example 6.3 Solution

Since V depends only on ϕ , Laplace's equation in cylindrical coordinates becomes

$$\nabla^2 V = 0 \rightarrow \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial V}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} = 0 \rightarrow \boxed{\nabla^2 V = \frac{1}{\rho^2} \frac{d^2 V}{d\phi^2} = 0}$$

since $\rho=0$ is excluded due to the insulating gap, we can multiply by ρ^2 to get $\frac{d^2 V}{d\phi^2} = 0$

Integrating twice gives $V=A\phi+B$.

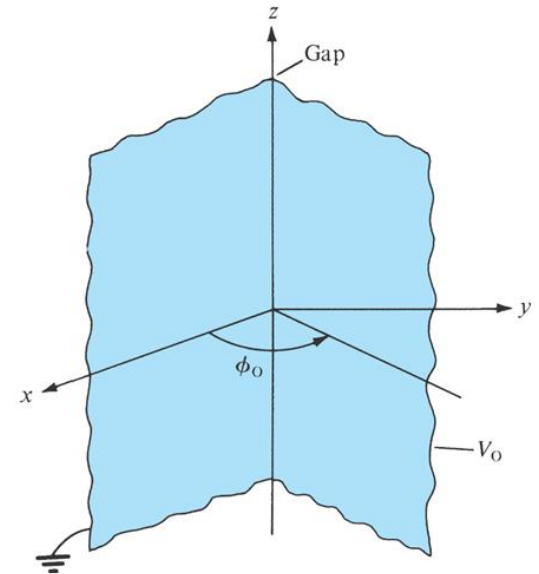
Apply boundary conditions:

when $\phi = 0, V = 0 \rightarrow 0=0+B \rightarrow B=0$

when $\phi = \phi_0, V = V_0 \rightarrow V_0=A\phi_0 \rightarrow A = \frac{V_0}{\phi_0}$, Hence

$$\boxed{V = \frac{V_0}{\phi_0} \phi}$$

$$E = -\nabla V = -\frac{1}{\rho} \frac{dV}{d\phi} a_\phi = -\frac{V_0}{\rho\phi_0} a_\phi$$

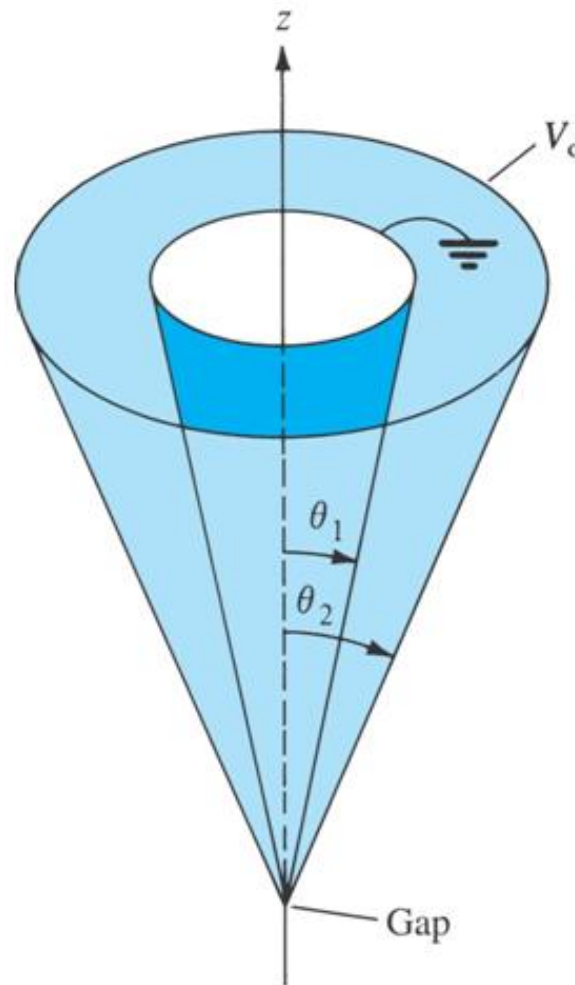


Substituting $V_0=100$ and $\phi_0 = \pi / 6$ gives: $V = \frac{600}{\pi} \phi$ and $E = \frac{-600}{\pi\rho} a_\phi$

check $V(\phi = 0)=0$, $V(\phi = \pi / 6)=100$, $\nabla^2 V = 0$

Example 6.4

Two conducting cones ($\theta = \pi/10$ and $\theta = \pi/6$) of infinite extent are separated by an infinitesimal gap at $r = 0$. If $V(\theta = \pi/10) = 0$ and $V(\theta = \pi/6) = 50$ V, Find \mathbf{V} and \mathbf{E} between the cones.



Example 6.4 Solution

Since V depends only on θ , Laplace's equation in spherical

coordinates becomes:
$$\nabla^2 V = \frac{1}{r^2 \sin \theta} \frac{d}{d\theta} \left[\sin \theta \frac{dV}{d\theta} \right] = 0$$

since $r=0$ and $\theta=0, \pi$ are excluded, we can multiply by $r^2 \sin \theta$ to get

$$\frac{d}{d\theta} \left[\sin \theta \frac{dV}{d\theta} \right] = 0$$

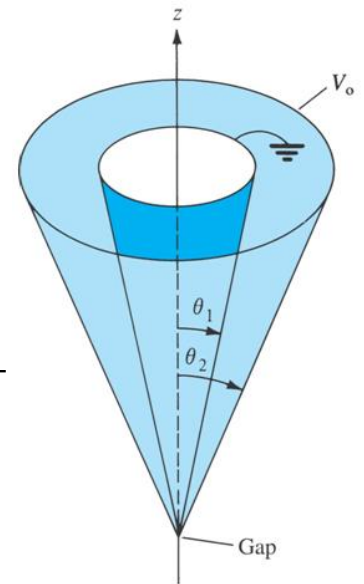
Integrating once gives:
$$\sin \theta \frac{dV}{d\theta} = A \rightarrow \frac{dV}{d\theta} = \frac{A}{\sin \theta}$$

Integrating this results in
$$V = A \int \frac{d\theta}{\sin \theta} = A \int \frac{d\theta}{2 \cos(\theta/2) \sin(\theta/2)}$$

$$V = A \int \frac{d\theta}{2 \cos(\theta/2) \sin(\theta/2)} \times \frac{\cos(\theta/2)}{\cos(\theta/2)} = A \int \frac{1}{2} \frac{\sec^2(\theta/2) d\theta}{\tan(\theta/2)}$$

Let $u = \tan \theta / 2$, $du = 1/2 \sec^2(\theta/2) d\theta$

$$\rightarrow V = A \int \frac{du}{u} = A \ln(u) + B \rightarrow \boxed{V = A \ln(\tan \theta / 2) + B}$$



Example 6.4 Continued

$$V = A \ln(\tan \theta / 2) + B$$

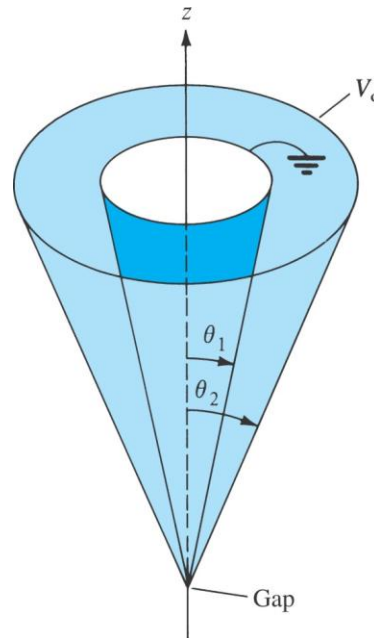
Apply boundary conditions to determine the integration constants A and B .

$$V(\theta = \theta_1) = 0 \rightarrow 0 = A \ln(\tan \theta_1 / 2) + B \rightarrow B = -A \ln(\tan \theta_1 / 2)$$

$$\text{Hence } V = A \ln(\tan \theta / 2) - A \ln(\tan \theta_1 / 2) \rightarrow V = A \ln \left[\frac{\tan \theta / 2}{\tan \theta_1 / 2} \right]$$

$$\text{Also } V(\theta = \theta_2) = V_0 \rightarrow V_0 = A \ln \left[\frac{\tan \theta_2 / 2}{\tan \theta_1 / 2} \right] \rightarrow A = \frac{V_0}{\ln \left[\frac{\tan \theta_2 / 2}{\tan \theta_1 / 2} \right]}$$

$$\text{Thus } V = \frac{V_0 \ln \left[\frac{\tan \theta / 2}{\tan \theta_1 / 2} \right]}{\ln \left[\frac{\tan \theta_2 / 2}{\tan \theta_1 / 2} \right]}$$



Example 6.4 Continued

$$\frac{dV}{d\theta} = \frac{A}{\sin \theta}$$

$$E = -\nabla V = -\frac{1}{r} \frac{dV}{d\theta} a_\theta = -\frac{A}{r \sin \theta} a_\theta = -\frac{V_0}{r \sin \theta \ln \left[\frac{\tan \theta_2 / 2}{\tan \theta_1 / 2} \right]} a_\theta$$

Taking $\theta_1 = \frac{\pi}{10}$, $\theta_2 = \frac{\pi}{6}$, and $V_0 = 50$ gives

$$V = \frac{50 \ln \left[\frac{\tan \theta / 2}{\tan \pi / 20} \right]}{\ln \left[\frac{\tan \pi / 12}{\tan \pi / 20} \right]} = 95.1 \ln \left[\frac{\tan \theta / 2}{0.1584} \right] \quad V$$

and

$$E = -\frac{95.1}{r \sin \theta} a_\theta \quad \text{V/m}$$

Check: $V(\theta = \pi / 10) = 0$, $V(\theta = \pi / 6) = V_0$, $\nabla^2 V = 0$

$$V = A \ln \left[\frac{\tan(\theta / 2)}{\tan(\theta_1 / 2)} \right]$$

$$A = \frac{V_0}{\ln \left[\frac{\tan(\theta_2 / 2)}{\tan(\theta_1 / 2)} \right]}$$

$$V = \frac{V_0 \ln \left[\frac{\tan(\theta / 2)}{\tan \theta_1 / 2} \right]}{\ln \left[\frac{\tan(\theta_2 / 2)}{\tan(\theta_1 / 2)} \right]}$$

6.5 Resistance and Capacitance

- For conductor with uniform cross section, the resistance is

$$R = \frac{l}{\sigma S}$$

- If the cross section of the conductor is not uniform, the resistance is obtained from:

$$R = \frac{V}{I} = \frac{\int \mathbf{E} \cdot d\mathbf{l}}{\int \sigma \mathbf{E} \cdot d\mathbf{S}}$$

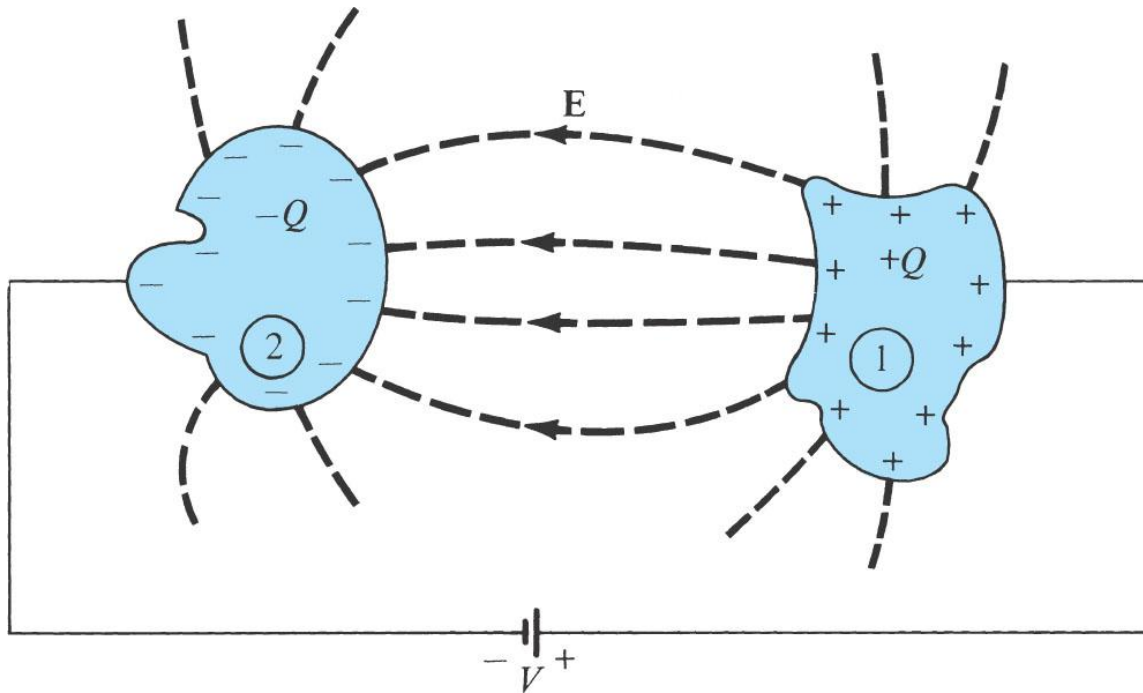
Resistance

The resistance R (or conductance $G=1/R$) of a conducting material can be found by the following steps:

1. Choose a suitable coordinate system.
2. Assume V_0 as the potential difference between conductor terminals.
3. Solve Laplace's equation $\nabla^2 V = 0$ to obtain V then determine E from $E = -\nabla V$ and find I from $I = \int \sigma E \cdot dS$.
4. Finally, obtain R as V_0/I .

Capacitance

- To have capacitance, we must have two (or more) conductors carrying **equal** but **opposite** charges.
- The conductors may be separated by free space or dielectric.



- The conductors are maintained at a potential difference:
- $V = -\int_2^1 \mathbf{E} \cdot d\mathbf{l}$ (note \mathbf{E} is always normal to conducting surfaces)

Capacitance

The capacitance **C** of the capacitor is defined as the ratio of the magnitude of the charge on one of the plates to the potential difference between them:

$$C = \frac{Q}{V} = \frac{\epsilon \int \mathbf{E} \cdot d\mathbf{S}}{\int \mathbf{E} \cdot d\mathbf{l}}$$

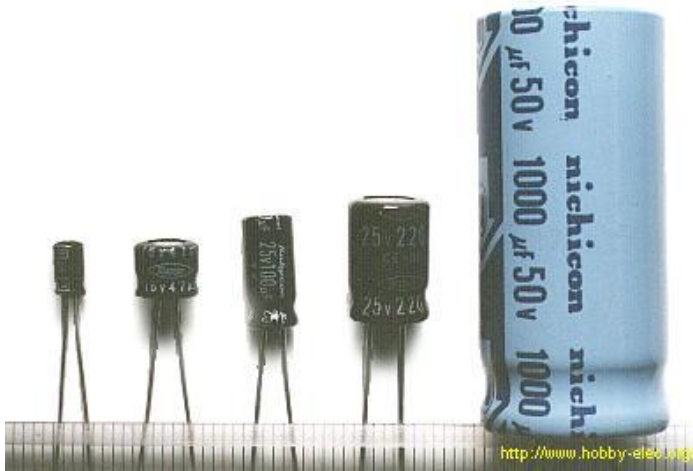
Obtain **C** for any two conductor capacitors by either:

1. Assume **Q** and determine **V** in terms of **Q** (involves **Gauss Law**).
2. Assume **V** and determine **Q** in terms of **V** (involves **Laplace's equation**)

Method (1) involves the following:

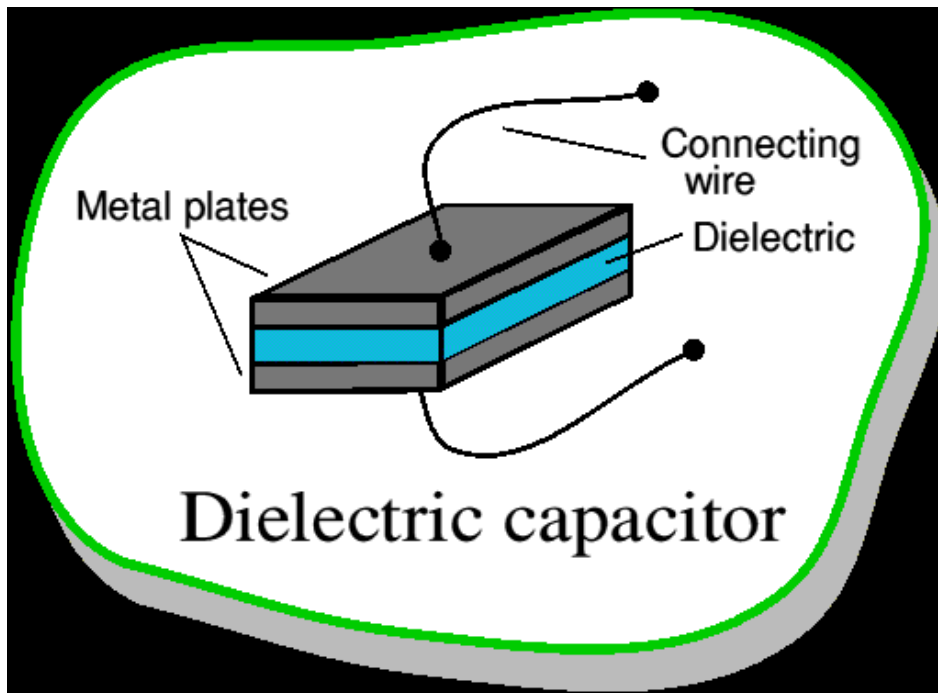
1. Choose a suitable coordinate system.
2. Let the two conducting plates carry charges +Q and -Q.
3. Determine **E** using Coulomb or Gauss law and find V from $V = -\int \mathbf{E} \cdot d\mathbf{l}$
4. Finally, obtain C from $C=Q/V$.

Capacitors



Storing Charges- Capacitors:

A capacitor consists of 2 conductors of any shape placed near one another without touching. It is common; to fill up the region between these 2 conductors with an insulating material called a dielectric. We charge these plates with opposing charges to set up an electric field.



Capacitance

or

Capacitance

Resistance

$$C = \frac{Q}{V}$$

↓

$$V = -\int_2^1 \mathbf{E} \cdot d\mathbf{l}$$

↓

$\mathbf{E} \rightarrow$ Gauss or
Coulombs law
(in terms of Q)

$$C = \frac{Q}{V_0}$$

$$Q = \oint \mathbf{D} \cdot d\mathbf{S}$$

↓

$$\mathbf{D} = \epsilon \mathbf{E}$$

↓

$$\mathbf{E} = -\nabla V$$

↓

$$\nabla^2 V = 0$$

or $\nabla^2 V = -\rho_V / \epsilon$

$$R = \frac{V_0}{I}$$

↓

$$I = \int \mathbf{J} \cdot d\mathbf{S}$$

↓

$$\mathbf{J} = \sigma \mathbf{E}$$

↓

$$\mathbf{E} = -\nabla V$$

↓

$$\nabla^2 V = 0$$

or $\nabla^2 V = -\rho_V / \epsilon$

Parallel Plate Capacitor

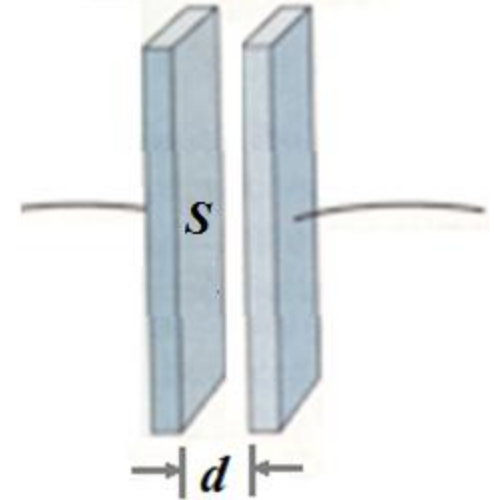
Suppose each plate has an area **S** and the plates are separated by distance **d**. Assume plates 1 and 2 carry uniformly distributed charges +Q and -Q

$$\rho_s = \frac{Q}{S}, \quad D = -\rho_s a_x \quad \text{or} \quad E = \frac{\rho_s}{\epsilon} (-a_x) = -\frac{Q}{\epsilon S} a_x$$

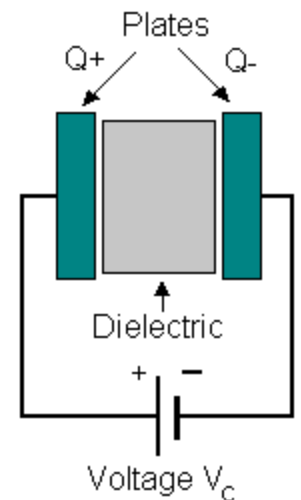
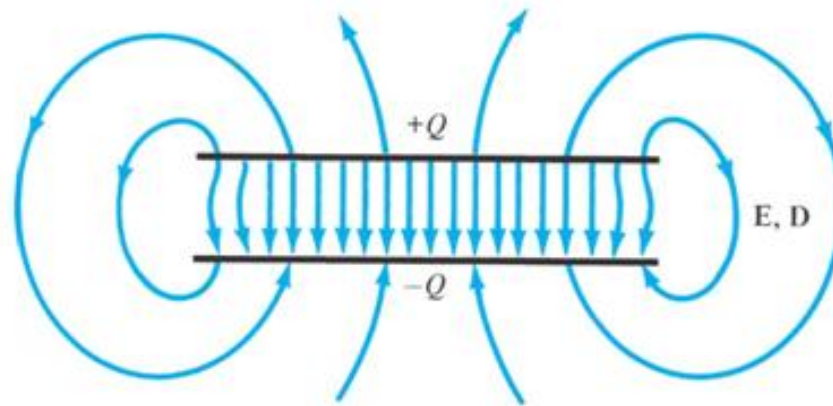
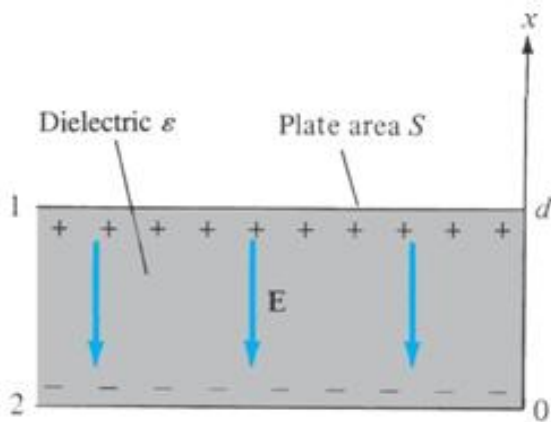
$$\text{Hence } V = -\int_2^1 \mathbf{E} \cdot d\mathbf{l} = -\int_0^d \left[-\frac{Q}{\epsilon S} a_x \right] \cdot dx a_x = \frac{Qd}{\epsilon S}$$

and thus for a parallel-plate capacitor

$$C = \frac{Q}{V} = \frac{\epsilon S}{d}$$



The energy stored in a capacitor is $W_E = \frac{1}{2} CV^2 = \frac{1}{2} QV = \frac{Q^2}{2C}$



Coaxial Capacitor

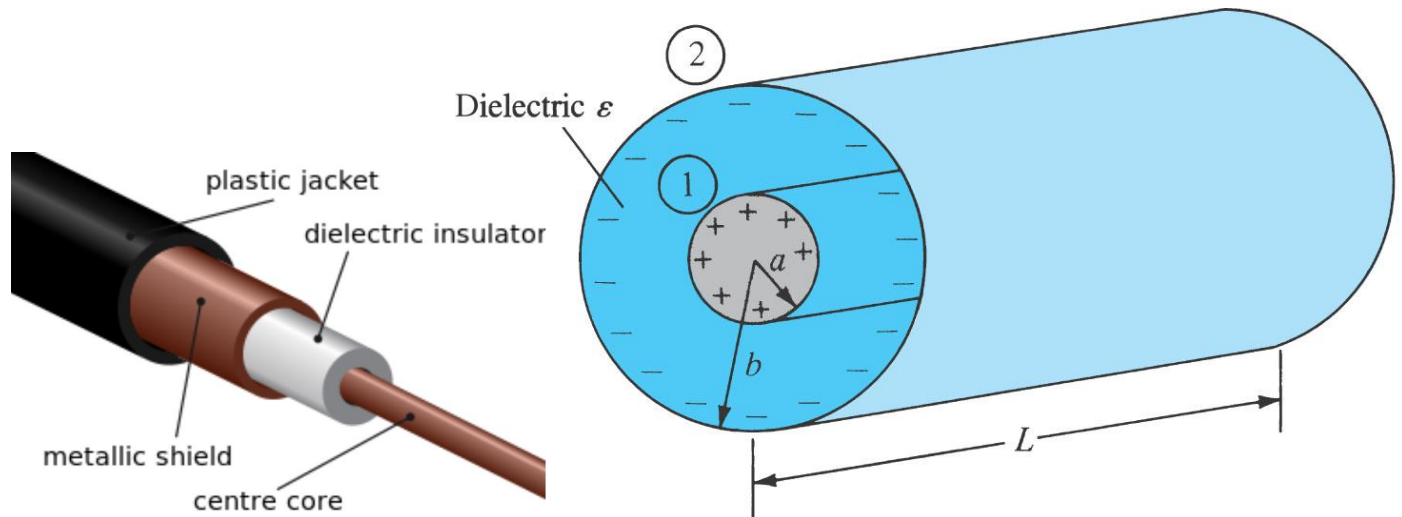
Assume Conductors 1 and 2 carry uniformly distributed charges $+Q$ and $-Q$. Apply Gauss law to arbitrary Gaussian cylindrical surface of radius ρ such that $a < \rho < b$,

$$Q = \epsilon \oint \mathbf{E} \cdot d\mathbf{S} = \epsilon E_{\rho} 2\pi\rho L \rightarrow \mathbf{E} = \frac{Q}{2\pi\epsilon\rho L} \mathbf{a}_{\rho}$$

$$V = -\int_2^1 \mathbf{E} \cdot d\mathbf{l} = -\int_b^a \left[\frac{Q}{2\pi\epsilon\rho L} \mathbf{a}_{\rho} \right] \cdot d\rho \mathbf{a}_{\rho} = \frac{Q}{2\pi\epsilon L} \ln \frac{b}{a}$$

Thus the capacitance of a coaxial cylinder is given by

$$C = \frac{Q}{V} = \frac{2\pi\epsilon L}{\ln \frac{b}{a}}$$



Spherical Capacitor

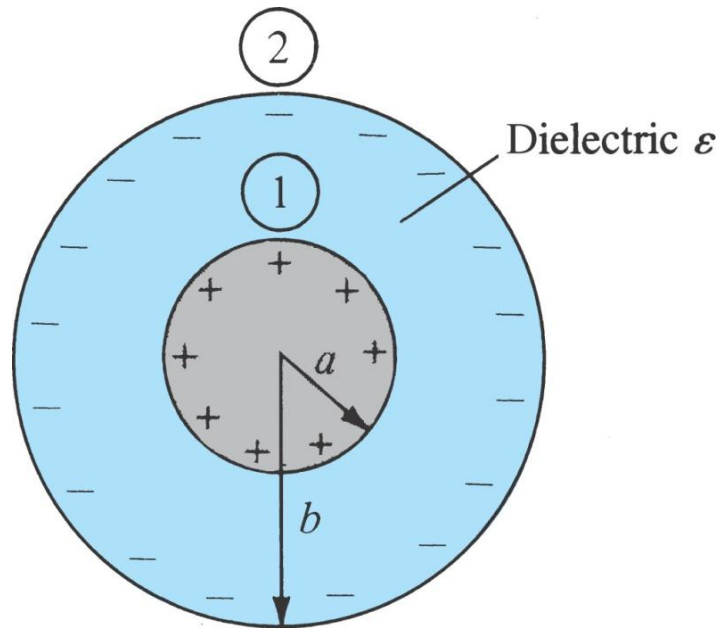
Assume charges $+Q$ and $-Q$ are on inner and outer spheres. Apply Gauss law to arbitrary Gaussian spherical surface of radius r such that $a < r < b$,

$$Q = \epsilon \oint \mathbf{E} \cdot d\mathbf{S} = \epsilon E_r 4\pi r^2 \rightarrow \mathbf{E} = \frac{Q}{4\pi\epsilon r^2} \mathbf{a}_r$$

$$V = -\int_2^1 \mathbf{E} \cdot d\mathbf{l} = -\int_b^a \left[\frac{Q}{4\pi\epsilon r^2} \mathbf{a}_r \right] \cdot dr \mathbf{a}_r = \frac{Q}{4\pi\epsilon} \left[\frac{1}{a} - \frac{1}{b} \right]$$

Thus the capacitance of a spherical capacitor is given by

$$C = \frac{Q}{V} = \frac{4\pi\epsilon}{\frac{1}{a} - \frac{1}{b}}$$



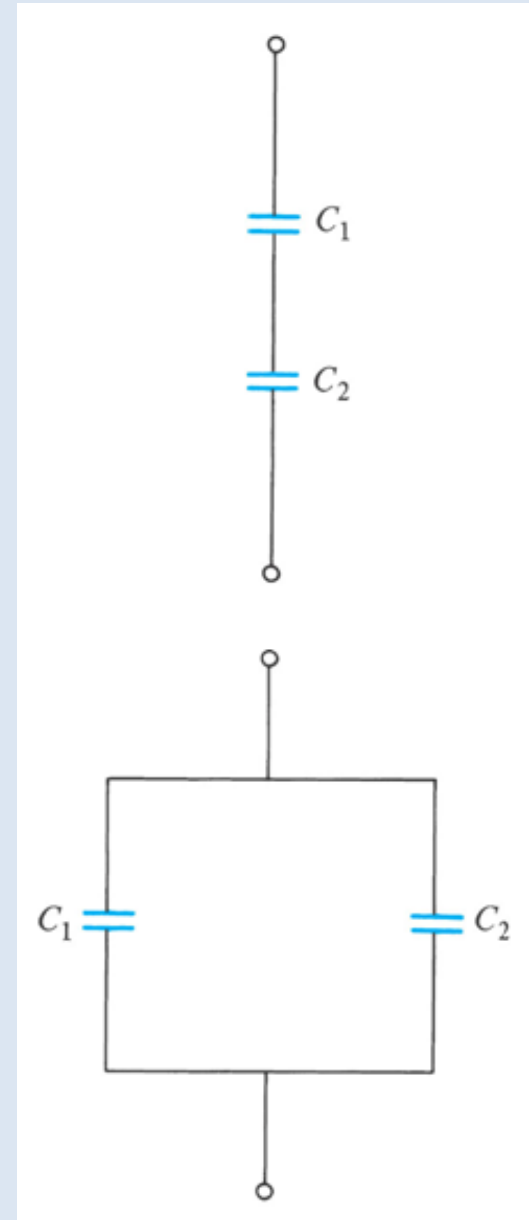
Series and Parallel Capacitors

•If two capacitors with capacitance C_1 and C_2 are in series (have same charge on them), the total capacitance is:

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} \quad \text{or} \quad C = \frac{C_1 C_2}{C_1 + C_2}$$

•If the capacitors are in Parallel (have same voltage across them), the total capacitance is:

$$C = C_1 + C_2$$



It has been shown that:

$$R = \frac{V}{I} = \frac{\int \mathbf{E} \cdot d\mathbf{l}}{\int \sigma \mathbf{E} \cdot d\mathbf{S}}$$

$$C = \frac{Q}{V} = \frac{\varepsilon \int \mathbf{E} \cdot d\mathbf{S}}{\int \mathbf{E} \cdot d\mathbf{l}}$$

The product of these expressions yields

$$\boxed{RC = \frac{\varepsilon}{\sigma}}$$

Which is the relaxation time T_r of the medium separating the conductors.

Assuming homogeneous media, the resistance of various capacitors mentioned earlier can be obtained using $\mathbf{RC = \epsilon/\sigma}$:

For a parallel-plate capacitor $C = \frac{\epsilon S}{d}$, $R = \frac{d}{\sigma S}$

For a cylindrical capacitor $C = \frac{2\pi\epsilon L}{\ln \frac{b}{a}}$, $R = \frac{\ln \frac{b}{a}}{2\pi\sigma L}$

For a spherical capacitor $C = \frac{4\pi\epsilon}{\frac{1}{a} - \frac{1}{b}}$, $R = \frac{\frac{1}{a} - \frac{1}{b}}{4\pi\sigma}$

Note: \mathbf{R} in these equations is **NOT** the resistance of the capacitor plate, but the leakage resistance between the plates. Therefore, σ is the conductivity of the dielectric medium separating the plates.

Example 6.8

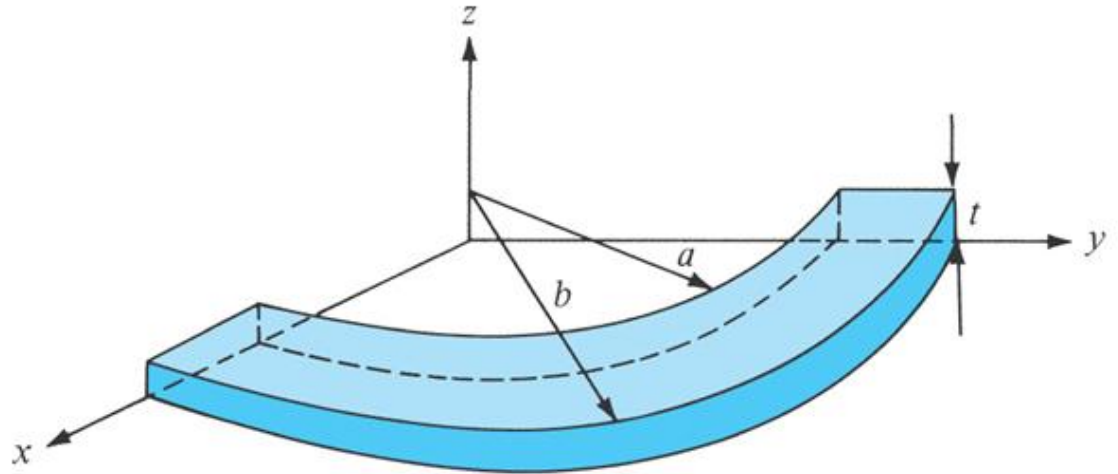
A metal bar of conductivity σ is bent to form a flat 90° sector of inner radius a , outer radius b , and thickness t as shown in the Figure. Show that :-

(a) The resistance of the bar between the vertical curved surfaces at $r = a$ and $r = b$ is:

$$R = \frac{2 \ln \frac{b}{a}}{\sigma \pi t}$$

(b) The resistance between the two horizontal surfaces at $z=0$ and $z=t$ is

$$R' = \frac{4t}{\sigma \pi (b^2 - a^2)}$$



Example 6.8 Solution

(a) Between the vertical curved ends located at $\rho=a$ and $\rho=b$, the bar has a nonuniform cross section. Let a potential difference V_0 be maintained between the curved surfaces at $\rho=a$ and $\rho=b$ so that $V(\rho=a)=0$ and $V(\rho=b)=V_0$. Solve V using Laplace's equation in cylindrical coordinates,

$$\text{since } V=V(\rho), \quad \nabla^2 V = \frac{1}{\rho} \frac{d}{d\rho} \left(\rho \frac{dV}{d\rho} \right) = 0$$

As $\rho = 0$ is excluded, multiply by ρ and

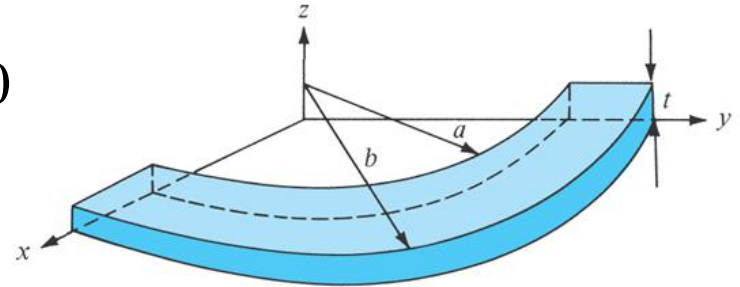
$$\text{integrate once, } \rho \frac{dV}{d\rho} = A \rightarrow \frac{dV}{d\rho} = \frac{A}{\rho}, \text{ integrating again yields,}$$

$V=A \ln \rho+B$, where A and B are constants of integration.

Use the boundary conditions:

$$V(\rho=a)=0 \rightarrow 0=A \ln a+B \text{ or } B = -A \ln a$$

$$V(\rho=b)=V_0 \rightarrow V_0=A \ln b+B=A \ln b - A \ln a = A \ln \frac{b}{a} \text{ or } A = \frac{V_0}{\ln \frac{b}{a}}$$



Example 6.8 Continued

Hence,

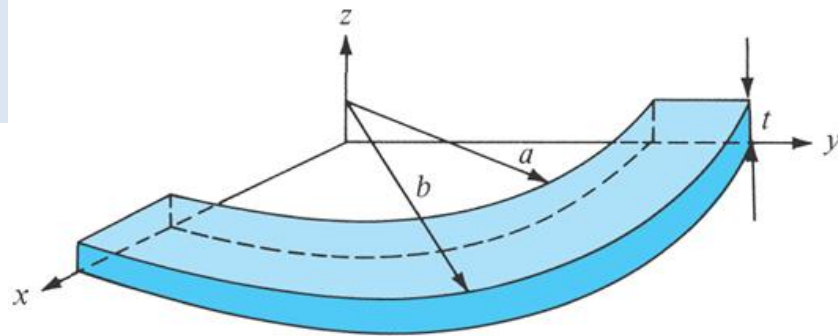
$$V = A \ln \rho - A \ln a = A \ln \frac{\rho}{a} = \frac{V_0}{\ln \frac{b}{a}} \ln \frac{\rho}{a}$$

$$E = -\nabla V = -\frac{dV}{d\rho} a_\rho = -\frac{A}{\rho} a_\rho = -\frac{V_0}{\rho \ln \frac{b}{a}} a_\rho$$

$$J = \sigma E, \quad dS = -\rho d\phi dz a_\rho$$

$$I = \int J \cdot dS = \int_{\phi=0}^{\pi/2} \int_{z=0}^t \frac{V_0 \sigma}{\rho \ln \frac{b}{a}} dz \rho d\phi = \frac{\pi}{2} \frac{t V_0 \sigma}{\ln \frac{b}{a}}$$

$$\text{Thus } R = \frac{V_0}{I} = \frac{2 \ln \frac{b}{a}}{\sigma \pi t} \text{ as required}$$



$$V = A \ln \rho + B$$

$$A = \frac{V_0}{\ln \frac{b}{a}}$$

$$B = -A \ln a$$

Example 6.8 Continued

(b)

Let V_0 be the potential difference between the two horizontal surfaces so that $V(z=0)=0$ and $V(z=t)=V_0$.

since $V = V(z)$, Laplace's equation becomes

$$\frac{d^2V}{dz^2} = 0, \text{ integrating twice gives}$$

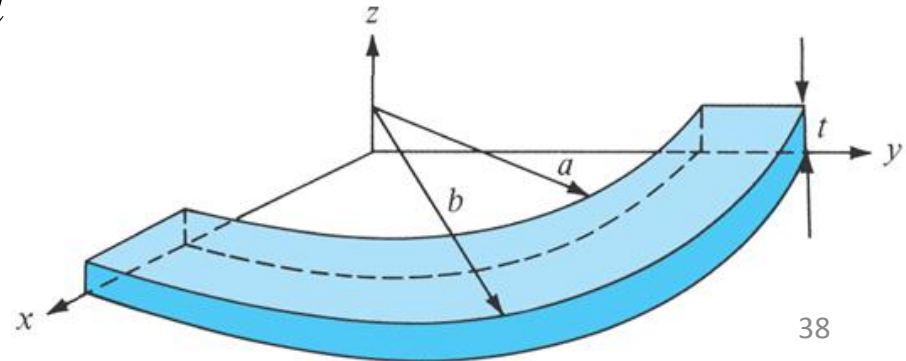
$V = Az + B$, apply boundary conditions to get A and B ,

$$V(z=0)=0 \rightarrow 0 = 0 + B \text{ or } B = 0$$

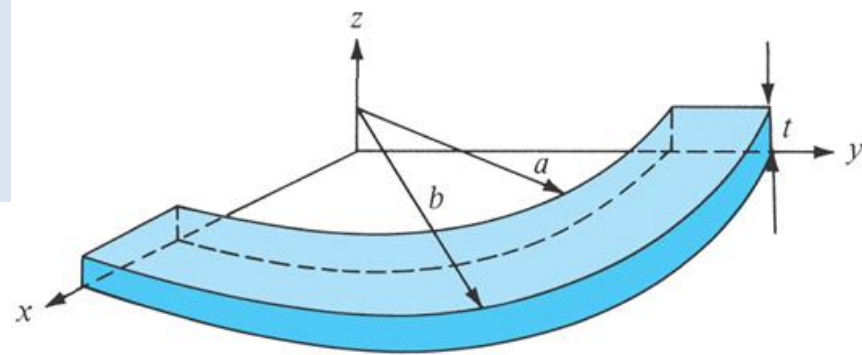
$$V(z=t)=V_0 \rightarrow V_0 = At \text{ or } A = \frac{V_0}{t}$$

$$\text{Hence, } V = \frac{V_0}{t} z$$

$$E = -\nabla V = -\frac{dV}{dz} a_z = -\frac{V_0}{t} a_z$$



Example 6.8 Continued



$$\mathbf{J} = \sigma \mathbf{E} = -\frac{\sigma V_0}{t} \mathbf{a}_z, \quad d\mathbf{S} = -\rho d\phi d\rho \mathbf{a}_z$$

$$I = \int \mathbf{J} \cdot d\mathbf{S} = \int_{\rho=a}^b \int_{\phi=0}^{\pi/2} \frac{V_0 \sigma}{t} \rho d\phi d\rho = \frac{V_0 \sigma}{t} \cdot \frac{\pi}{2} \frac{\rho^2}{2} \Big|_a^b = \frac{V_0 \sigma \pi (b^2 - a^2)}{4t}$$

$$\text{Thus } R' = \frac{V_0}{I} = \frac{4t}{\sigma \pi (b^2 - a^2)}$$

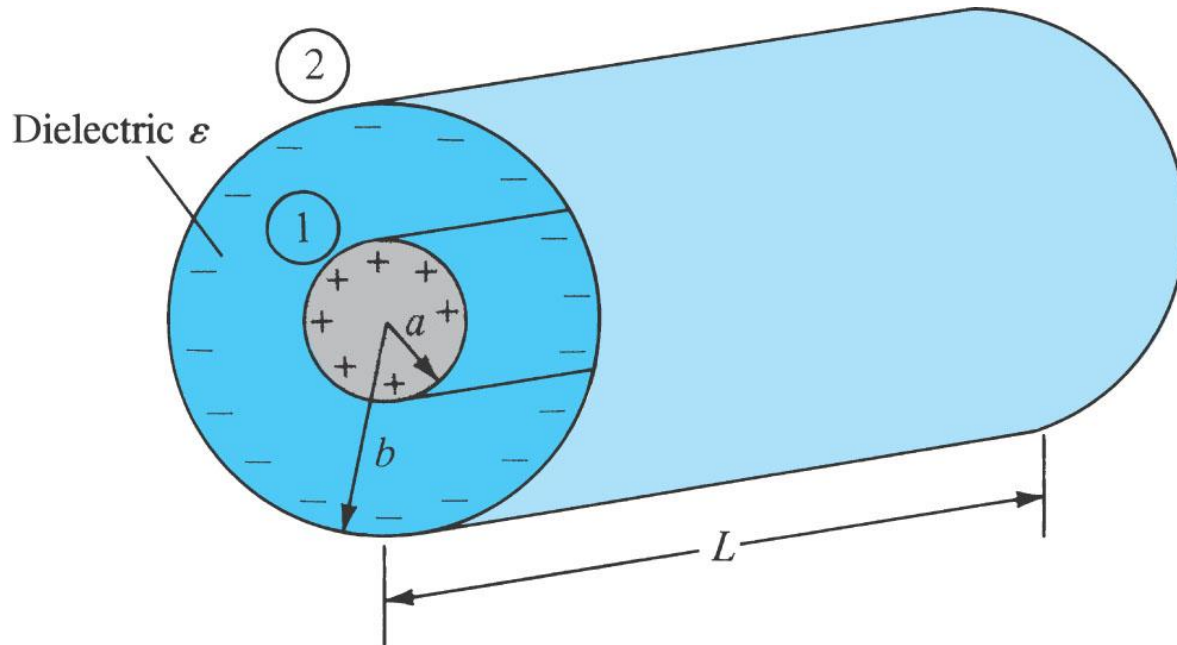
Alternatively, for this case, the cross section of the bar is uniform between the horizontal surfaces at $z=0$ and $z=t$, then

$$R' = \frac{l}{\sigma S} = \frac{t}{\sigma \frac{\pi}{4} (b^2 - a^2)} = \frac{4t}{\sigma \pi (b^2 - a^2)}$$

Example 6.9

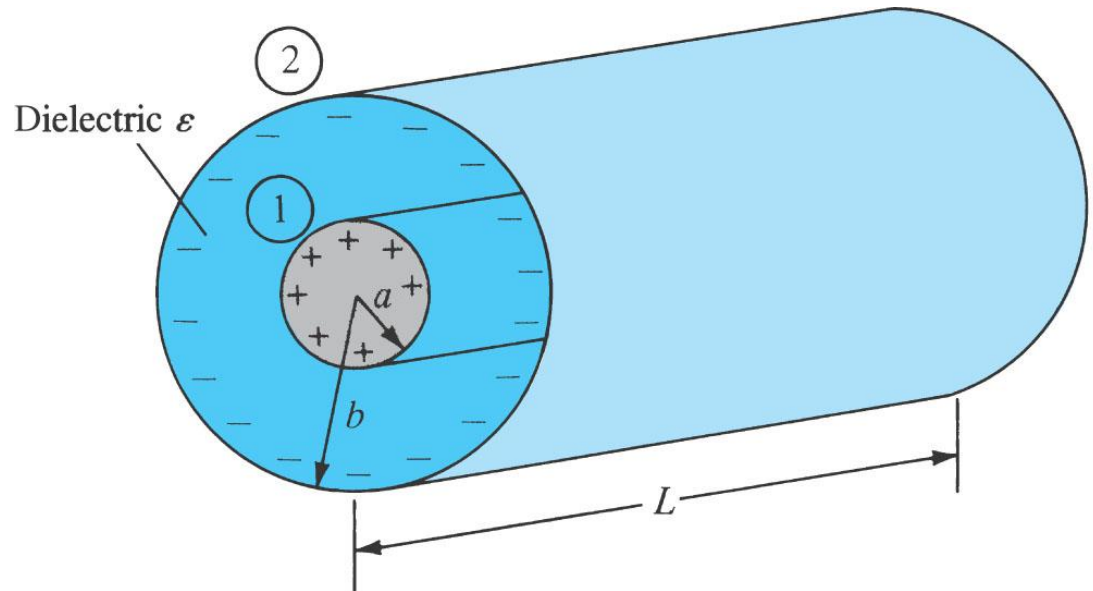
A coaxial cable contains an insulating material of conductivity σ . If the radius of the central wire is a and that of the sheath is b , show that the conductance of the cable per unit length is:

$$G = \frac{2\pi\sigma}{\ln \frac{b}{a}}$$

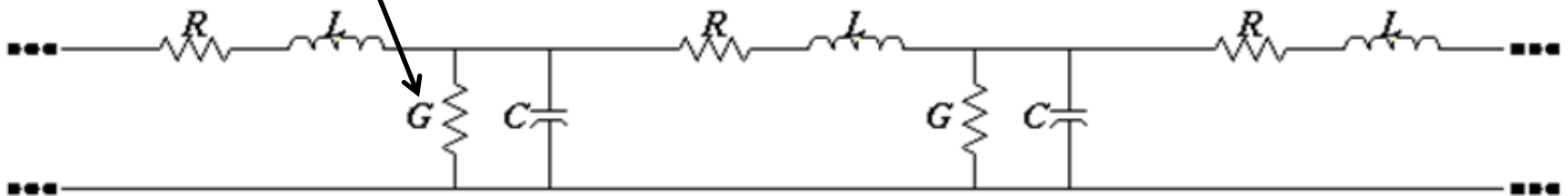


Example 6.9

$$G = \frac{2\pi\sigma}{\ln \frac{b}{a}}$$



Transmission Line Equivalent circuit: (See Chapter 11)



G=Conductance per unit length for dielectric.
G=1/(Leakage Resistance of Dielectric)

Example 6.9 Solution

Let V_0 be the potential difference between the inner and outer conductors, so that $V(\rho = a) = 0$ and $V(\rho = b) = V_0$

Solve $\nabla^2 V = \frac{1}{\rho} \frac{d}{d\rho} \left(\rho \frac{dV}{d\rho} \right) = 0$. From example 6.8, we got:

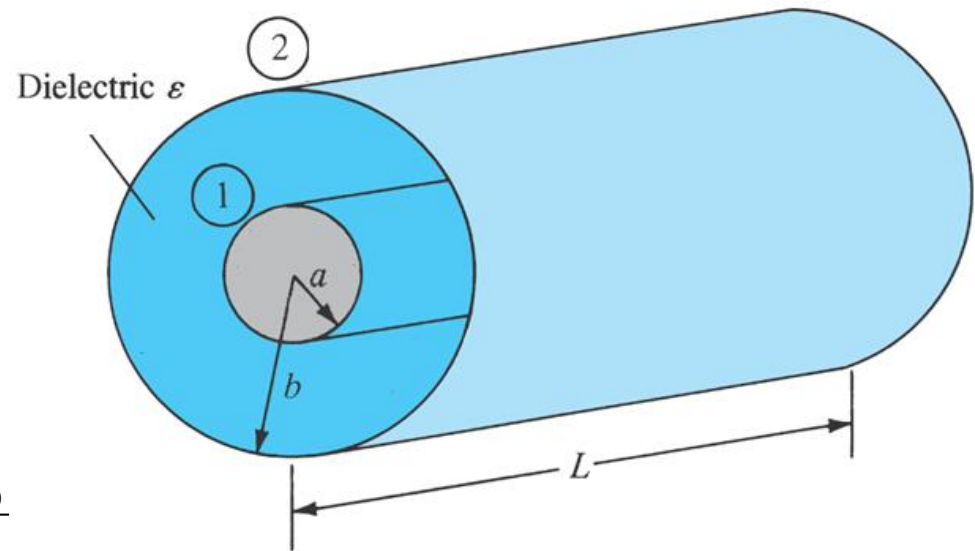
$$V = \frac{V_0}{\ln \frac{b}{a}} \ln \frac{\rho}{a} \quad \text{and} \quad E = -\frac{V_0}{\rho \ln \frac{b}{a}} a_\rho$$

$$J = \sigma E = -\frac{\sigma V_0}{\rho \ln \frac{b}{a}} a_\rho, \quad dS = -\rho d\phi dz a_\rho$$

$$I = \int J \cdot dS = \int_{\phi=0}^{2\pi} \int_{z=0}^l \frac{V_0 \sigma}{\rho \ln \frac{b}{a}} \rho dz d\phi = \frac{2\pi L \sigma V_0}{\ln \frac{b}{a}}$$

The resistance per unit length is $R = \frac{V_0}{I} \cdot \frac{1}{L} = \frac{\ln \frac{b}{a}}{2\pi\sigma}$

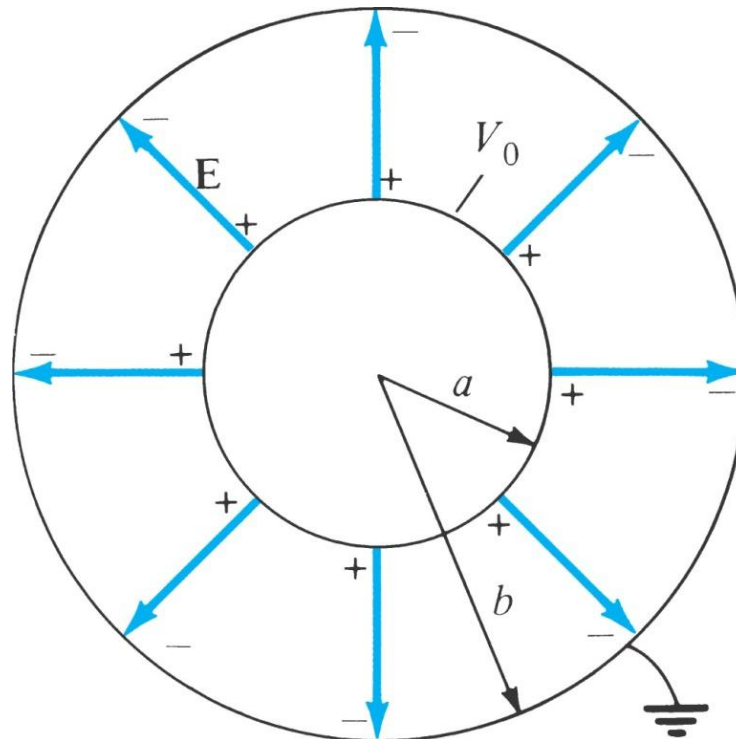
and the conductance per unit length is $G = \frac{1}{R} = \frac{2\pi\sigma}{\ln \frac{b}{a}}$



Example 6.10

Conducting spherical shells with radii $a = 10$ cm and $b = 30$ cm are maintained at a potential difference of 100 V such that $V(r=b) = 0$ and $V(r=a) = 100$ V. Determine \mathbf{V} and \mathbf{E} in the region between the shells.

if $\epsilon_r = 2.5$ in the region, determine the total charge induced on the shells and the capacitance of the capacitor.



Example 6.10 Solution

$$\nabla^2 V = \frac{1}{r^2} \frac{d}{dr} \left[r^2 \frac{dV}{dr} \right] = 0$$

since $r \neq 0$ in the region of interest, we multiply through by r^2 to obtain

$$\frac{d}{dr} \left[r^2 \frac{dV}{dr} \right] = 0, \quad \text{Integrating once gives: } r^2 \frac{dV}{dr} = A \quad \text{or} \quad \frac{dV}{dr} = \frac{A}{r^2}$$

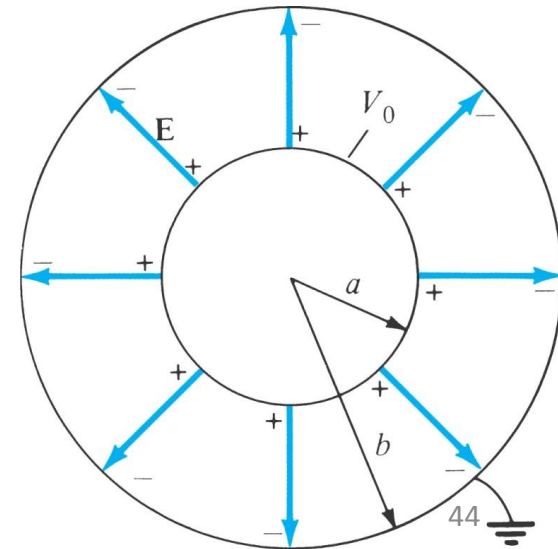
Integrating again: $V = -\frac{A}{r} + B$ (use boundary conditions to find A and B)

$$\text{when } r = b, V = 0 \rightarrow 0 = -\frac{A}{b} + B \quad \text{or} \quad B = \frac{A}{b}, \quad \text{Hence } V = A \left[\frac{1}{b} - \frac{1}{r} \right]$$

$$\text{when } r = a, V = V_0 \rightarrow V_0 = A \left[\frac{1}{b} - \frac{1}{a} \right], \quad \text{or} \quad A = \frac{V_0}{\left[\frac{1}{b} - \frac{1}{a} \right]}$$

Thus

$$V = V_0 \frac{\left[\frac{1}{r} - \frac{1}{b} \right]}{\left[\frac{1}{a} - \frac{1}{b} \right]}$$



Example 6.10 Continued

$$\mathbf{E} = -\nabla V = -\frac{dV}{dr} \mathbf{a}_r = -\frac{A}{r^2} \mathbf{a}_r = \frac{V_0}{r^2} \left[\frac{1}{a} - \frac{1}{b} \right] \mathbf{a}_r$$

To determine capacitance, first find Q on either shell

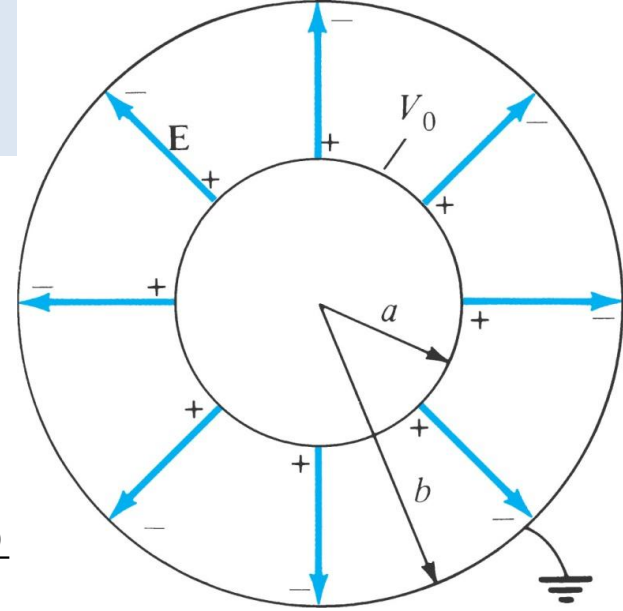
$$Q = \int \varepsilon \mathbf{E} \cdot d\mathbf{S} = \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \frac{\varepsilon_0 \varepsilon_r V_0}{r^2} \left[\frac{1}{a} - \frac{1}{b} \right] r^2 \sin \theta d\phi d\theta = \frac{4\pi \varepsilon_0 \varepsilon_r V_0}{\frac{1}{a} - \frac{1}{b}}$$

Or, $Q = \int \rho_S dS \rightarrow$ For inner shell, $Q = \rho_S [4\pi a^2]$, but $\rho_S = D_n|_{r=a}$

$$D_n|_{r=a} = \varepsilon E_n|_{r=a} = \frac{\varepsilon_0 \varepsilon_r V_0}{a^2 \left[\frac{1}{a} - \frac{1}{b} \right]} \rightarrow Q = \frac{\varepsilon_0 \varepsilon_r V_0}{a^2 \left[\frac{1}{a} - \frac{1}{b} \right]} \cdot 4\pi a^2 = \frac{4\pi \varepsilon_0 \varepsilon_r V_0}{\left[\frac{1}{a} - \frac{1}{b} \right]}$$

The capacitance is easily determined as

$$C = \frac{Q}{V_0} = \frac{4\pi \varepsilon}{\frac{1}{a} - \frac{1}{b}}$$



$$V = A \left[\frac{1}{b} - \frac{1}{r} \right]$$

$$A = \frac{V_0}{\left[\frac{1}{b} - \frac{1}{a} \right]}$$

$$V = V_0 \frac{\left[\frac{1}{r} - \frac{1}{b} \right]}{\frac{1}{a} - \frac{1}{b}}$$

Example 6.10 Continued

$$V = V_0 \left[\frac{1}{r} - \frac{1}{b} \right]$$

Substituting $a=0.1$ m, $b=0.3$ m, $V_0=100$ V yields,

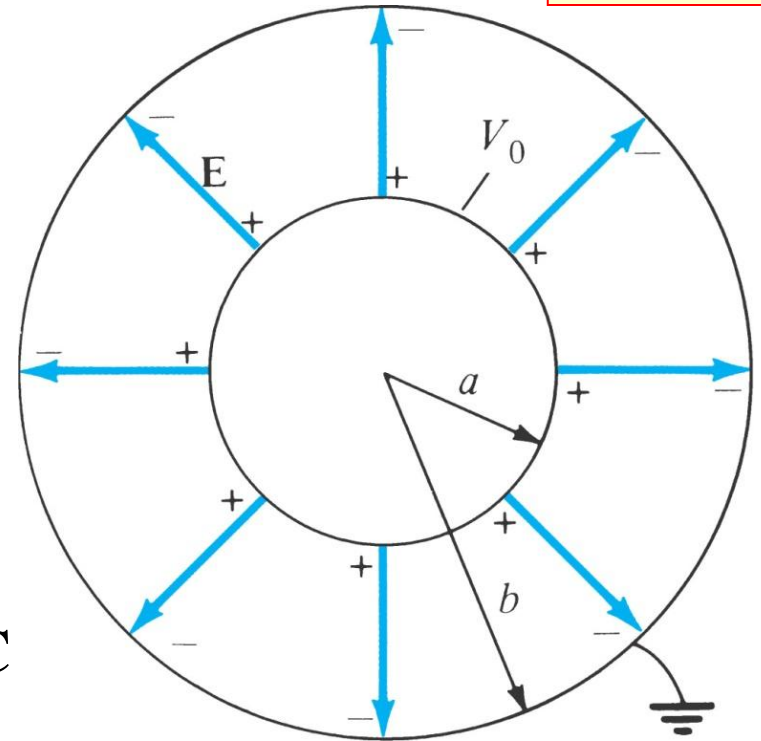
$$V = 100 \frac{\left[\frac{1}{r} - \frac{10}{3} \right]}{10 - 10/3} = 15 \left[\frac{1}{r} - \frac{10}{3} \right] V$$

$$E = \frac{100}{r^2 [10 - 10/3]} a_r = \frac{15}{r^2} a_r \text{ V/m}$$

$$Q = \pm \frac{4\pi \left(\frac{10^{-9}}{36\pi} \right) (2.5)(100)}{[10 - 10/3]} = \pm 4.167 \text{ nC}$$

positive charge is induced on the inner shell, negative charge is induced on the outer shell.

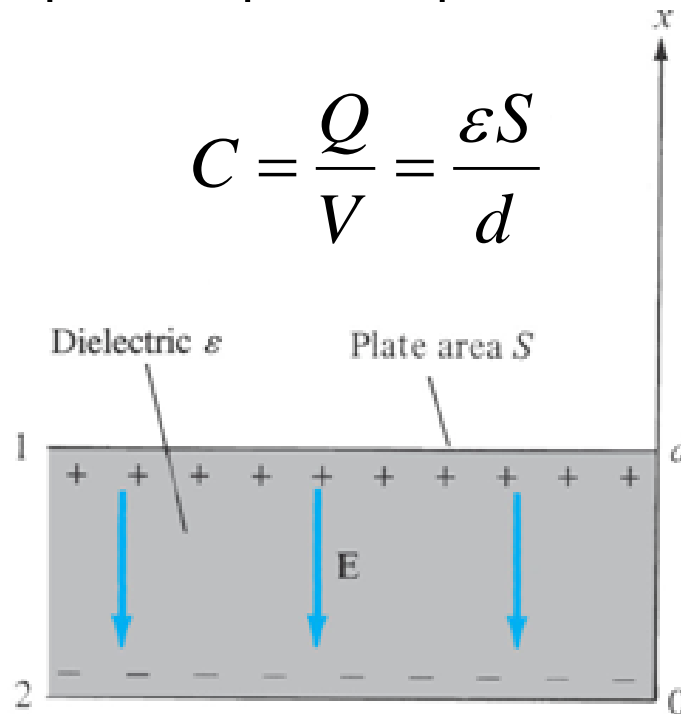
$$C = \frac{|Q|}{V_0} = \frac{4.167 \times 10^{-9}}{100} = 41.67 \text{ pF}$$



Example 6.11

In Section 6.5, it was mentioned that the capacitance $C=Q/V$ of a capacitor can be found by either assuming Q and finding V or by assuming V and finding Q . The former approach was used in Section 6.5 while we have used the latter method in the last example. Using the latter method, derive capacitance for parallel plate capacitor:

$$C = \frac{Q}{V} = \frac{\epsilon S}{d}$$



Assuming the potential difference between the parallel plates is V_0 so that $V(x=0)=0$ and $V(x=d)=V_0$.

Example 6.11 Solution

Solve Laplace's equation: $\nabla^2 V = \frac{d^2 V}{dx^2} = 0$

Integrating twice gives $V = Ax + b$

At $x = 0$, $V = 0 \rightarrow 0 = 0 + B$ or $B = 0$

At $x = d$, $V = V_0 \rightarrow V_0 = Ad + 0$ or $A = V_0 / d$. Hence $V = \frac{V_0}{d} x$

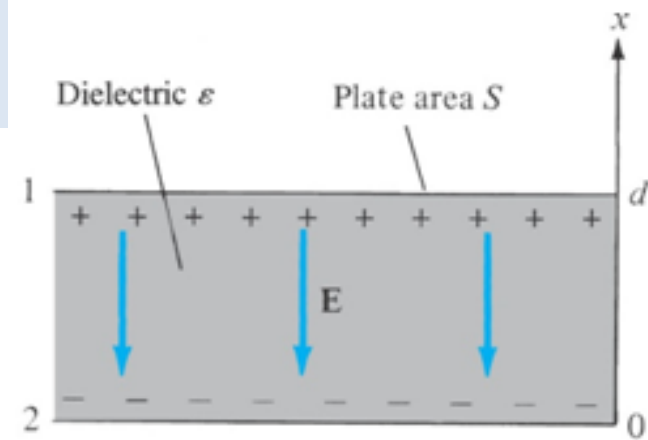
To find the capacitance, first find the charge on either plate $\rightarrow Q = \int \rho_S dS$

But $\rho_S = \mathbf{D} \cdot \mathbf{a}_n = \epsilon \mathbf{E} \cdot \mathbf{a}_n$, where $\mathbf{E} = -\nabla V = -\frac{dV}{dx} \mathbf{a}_x = -A \mathbf{a}_x = -\frac{V_0}{d} \mathbf{a}_x$

On the lower plate $\mathbf{a}_n = \mathbf{a}_x$, So $\rho_S = -\frac{\epsilon V_0}{d}$ and $Q = -\frac{\epsilon V_0 S}{d}$

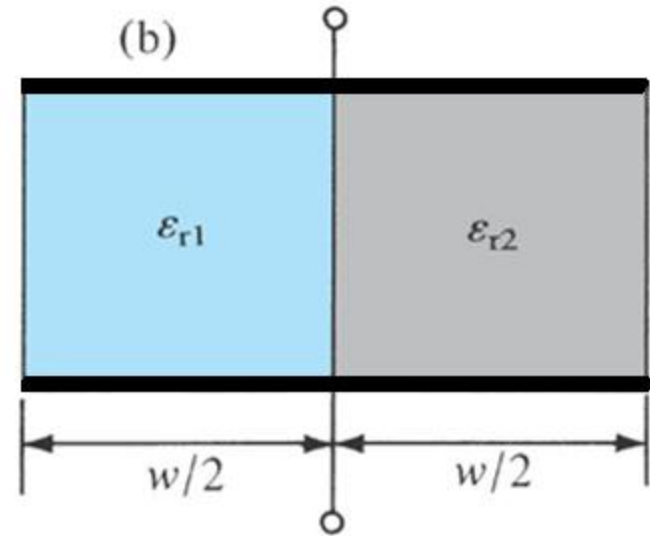
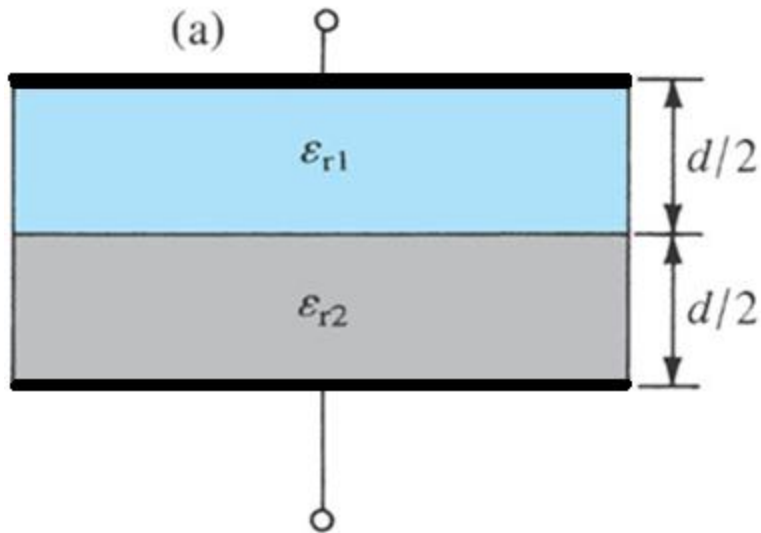
On the upper plate $\mathbf{a}_n = -\mathbf{a}_x$, So $\rho_S = \frac{\epsilon V_0}{d}$ and $Q = \frac{\epsilon V_0 S}{d}$

(Q is equal but opposite on each plate). $C = \frac{|Q|}{V_0} = \frac{\epsilon S}{d}$



Example 6.12

Determine the capacitance of each of the capacitors in the Figure. Take $\epsilon_{r1}=4$, $\epsilon_{r2}=6$, $d=5$ mm, $S=30$ Cm².



Example 6.12 Solution

(a) Since D and E are normal to the dielectric interface, the capacitor in Fig. (a) can be treated as two capacitors C_1 and C_2 in series.

$$C_1 = \frac{\epsilon_0 \epsilon_{r1} S}{d/2} = \frac{2\epsilon_0 \epsilon_{r1} S}{d}, \quad C_2 = \frac{2\epsilon_0 \epsilon_{r2} S}{d}$$

Example 6.12 Solution Continued

The total capacitor is given by

$$C = \frac{C_1 C_2}{C_1 + C_2} = \frac{2\epsilon_0 S (\epsilon_{r1} \epsilon_{r2})}{d (\epsilon_{r1} + \epsilon_{r2})} = 2 \cdot \frac{10^{-9}}{36\pi} \cdot \frac{30 \times 10^{-4}}{5 \times 10^{-3}} \cdot \frac{4 \times 6}{10} \rightarrow C = 25.46 \text{ pF}$$

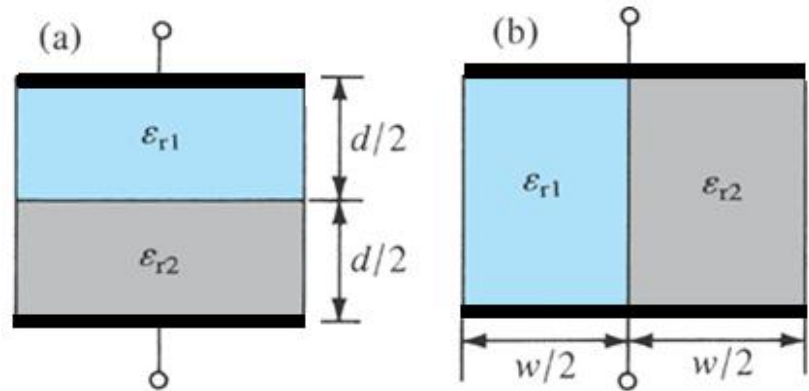
(b)

D and E are parallel to the dielectric interface, the capacitor in Fig. (b) can be treated as two capacitors C_1 and C_2 in parallel (the same voltage across C_1 and C_2)

$$C_1 = \frac{\epsilon_0 \epsilon_{r1} S / 2}{d} = \frac{\epsilon_0 \epsilon_{r1} S}{2d}, \quad C_2 = \frac{\epsilon_0 \epsilon_{r2} S}{2d}$$

The total capacitance is

$$C = C_1 + C_2 = \frac{\epsilon_0 S}{2d} (\epsilon_{r1} + \epsilon_{r2}) = \frac{10^{-9}}{36\pi} \cdot \frac{30 \times 10^{-4}}{(5 \times 10^{-3})} \cdot 10 = 26.53 \text{ pF}$$



Example 6.13

A cylindrical capacitor has radii $a = 1$ cm and $b = 2.5$ cm. If the space between the plates is filled with an inhomogeneous dielectric with $\epsilon_r = (10 + \rho)/\rho$, where ρ is in centimeters, find the capacitance per meter of the capacitor.

$$Q = \epsilon \oint \mathbf{E} \cdot d\mathbf{S} = \epsilon E_\rho 2\pi\rho L. \quad \text{Hence } E = \frac{Q}{2\pi\epsilon\rho L} a_\rho$$

$$V = -\int_b^a \frac{Q}{2\pi\epsilon_0\epsilon_r\rho L} d\rho = -\frac{Q}{2\pi\epsilon_0 L} \int_b^a \frac{d\rho}{\rho \left(\frac{10 + \rho}{\rho} \right)} = -\frac{Q}{2\pi\epsilon_0 L} \int_b^a \frac{d\rho}{10 + \rho}$$

$$V = \frac{-Q}{2\pi\epsilon_0 L} \ln(10 + \rho) \Big|_b^a = \frac{Q}{2\pi\epsilon_0 L} \ln \frac{10 + b}{10 + a}$$

Thus the capacitance per meter is ($L=1$ m)

$$C = \frac{Q}{V} = \frac{2\pi\epsilon_0}{\ln \frac{10 + b}{10 + a}} = 2\pi \cdot \frac{10^{-9}}{36\pi} \cdot \frac{1}{\ln \frac{12.5}{11.0}} = 434.6 \text{ pF/m}$$

