

Laplace's and Poisson's Equations

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Poisson's and Laplace's Equations

- These can be derived from Gauss's law:

$$\nabla \cdot \epsilon \mathbf{E} = \rho_v$$

$$\nabla \cdot -\epsilon \nabla V = \rho_v$$

- For a homogeneous medium:

$$\nabla^2 V = -\frac{\rho_v}{\epsilon}$$



Poisson's Equation

- Relates the potential at a point to the source at that point
- For a source free region

$$\nabla^2 V = 0$$



Laplace's Equation

- These two are the governing equations to solve electrostatic problems

- ∇^2 is a linear differential operator which can be derived using the expression of ∇

- In Cartesian coordinate system

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

- In the cylindrical coordinate system

$$\nabla^2 = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2}$$

- In the spherical coordinate system

$$\nabla^2 = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2}$$

- Uniqueness Theorem:

- Exact solution of Poisson's equation and Laplace's equation for a given geometry can be determined by applying appropriate boundary conditions
- For a given geometry and boundary conditions, the solution of these equations is unique

General Procedure for Solving Poisson's or Laplace's Equations

- Direct integration: If the domain geometry is one dimensional and simple
- Separation of variables: For two- or three-dimensional problems
- Numerical techniques: Can handle complex two- or three-dimensional problems with anisotropy/inhomogeneity

Resistance and Capacitance

Resistance:

$$R = \frac{V}{I} = \frac{\int_L \mathbf{E} \cdot d\mathbf{l}}{\int_S \sigma \mathbf{E} \cdot d\mathbf{S}}$$

- Resistance for complicated structures can be determined using the following steps:
 1. Choose a suitable coordinate system.
 2. Assume V_0 as the potential difference between conductor terminals.
 3. Solve Laplace's equation $\nabla^2 V = 0$ to obtain V . Then determine \mathbf{E} from $\mathbf{E} = -\nabla V$ and find I from $I = \int_S \sigma \mathbf{E} \cdot d\mathbf{S}$.
 4. Finally, obtain R as V_0/I .

Capacitance:

$$C = \frac{Q}{V} = \frac{\epsilon \int_S \mathbf{E} \cdot d\mathbf{S}}{\int_L \mathbf{E} \cdot d\mathbf{l}}$$

- Two approaches:
 1. Assuming Q and determining V in terms of Q (involving Gauss's law)
 2. Assuming V and determining Q in terms of V (involving solving Laplace's equation)
- The capacitance value depends on material properties and objects in vicinity

Method of images and its applications

The **image theory** states that a given charge configuration above an infinite grounded perfect conducting plane may be replaced by the charge configuration itself, its image, and an equipotential surface in place of the conducting plane.

- Solving Laplace's and Poisson's equation is avoided.

The following conditions must be satisfied to apply the image method:

- The image charge(s) must be located in the conducting region
- The image charge(s) must be located such that on the conducting surface(s) the potential is zero or constant

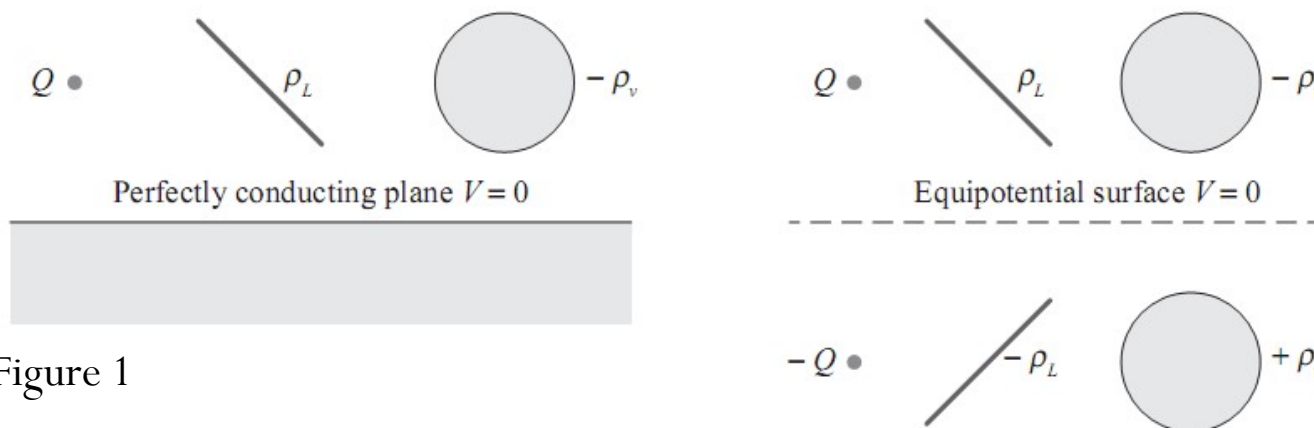


Figure 1

Application of Poisson's and Laplace's Equations

- They are useful for finding out surface charge densities and electric field in the regions of interest.
- They can be used to find out the capacitance of different structures by the application of boundary conditions appropriately.