

Convection and Conduction Currents

- Current (in amperes) through a given area is the electric charge passing through the area per unit time

$$\text{Current} \quad I = \frac{dQ}{dt}$$

- Current density is the amount of current flowing through a surface, A/m², or the current through a unit normal area at that point

$$\text{Current density} \quad J = \frac{\Delta I}{\Delta S} \quad \text{where} \quad I = \int_s \vec{J} \cdot d\vec{S}$$

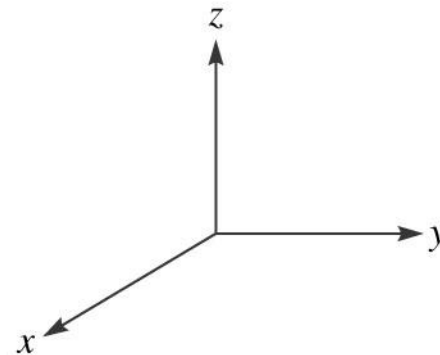
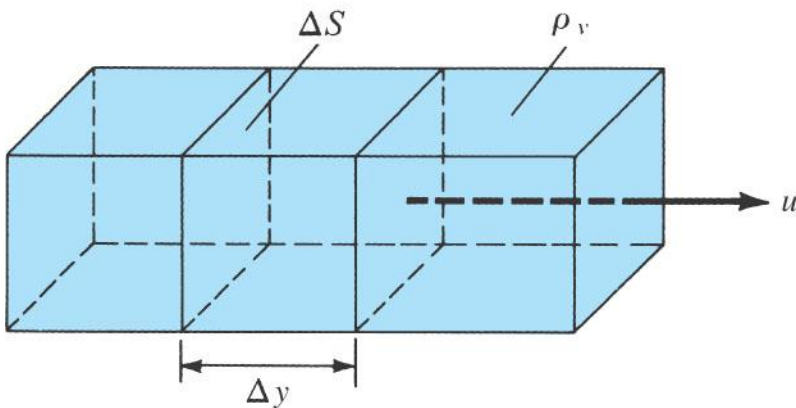
- Depending on how the current is produced, there are different types of current density
 - Convection current density
 - Conduction current density
 - Displacement current density (Chapter 9)
 - Current generated by a magnetic field

Convection Current Density

- Convection current density
 - Does not involve conductors and does not obey Ohm's law
 - Occurs when current flows through an insulating medium such as liquid, gas, or vacuum

$$\Delta I = \frac{\Delta Q}{\Delta t} = \rho_v \Delta S \frac{\Delta y}{\Delta t} = \rho_v \Delta S u_y$$

Where u is the velocity vector of the fluid



$$J_y = \frac{\Delta I}{\Delta S} = \rho_v u_y$$
$$\vec{J} = \rho_v \vec{u}$$

Conduction Current Density

- Conduction current density
 - Current in a conductor
 - Obeys Ohm's law
- Consider a large number of free electrons traveling in a metal with mass (m), velocity (u), and scattering time (time between electron collisions), τ .

$$\vec{F} = -q\vec{E} = \frac{m\vec{u}}{\tau}$$

- The carrier density is determined by the number of electrons, n , with charge, e

$$\varphi_v = -ne$$

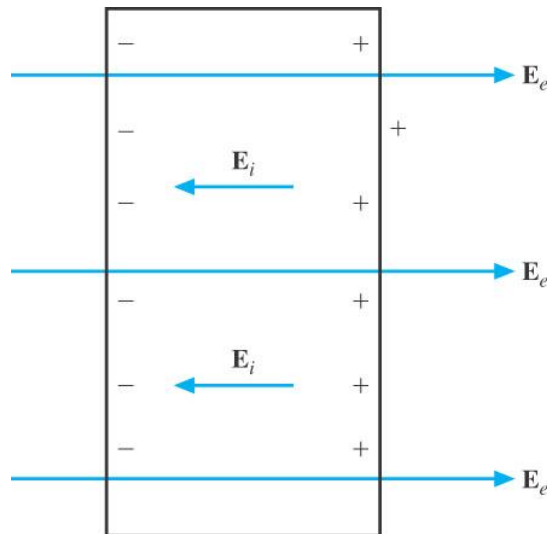
- Conduction current density can then be calculate as

$$\vec{J} = \varphi_v \vec{u} = \frac{ne^2\tau}{m} \vec{E} = \sigma \vec{E}$$

- Where σ is the conductivity of the conductor
- This relationship between current concentration and electric field is known as Ohm's Law

Conductors

- Conductors are materials with an abundance of free moving charges
- Convention states that when an electric field is applied to a conductor, the positive free charges are pushed along the same direction as the applied field, while the negative charges move in the opposite direction
- The free charges do two things
 - They accumulate on the surface of the conductor to form an induced surface charge
 - The induced charges set up internal induced field E_i , which cancels the externally applied field inside the material
- Shielding of a conductor by an induced field generates current within the material



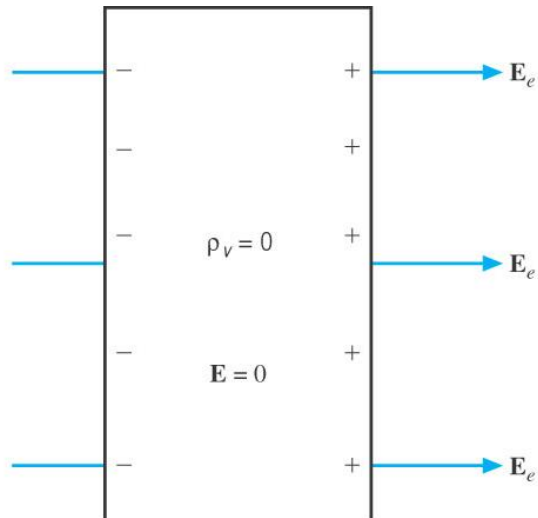
Good Conductor:

Reduced electric field inside vs. that incident on the material.

Conductors (2)

- Perfect conductor is a conductor in which no electrostatic field may enter, because the induced surface charges match the external field exactly eliminating all fields within the material
- Such conductors are called equipotential bodies, because the potential is the same everywhere within the conductor based on the fact that $E = -\text{Grad}(V)=0$
 - In reality metals are very good conductors in which the electric field below the skin depth of the conductor is indeed zero. However the skin depth is a frequency dependent function that is usually observed only in high frequency applications. If indeed the skin depth is considered in a problem, then the electric field below the skin depth of carrier conduction within the material is zero, and current is generated only on the surface.

Perfect Conductor:
No electric field inside



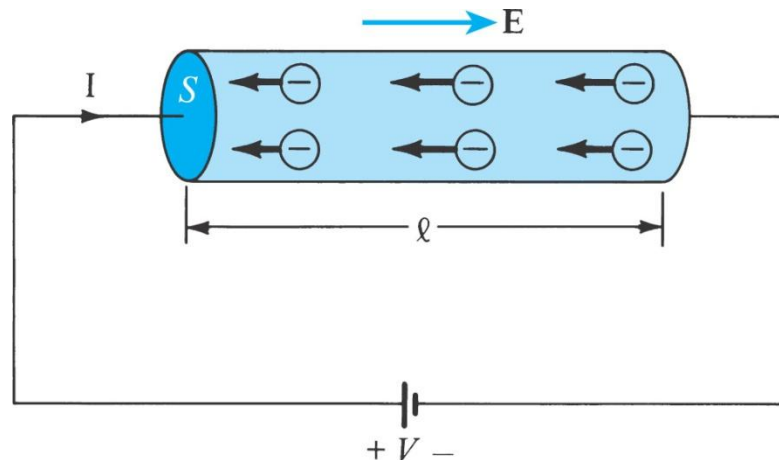
Skin Depth:

- The depth beneath the surface of a conductor at which the current drops to e^{-1} below the current density on the surface.
- This term is quite commonly used to determine the depth of high frequency electromagnetic waves incident on a surface or propagating along a metallic wire.

Electrical Resistivity

- Consider a conductor whose ends are maintained at a potential difference (i.e. the electric field within the conductor is nonzero and a field is passed through the material.)
- Note that there is no static equilibrium in this system. The conductor is being fed energy by the application of the electric field (bias potential)
- As electrons move within the material to set up induction fields, they scatter and are therefore damped. This damping is quantified as the **resistance, R**, of the material.
- For this example assume:
 - a uniform cross sectional area S, and length l.
 - The direction of the electric field, E, produced is the same as the direction of flow of positive charges (or the same as the current, I).

$$R = \frac{V}{I} = \frac{\int_v \vec{E} \cdot d\vec{l}}{\int_s \sigma \vec{E} \cdot d\vec{S}}$$



$$E = \frac{V}{l}$$

$$J = \frac{I}{S} = \sigma E = \sigma \frac{V}{l}$$

$$R = \frac{V}{I} = \frac{l}{\sigma S} = \frac{\rho_c l}{S}$$

Electrical Power

- Power is defined either as the rate change of energy (Joules) or force times velocity

$$P = \int_v \phi_v \vec{E} \cdot \vec{u} dv = \int_v \vec{E} \cdot \vec{J} dv \quad \text{Joule's Law}$$

$$w_p = \frac{dP}{dv} = \vec{E} \cdot \vec{J} = \sigma |\vec{E}|^2 \quad \text{Power density}$$

- For a conductor with uniform cross section

$$P = \int_L E dl \int_S J dS = VI = I^2 R$$

Polarization in Dielectrics

- The main difference between a conductor and a dielectric is the availability of free electrons in the outermost atomic shells to conduct current
- Carriers in a dielectric are bound by finite forces and as such, electric displacement occurs when external forces are applied
- Such displacements are produced when an applied electric field, E , creates dipoles within the media that polarize it
- Polarized media are evaluated by summing the original charge distribution and the dipole moment induced
- One may also define the polarization, P , of the material as the dipole moment per unit volume

$$\vec{P} = \lim_{\Delta v \rightarrow 0} \frac{\sum_{k=1}^n q_k \vec{d}_k}{\Delta v} = \lim_{\Delta v \rightarrow 0} \frac{\sum_{k=1}^n \vec{p}_k}{\Delta v}$$

See slides 39-40 for more on E fields, electrostatic potential, and dipoles

- Two types of dielectrics exist in nature: polar and nonpolar
 - Nonpolar dielectrics do not possess dipole moments until a strong electric field is applied
 - Polar dielectrics such as water, possess permanent dipole moments that further align (if possible) in the presence of an external field

Polarization in Dielectrics (2)

- Potential due to a dipole moment

$$V = \frac{\vec{p} \cdot \vec{a}_r}{4\pi\epsilon_o R^2} = \frac{\vec{p}}{4\pi\epsilon_o} \cdot \frac{(\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3}$$

$$V = \int_v \frac{(\vec{p} \cdot \vec{a}_r) dv}{4\pi\epsilon_o R^2}$$

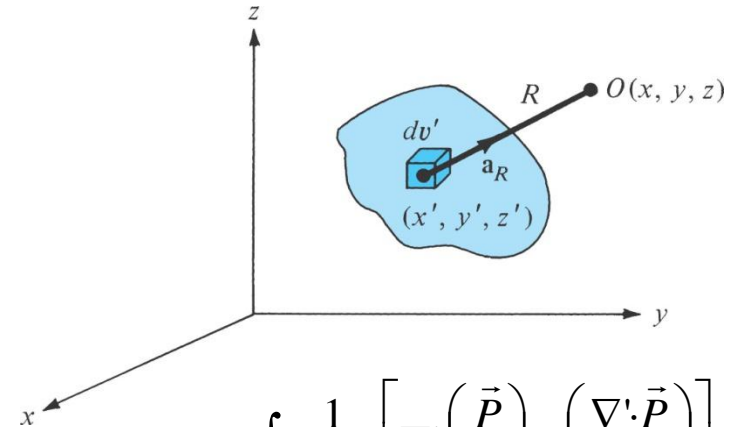
Where,

$$R^2 = |\vec{r} - \vec{r}'|^2 = (x - x')^2 + (y - y')^2 + (z - z')^2$$

$$\nabla' \left(\frac{1}{R} \right) = - \frac{(x - x')\vec{a}_x + (y - y')\vec{a}_y + (z - z')\vec{a}_z}{[(x - x')^2 + (y - y')^2 + (z - z')^2]^{3/2}} = - \frac{\vec{R}}{R^3} = \frac{\vec{a}_r}{R^2}$$

Where the ∇' operator is with respect to (x', y', z')

$$\frac{\vec{P} \cdot \vec{a}_r}{R^2} = \vec{P} \cdot \nabla' \left(\frac{1}{R} \right) = \nabla' \cdot \left(\frac{\vec{P}}{R} \right) - \left(\frac{\nabla' \cdot \vec{P}}{R} \right)$$



$$V = \int_v \frac{1}{4\pi\epsilon_o} \left[\nabla' \cdot \left(\frac{\vec{P}}{R} \right) - \left(\frac{\nabla' \cdot \vec{P}}{R} \right) \right] dv'$$

$$V = \int_v \frac{\vec{P} \cdot \vec{a}'_n}{4\pi\epsilon_o R} dS' + \int_v \frac{-\nabla' \cdot \vec{P}}{4\pi\epsilon_o R} dv'$$

\Rightarrow

$\phi_{ps} = \vec{P} \cdot \vec{a}'_n$ $\phi_{pv} = -\nabla' \cdot \vec{P}$

When polarization occurs, an equivalent volume charge density, ϕ_{pv} , is formed throughout the dielectric, while an equivalent surface charge density, ϕ_{ps} , is formed over the surface.

Polarization in Dielectrics(3)

- When polarization occurs, an equivalent volume charge density, φ_{pv} , is formed throughout the dielectric, while an equivalent surface charge density, φ_{ps} , is formed over the surface.

$$\varphi_{ps} = \vec{P} \cdot \vec{a}_n$$

$$\varphi_{pv} = -\nabla \cdot \vec{P}$$

- For nonpolar dielectrics with no added free charge

$$Q_{total} = \oint_S \varphi_{ps} dS + \oint_V \varphi_{pv} dv = 0$$

- For cases in which the dielectric contains free charge density, φ_v

$$\varphi_t = \varphi_v + \varphi_{pv} = \nabla \cdot \epsilon_o \vec{E}$$

Hence

$$\begin{aligned} \varphi_v &= \nabla \cdot \epsilon_o \vec{E} - \varphi_{pv} \\ &= \nabla \cdot \epsilon_o \vec{E} + \nabla \cdot \vec{P} \\ &= \nabla \cdot (\epsilon_o \vec{E} + \vec{P}) = \nabla \cdot \vec{D} \end{aligned}$$

This redefines our Electric Displacement definition from chapter 4 on slide 24 to include polarized media. Our previous definition is the special case in which the polarization of the material is zero

The Dielectric Constant

- It is important to note that up to this point, we have not committed ourselves to the cause of the polarization, \vec{P} . We dealt only with its effects. We have stated that the polarization of a dielectric results from an electric field which lines up the atomic or molecular dipoles.
- In many substances, experimental evidence shows that the polarization is proportional to the electric field, provided that E is not too strong. These substances are said to have a linear, isotropic dielectric constant
- This proportionality constant is called the **electric susceptibility**, χ_e . The convention is to extract the permittivity of free space from the electric susceptibility to make the units dimensionless. Thus we have

$$\vec{P} = \epsilon_o \chi_e \vec{E}$$

- From the previous slide

$$\vec{D} = \epsilon_o \vec{E} + \vec{P} = \epsilon_o (1 + \chi_e) \vec{E}$$

$$\vec{D} = \epsilon_o \epsilon_r \vec{E}$$

$$\vec{D} = \epsilon \vec{E}$$

The **dielectric constant** (or relative permittivity) of the material, ϵ_r , is the ratio of the permittivity to that of free space

- If the electric field is too strong, then it begins to strip electrons completely from molecules leading to short term conduction of electrons within the media. This is called **dielectric breakdown**.
- The maximum strength of the electric field that a dielectric can tolerate prior to which breakdown occurs is called the **dielectric strength**.

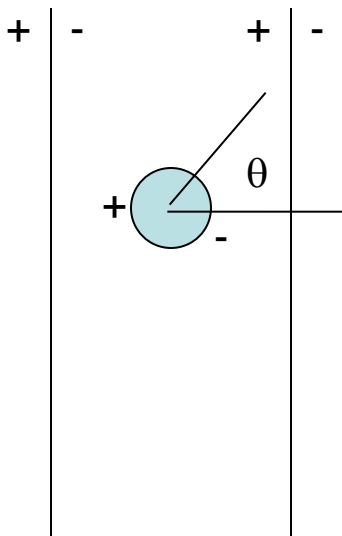
Linear, Isotropic, and Homogeneous Dielectrics

- In **linear dielectrics**, the permittivity, ϵ , does not change with applied field, E .
- **Homogenous dielectrics** do not change their permittivity from point to point within the material
- **Isotropic dielectrics** do not change their dielectric constant with respect to direction within the material
- Most commercial dielectrics are linear over some range, but may not be homogenous over large areas, and may not be isotropic.
- Inhomogeneity is most commonly due to local concentrations of one type of material verses another in an alloy, or simply from machine tolerance error on the thickness of a dielectric from point to point. These are commonly processing issues that need to be evaluated by the engineer when choosing the appropriate material and manufacturing process for the job.
- Isotropy is a material property. Many materials, such as single crystals, plasmas and magneto active materials possess anisotropic dielectric constants. These may be taken advantage of for specific engineering applications.
- For linear, homogeneous anisotropic materials:

$$\begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} = \begin{bmatrix} \epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{yx} & \epsilon_{yy} & \epsilon_{yz} \\ \epsilon_{zx} & \epsilon_{zy} & \epsilon_{zz} \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}$$

Note that these same concepts can be used to expand on anisotropic conduction and resistance properties as well

Nonpolar Molecules in a Poled Dielectric



Field at the center of the cavity is

$$\vec{E}_m = \vec{E}_x + \vec{E}_d + \vec{E}_s + \vec{E}'$$

\vec{E}_x is the primary field

\vec{E}_d is the depolarizing field due to polarization charge

\vec{E}_s is the polarization charge on the cavity surface, S

\vec{E}' is due to dipoles inside the cavity surface S

$$\vec{E}_m = \frac{\varphi_s}{\epsilon_o} \vec{a}_z - \frac{1}{\epsilon_o} \vec{P} - \int \frac{(\vec{P} \cos \theta) \vec{r} da}{4\pi \epsilon_o r^3} - 0$$

$$= \frac{\varphi_s}{\epsilon_o} \vec{a}_z - \frac{1}{\epsilon_o} \vec{P} + \frac{-\vec{P}}{4\pi \epsilon_o} \int_0^{2\pi} d\phi \int_0^\pi \cos \theta (-\cos \theta) \sin \theta d\theta$$

$$= \frac{\varphi_s}{\epsilon_o} \vec{a}_z - \frac{1}{\epsilon_o} \vec{P} + \frac{1}{3\epsilon_o} \vec{P} = \vec{E} + \frac{1}{3\epsilon_o} \vec{P} \quad \text{Take only the direction along } P$$

Molecular Polarizability, α

$$\vec{P}_m = \alpha \vec{E}_m$$

For N molecules per unit volume, the polarization is $\vec{P} = N \vec{P}_m$

$$\vec{P} = N \alpha \left(\vec{E} + \frac{1}{3\epsilon_o} \vec{P} \right)$$

$$\alpha = \frac{\chi_e \epsilon_o \vec{E}}{N \left(\vec{E} + \frac{1}{3\epsilon_o} \chi_e \epsilon_o \vec{E} \right)}$$

$$\chi_e = (\epsilon_r - 1)$$

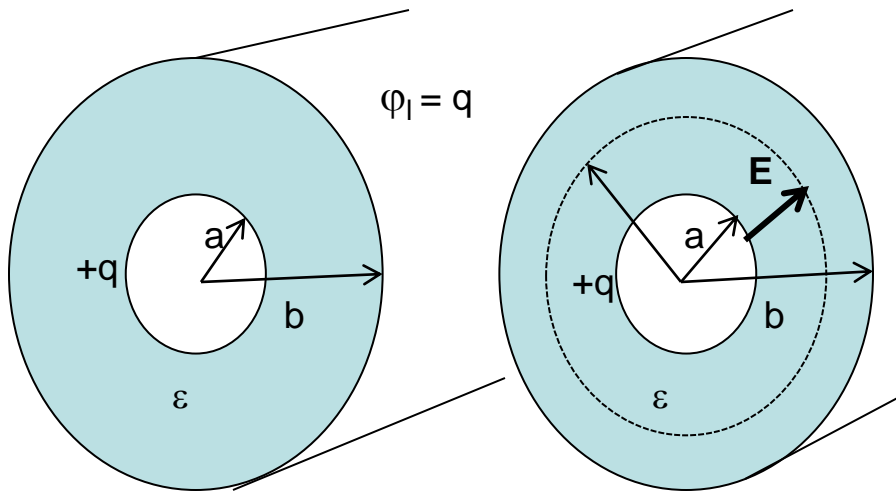
$$\alpha = \frac{3\epsilon_o}{N} \frac{\epsilon_r - 1}{\epsilon_r - 2}$$

Clausius-Mossotti eqn.

$$\vec{P} = \chi_e \epsilon_o \vec{E}$$

$$= \frac{3\chi_e}{N(3 + \chi_e)}$$

Polarization Vector in a Coaxial Cable



Assume that there is a total charge Q distributed across the conduct length of the inner shell

$$\int_L \vec{D} \cdot d\vec{S} = D2\pi\rho L = Q_{enc} = \int q dl = qL$$

$$\vec{D} = \frac{q}{2\pi\rho} \hat{\rho} \quad \vec{E} = \frac{q}{2\pi\rho\epsilon} \hat{\rho}$$

$$\vec{D} = \epsilon_o \vec{E} + \vec{P}$$

$$\vec{P} = \frac{q}{2\pi\rho} \hat{\rho} - \epsilon_o \frac{q}{2\pi\rho\epsilon} \hat{\rho} = \frac{q}{2\pi\rho} \left(1 - \frac{\epsilon_o}{\epsilon}\right) \hat{\rho} = \vec{P} = \vec{D} \left(1 - \frac{1}{\epsilon_r}\right)$$

One can show that the volume charge density in the dielectric is zero

$$\rho_v = -\nabla \cdot \vec{P} = -\frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho \vec{P}) = -\frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\frac{q}{2\pi} \left(1 - \frac{\epsilon_o}{\epsilon}\right) \right) = 0$$

Thus, the surface charge densities due to polarization of the dielectric must be equal and opposite at surfaces a and b

\Rightarrow

$$\varphi_{s,a} = \vec{P} \cdot \hat{n} = \frac{q}{2\pi\rho} \left(1 - \frac{\epsilon_o}{\epsilon}\right) \hat{\rho} \cdot (-\hat{\rho}) = -\frac{q}{2\pi\rho} \left(1 - \frac{\epsilon_o}{\epsilon}\right)$$

$$\varphi_{s,b} = \vec{P} \cdot \hat{n} = \frac{q}{2\pi\rho} \left(1 - \frac{\epsilon_o}{\epsilon}\right) \hat{\rho} \cdot (\hat{\rho}) = \frac{q}{2\pi\rho} \left(1 - \frac{\epsilon_o}{\epsilon}\right)$$

Continuity Equation

- Remembering that all charge is conserved, the time rate of decrease of charge within a given volume must be equal to the net outward flow through the surface of the volume
- Thus, the current out of a closed surface is

$$I = \oint_S \vec{J} \cdot d\vec{S} = -\frac{dQ_{\text{enclosed}}}{dt} = \int_v -\frac{\partial \rho_v}{\partial t} dv$$

Applying Stokes Theorem

$$\oint_S \vec{J} \cdot d\vec{S} = \int_v \nabla \cdot \vec{J} dv = -\int_v \frac{\partial \rho_v}{\partial t} dv$$

$$\nabla \cdot \vec{J} = -\frac{\partial \rho_v}{\partial t} \quad \text{Continuity Equation}$$

- For steady state problems, the derivative of charge with respect to time equals zero, and thus the gradient of current density at the surface is zero, showing that there can be no net accumulation of charge.

Relaxation Time Constant

- Utilizing the continuity equation and material properties such as permittivity and conductivity, one can derive a time constant (in seconds) by which to measure the relaxation time associated with the decay of charge from the point at which it was introduced within a material to the surface of that material
- We start with Ohm's and Gauss' Laws

$$\vec{J} = \sigma \vec{E}$$

$$\nabla \cdot \vec{E} = \frac{\rho_v}{\epsilon}$$

$$\nabla \cdot \vec{J} = \nabla \cdot \sigma \vec{E} = \frac{\sigma \rho_v}{\epsilon} = -\frac{\partial \rho_v}{\partial t}$$

$$\frac{\sigma \rho_v}{\epsilon} + \frac{\partial \rho_v}{\partial t} = 0$$

$$\frac{\partial \phi_v}{\partial t} = -\frac{\sigma}{\epsilon} \phi_v$$

$$\ln \phi_v = -\frac{\sigma t}{\epsilon} + \ln \phi_{v0}$$

$$\phi_v = \phi_{v0} e^{-t/T_r}$$

$$T_r = \frac{\epsilon}{\sigma} \quad \text{Time Constant (s)}$$

- The **relaxation time** is the time it takes a charge placed in the interior of a material to drop by e^{-1} (=36.8%) of its initial value.
- For good conductors T_r is approx. $2 \cdot 10^{-19}$ s.
- For good insulators T_r can be days

Electrostatic Boundary Conditions

- So far we have considered electric fields in a single medium
- If the field exist in two mediums
 - The fields within each medium obey the same theorems previously stated
 - An additional set of boundary conditions exist to match the two fields at the interface
- We shall consider boundary conditions separating
 - Dielectric media with two different permittivities
 - Conductors and dielectric media
 - Conductors and free space (one of the dielectric constants is equal to 1)
- To complete this analysis we will need both of Maxwell's Equations for Electrostatics

$$\nabla \cdot \vec{D} = \varphi_v \qquad \nabla \times \vec{E} = 0$$

- We will also need to break the electric field intensity into two orthogonal components (tangential and normal)

$$\vec{E} = \vec{E}_t + \vec{E}_n$$

Dielectric-Dielectric Boundary

- Two different dielectrics characterized by ϵ_1 and ϵ_2 .

Apply $\nabla \times \vec{E} = 0 = \oint \vec{E} \cdot d\vec{l}$ Around the patch abcd that encloses the boundary of both dielectrics

$$\oint \vec{E} \cdot d\vec{l} = 0$$

$$= E_{1t} \Delta w - E_{1n} \frac{\Delta h}{2} - E_{2n} \frac{\Delta h}{2} - E_{2t} \Delta w + E_{2n} \frac{\Delta h}{2} + E_{1n} \frac{\Delta h}{2}$$

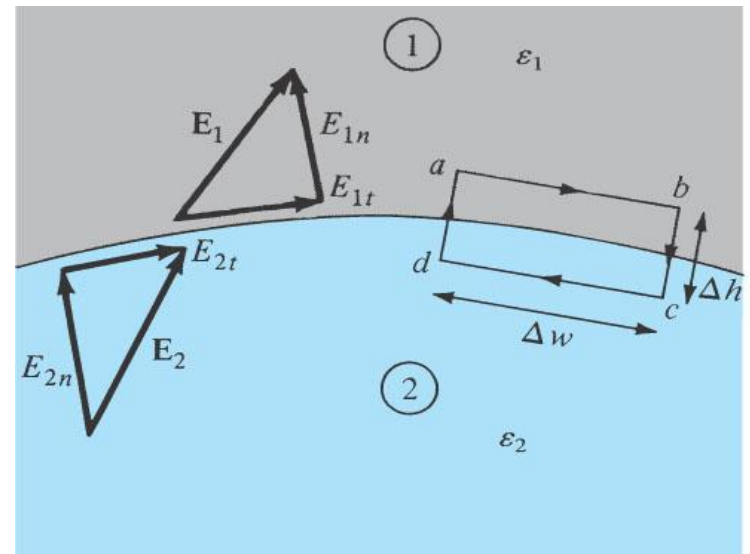
$$= E_{1t} \Delta w - E_{2t} \Delta w = (E_{1t} - E_{2t}) \Delta w$$

$$\Delta h \rightarrow 0$$

$$E_{1t} = E_{2t}$$

$$\frac{D_{1t}}{\epsilon_1} = E_{1t} = E_{2t} = \frac{D_{2t}}{\epsilon_2}$$

Tangential E undergoes no change and is continuous across the boundary condition
Tangential D on the other hand is discontinuous across the interface



Dielectric-Dielectric Boundary (2)

- Two different dielectrics characterized by ϵ_1 and ϵ_2 .

$$\text{Apply } \nabla \cdot \vec{D} = \varphi_v \Rightarrow \oint_S \vec{D} \cdot d\vec{S} = Q_{enc}$$

To a pillbox that encloses the boundary of both dielectrics

$$\Delta Q = \varphi_s \Delta S = D_{1n} \Delta S - D_{2n} \Delta S$$

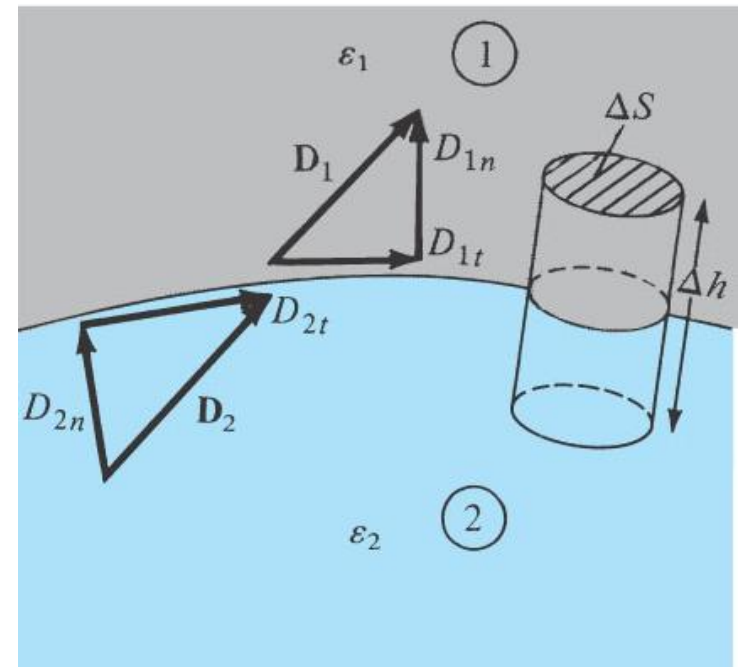
$$\varphi_s = D_{1n} - D_{2n}$$

$$\varphi_s \rightarrow 0$$

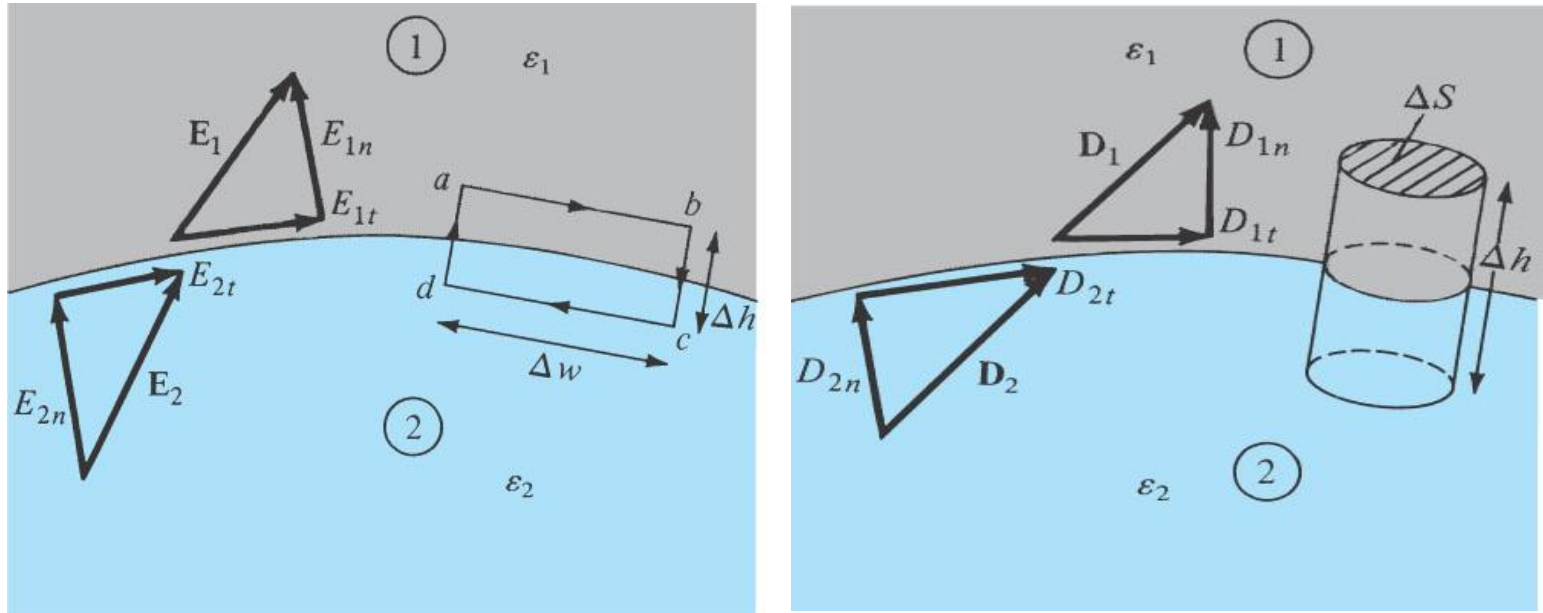
$$D_{1n} = D_{2n}$$

$$E_{1n} \epsilon_1 = D_{1n} = D_{2n} = E_{2n} \epsilon_2$$

Normal D undergoes no change and is continuous across the boundary condition
 Normal E on the other hand is discontinuous across the interface



Dielectric-Dielectric Boundary (3)



$$E_{1t} = E_{2t}$$

$$D_{1n} = D_{2n}$$

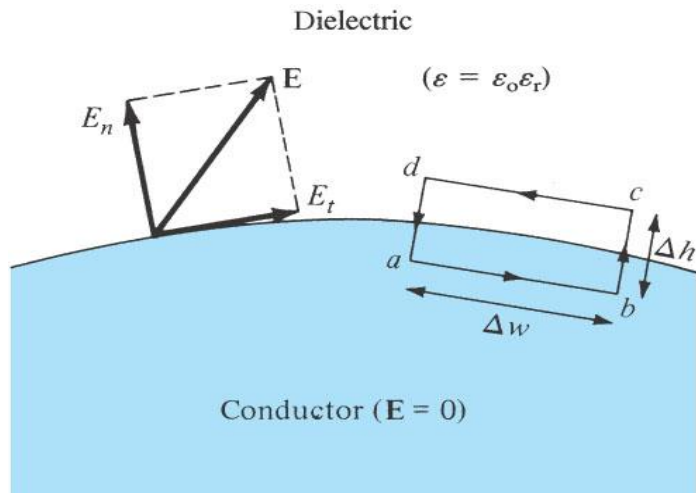
Conductor-Dielectric Boundary

- Perfect conductor with infinite conductivity (therefore no volume charge density, potential or electric field inside the conductor) and a dielectric, ϵ_2 .

Apply $\oint \vec{E} \cdot d\vec{l} = 0$

$$= 0\Delta w - E_{1n} \frac{\Delta h}{2} - E_{2n} \frac{\Delta h}{2} - E_t \Delta w + E_{2n} \frac{\Delta h}{2} + E_{1n} \frac{\Delta h}{2}$$

$$\Delta h \rightarrow 0, E_t = 0 = \frac{D_t}{\epsilon_2}$$



$$E_{2t} = 0$$

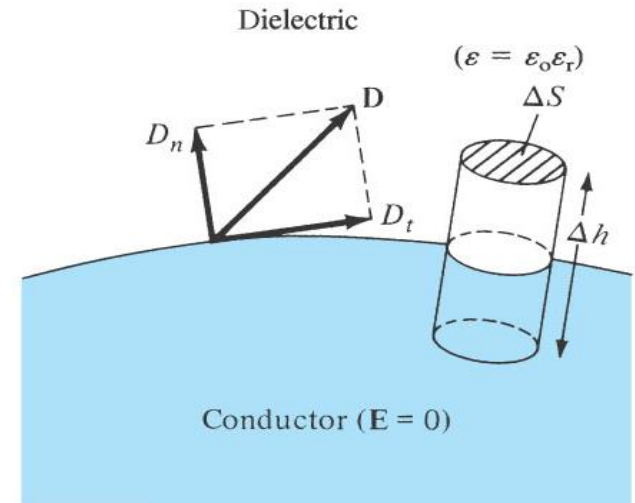
$$D_{2n} = \epsilon_2 E_{2n}$$

$$E_1 = 0$$

Apply $\oint_S \vec{D} \cdot d\vec{S} = Q_{enc}$

$$\Delta Q = \varphi_s \Delta S = D_n \Delta S - 0 \Delta S$$

$$\varphi_s = D_n = \epsilon_2 E_n$$



Snell's Law of Refraction

- Consider the boundary of two dielectrics, ϵ_1 and ϵ_2
- We can determine the refraction of the electric field across the interface using the dielectric boundary conditions provided

$$\oint \vec{E} \cdot d\vec{l} = 0$$

$$\Delta h \rightarrow 0$$

$$E_{1t} = E_{2t}$$

$$E_1 \sin \theta_1 = E_{1t} = E_{2t} = E_2 \sin \theta_2$$

$$E_1 \sin \theta_1 = E_2 \sin \theta_2$$

$$\oint_S \vec{D} \cdot d\vec{S} = Q_{enc}$$

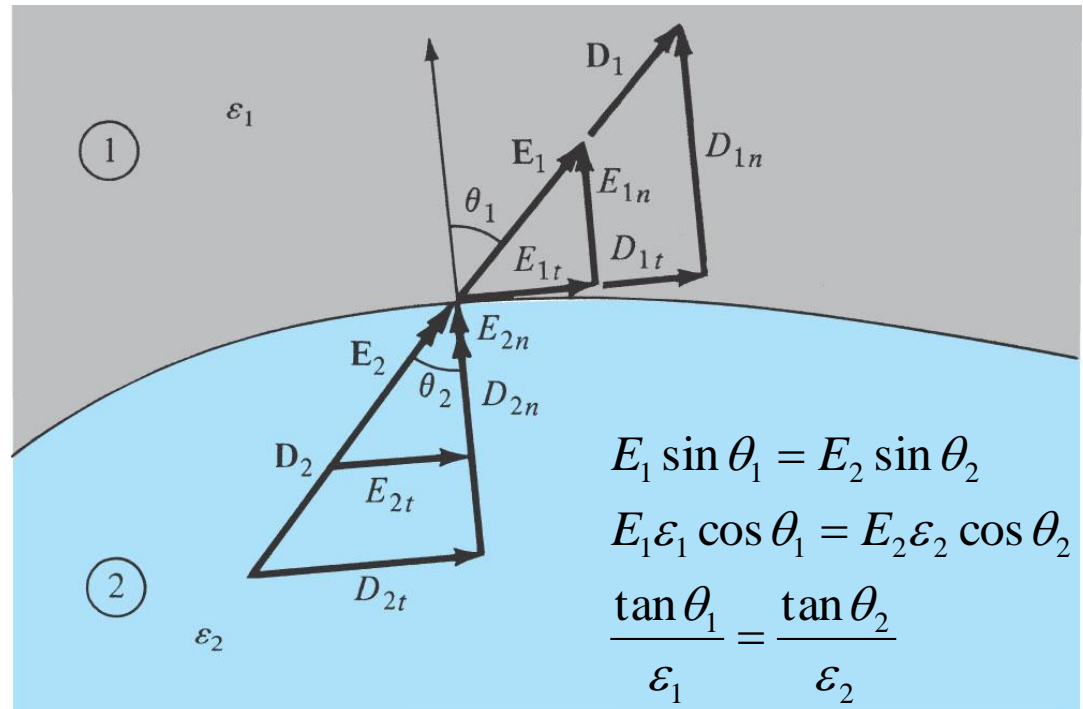
$$\varphi_s \rightarrow 0$$

$$D_{1n} = D_{2n}$$

$$E_{1n}\epsilon_1 = D_{1n} = D_{2n} = E_{2n}\epsilon_2$$

$$E_1\epsilon_1 \cos \theta_1 = D_{1n} = D_{2n} = E_2\epsilon_2 \cos \theta_2$$

$$E_1\epsilon_1 \cos \theta_1 = E_2\epsilon_2 \cos \theta_2$$



$$E_1 \sin \theta_1 = E_2 \sin \theta_2$$

$$E_1 \epsilon_1 \cos \theta_1 = E_2 \epsilon_2 \cos \theta_2$$

$$\frac{\tan \theta_1}{\epsilon_1} = \frac{\tan \theta_2}{\epsilon_2}$$

- Thus an interface between two dielectrics produces bending of flux lines as a result of unequal polarization charges that accumulate on the opposite sides of the interface