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# System Modeling & Control

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# Module VI

## Frequency Response Analysis

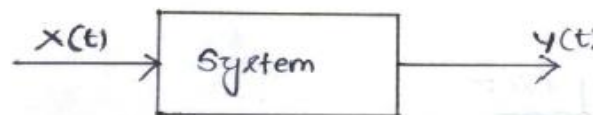


## Frequency Response Analysis:-

### → Frequency response:-

It is defined as the steady state response of a system when the input to the system is a sinusoidal signal.

Consider a linear time invariant system as shown in figure. Let  $x(t)$  be an input sinusoidal signal. The output or response  $y(t)$  is also a sinusoidal signal of same frequency but with different magnitude and frequency phase angle.



$$x(t) = x \sin \omega t$$

$$y(t) = y \sin(\omega t + \phi)$$

The magnitude and phase relationship between the sinusoidal input and the steady state output of a system is termed the frequency response. In linear time invariant systems the frequency response is independent of the amplitude and phase of the input signal.

The frequency response is normally obtained by varying the frequency of the input signal by keeping the magnitude of the input signal at a constant value.



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In the system transfer function  $T(s)$ , if  $s$  is replaced by  $j\omega$  then the resulting transfer function  $T(j\omega)$  is called sinusoidal transfer function.  $T(j\omega)$ . The frequency response of the system can be directly obtained from the sinusoidal transfer function  $T(j\omega)$  of the system. The transfer function  $T(j\omega)$  is a complex function of frequency. The magnitude and phase of  $T(j\omega)$  are function of frequency and can be ~~evaluated~~ evaluated for various values of frequency.

$$\text{So } G(j\omega) = G(s)|_{s=j\omega} = M(\omega) e^{j\theta\omega} = M \angle \theta$$

where  $M$  is the ratio of the amplitudes of the output and input sinusoid and is called as the 'magnitude ratio' or gain also.

$\theta$  is the angle by which the output leads the input. Both  $M$  and  $\theta$  are the functions of the angular frequency  $\omega$ .





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## \* Methods in frequency response:-

The magnitude function and phase function are the two plots in the frequency response characteristics. The angular frequency  $\omega$  is the common variable for both these plots.

The following curves are mainly used in frequency response analysis.

- i, polar plot:- The plot of the magnitude versus phase angle in polar coordinate where  $\omega$  is varied from 0 to  $\infty$  is called polar plot.
- ii, Bode plot:- The plots of both i, magnitude and ii, phase versus frequency  $\omega$  in logarithmic values are called Bode plot.



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### \* Advantages of frequency domain specifications:-

- i, Without the knowledge of the transfer function the frequency response of stable open loop system can be obtained experimentally.
- ii, These methods are easy to use for design of control systems and for finding absolute as well as relative stability of the system. calculations are simple.
- iii, The apparatus required for obtaining frequency response is simple and inexpensive and easy to use.
- iv, The transfer functions of complicated systems can be determined experimentally by frequency response tests.
- v, When system is designed by the use of frequency response analysis, the effects of noise disturbance and parameter variations are relatively easy to visualize and incorporate corrective measures.

\* Steady state response to sinusoidal input:- (Frequency response)

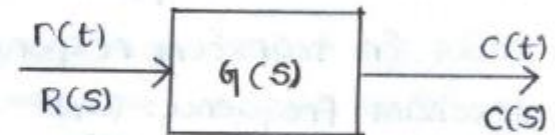
Consider a linear invariant system with its transfer function  $G(s)$  as shown.

Consider input  $r(t)$  to be purely sinusoidal.

$$r(t) = A \sin \omega t$$

$$\Rightarrow R(s) = \frac{A\omega}{s^2 + \omega^2}$$

$$\text{where } G(s) = \frac{N(s)}{(s+s_1)(s+s_2)\dots(s+s_n)}$$



$$\therefore \text{output} = C(s) = R(s)G(s) = \frac{A\omega}{s^2 + \omega^2} \cdot \frac{N(s)}{(s+s_1)(s+s_2)\dots(s+s_n)}$$

$$\Rightarrow C(s) = \frac{a_1}{s+j\omega} + \frac{a_2}{s-j\omega} + \frac{b_1}{s+s_1} + \dots + \frac{b_n}{s+s_n}$$

$$\Rightarrow C(t) = a_1 e^{-j\omega t} + a_2 e^{j\omega t} + b_1 e^{-s_1 t} + \dots + b_n e^{-s_n t}$$

$$\text{Steady state output} = \lim_{t \rightarrow \infty} C(t) = C_{ss}(t) = a_1 e^{-j\omega t} + a_2 e^{j\omega t}$$

$$a_1 = G(s) \times \frac{A\omega}{s^2 + \omega^2} \times (s+j\omega) \Big|_{s=-j\omega} = - \frac{A G(-j\omega)}{2j}$$





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$$a_2 = G(s) \times \frac{A\omega}{s^2 + \omega^2} (s - j\omega) \Big|_{s=j\omega} = \frac{A G(j\omega)}{2j}$$

$$G(j\omega) = |G(j\omega)| e^{j\phi}$$

$$\text{where } |G(j\omega)| = \sqrt{(\text{Real part})^2 + (\text{Imaginary part})^2}$$

$$\phi = \tan^{-1} \left\{ \frac{\text{Imaginary part of } G(j\omega)}{\text{Real part of } G(j\omega)} \right\} = \angle G(j\omega)$$

$$\text{Similarly } |G(-j\omega)| e^{-j\phi} = |G(j\omega)| e^{-j\phi}$$

$$\begin{aligned} C_{SS}(t) &= \frac{-A |G(-j\omega)|}{2j} e^{-j\phi} \cdot e^{-j\omega t} + \frac{A |G(j\omega)|}{2j} e^{j\phi} e^{j\omega t} \\ &= A |G(j\omega)| \left\{ \frac{e^{j(\omega t + \phi)} - e^{-j(\omega t + \phi)}}{2j} \right\} \end{aligned}$$

$$\boxed{C_{SS}(t) = A |G(j\omega)| \sin(\omega t + \phi)}$$

$$\begin{aligned} \therefore \sin \theta &= \frac{e^{j\theta} - e^{-j\theta}}{2j} \\ \cos \theta &= \frac{e^{j\theta} + e^{-j\theta}}{2j} \end{aligned}$$





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### \* Frequency domain specifications:-

The different frequency domain specifications are

- |                                    |                 |                  |
|------------------------------------|-----------------|------------------|
| i, Resonant peak ( $M_r$ ),        | ii, Bandwidth   | iii, Gain Margin |
| iv, Resonant frequency, $\omega_r$ | v, Cut-off rate | vi, Phase Margin |

#### i, Resonant peak:- ( $M_r$ )

The  $\max^m$  value of the magnitude of closed loop transfer function is called the resonant peak  $M_r$ . A large resonant peak corresponds to a large overshoot in transient response.

#### ii, Resonant frequency:- ( $\omega_r$ )

The frequency at which the resonant peak occurs is called resonant frequency  $\omega_r$ . This is related to the frequency of oscillation in the step response and thus it is indicative of the speed of the transient response.



→ Derivations of  $M_r$  and  $w_r$ :- (2nd order system)

Consider the closed loop transfer function of a second order system

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

The sinusoidal transfer function  $T(j\omega)$  is obtained by putting  $s = j\omega$

$$\begin{aligned}\therefore T(j\omega) &= \frac{C(j\omega)}{R(j\omega)} = \frac{\omega_n^2}{(j\omega)^2 + 2\zeta\omega_n(j\omega) + \omega_n^2} \\ &= \frac{\omega_n^2}{-\omega^2 + 2\zeta\omega_n j\omega + \omega_n^2} = \frac{1}{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right] + 2\zeta j \frac{\omega}{\omega_n}}\end{aligned}$$

replacing  $\frac{\omega}{\omega_n} = x$

$$\frac{C(j\omega)}{R(j\omega)} = \frac{1}{[1 - x^2] + 2\zeta j x}$$

$$\text{Magnitude } M = \frac{1}{\sqrt{(1-x^2)^2 + 4\zeta^2 x^2}}$$



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At  $x$ , which will max<sup>m</sup> magnitude, we can write

$$\begin{aligned}\frac{dM}{dx} = 0 &\Rightarrow \frac{dM}{dx} = \frac{d}{dx} \left\{ (1-x^2)^2 + 4\epsilon^2 x^2 \right\}^{-1/2} \\ &= -\frac{1}{2} \left[ (1-x^2)^2 + 4\epsilon^2 x^2 \right]^{-3/2} \frac{d}{dx} \left[ (1-x^2)^2 + 4\epsilon^2 x^2 \right] \\ &= -\frac{1}{2} \frac{1}{\left[ (1-x^2)^2 + 4\epsilon^2 x^2 \right]^{3/2}} [-4x + 4x^3 + 8\epsilon^2 x] = 0\end{aligned}$$

$$\therefore 4x [x^2 + 2\epsilon^2 - 1] = 0$$

$$\Rightarrow x = 0 \quad \text{or} \quad x^2 + 2\epsilon^2 - 1 = 0 \quad \text{i.e.} \quad x^2 = 1 - 2\epsilon^2$$

$$\Rightarrow x = \sqrt{1 - 2\epsilon^2}$$

$$\Rightarrow \frac{\omega}{\omega_n} = \sqrt{1 - 2\epsilon^2} \Rightarrow \boxed{\omega_r = \omega_n \sqrt{1 - 2\epsilon^2}}$$



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where  $\omega_r$  = Resonant frequency.

$$\Rightarrow \omega_r = \omega_n = \sqrt{1 - 2\xi^2}$$

$$\Rightarrow \frac{\omega}{\omega_n} = \sqrt{1 - 2\xi^2}$$

The ~~resonant~~ <sup>resonant</sup> peak is obtained by substituting  $\omega_r$  in expression of  $M_r$ .

$$\begin{aligned} M_r &= \frac{1}{\sqrt{[1 - (\sqrt{1 - 2\xi^2})^2]^2 + 4\xi^2(1 - 2\xi^2)}} \\ &= \frac{1}{\sqrt{4\xi^4 + 4\xi^2(1 + 2\xi^2)}} = \frac{1}{2\xi\sqrt{\xi^2 + 1 - 2\xi^2}} = \frac{1}{2\xi\sqrt{1 - \xi^2}} \end{aligned}$$

$$\Rightarrow \boxed{M_r = \frac{1}{2\xi\sqrt{1 - \xi^2}}}$$

where  $\omega_d = \omega_n\sqrt{1 - \xi^2}$

$$M_p = \frac{1}{2\xi\sqrt{1 - \xi^2}} e^{-\eta\xi/\sqrt{1 - \xi^2}}$$





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→ Bandwidth:-

It is defined as the range of frequencies for which the system gain is more than ~~-40 dB~~ -3 dB. The significance of Bandwidth is signal significant and also to measure its noise reduction. The input signal and also measures its noise rejection characteristics. It is indicative of rise time in transient response of a for a given damping factor.

As we know for 2nd order system

$$\begin{aligned}\frac{C(S)}{R(S)} &= \frac{\omega_n^2}{S^2 + 2\zeta\omega_n S + \omega_n^2} = \frac{1}{1 + \frac{2\zeta\omega_n S}{\omega_n} + \frac{\omega_n^2 S^2}{\omega_n^2}} \\ &= \frac{1}{1 + \frac{2\zeta}{\omega_n} j\omega + j^2 \frac{\omega^2}{\omega_n^2}} = \frac{1}{1 + \frac{j2\zeta\omega}{\omega_n} - \frac{\omega^2}{\omega_n^2}}\end{aligned}$$

$$\text{Let } \frac{\omega}{\omega_n} = x$$

$$\frac{C(j\omega)}{R(j\omega)} = \frac{1}{1 + 2\zeta jx - x^2} \Rightarrow \left| \frac{C(j\omega)}{R(j\omega)} \right| = \frac{1}{\sqrt{1 - 2\zeta(1-x^2) + 4\zeta^2 x^2}}$$

so for  $\left| \frac{C(j\omega)}{R(j\omega)} \right|$ , the corresponding dB value is  $20 \log \left| \frac{C(j\omega)}{R(j\omega)} \right|$  in dB

$$\therefore 20 \log \frac{1}{\sqrt{(1-x^2)^2 + 4\xi^2 x^2}} = 20 \log \frac{1}{\sqrt{(1-x^2)^2 + 4\xi^2 x^2}} = -3 \quad \text{or } 3 \text{ dB down}$$

$$\Rightarrow -20 \log \sqrt{(1-x^2)^2 + 4\xi^2 x^2} = -3$$

Taking <sup>antilog</sup> both sides

$$\sqrt{(1-x^2)^2 + 4\xi^2 x^2} = 1.4125 \quad \text{By squaring both the sides}$$

$$\Rightarrow (1-x^2)^2 + 4\xi^2 x^2 = 1.9952 \approx 2$$

$$\Rightarrow (1-x^2)^2 + 4\xi^2 x^2 = 2$$

$$\Rightarrow (1-2x^2+x^4) + 4\xi^2 x^2 = 2$$

$$\Rightarrow x^4 + x^2(4\xi^2 - 2) + 1 = 2$$

$$\Rightarrow x^4 + x^2(4\xi^2 - 2) - 1 = 0$$

$$\Rightarrow x^2 = \frac{-2(2\xi^2 - 1) \pm \sqrt{4(2\xi^2 - 1)^2 + 4}}{2}$$

$$\Rightarrow x^2 = \frac{-2(2\xi^2 - 1) \pm \sqrt{4(2\xi^2 - 1)^2 + 4}}{2}$$

$$\Rightarrow x^2 = \frac{-4\xi^2 + 2 \pm 2\sqrt{4\xi^4 - 4\xi^2 + 1 + 1}}{2} = -2\xi^2 \pm \sqrt{4\xi^4 - 4\xi^2 + 2}$$

$$\Rightarrow x = \left[ 1 - 2\xi^2 \pm \sqrt{2 - 4\xi^2 + 4\xi^4} \right]^{1/2}$$

$$\frac{\omega_b}{\omega_n} = x$$

$$\Rightarrow \text{Bandwidth } \frac{\omega_n}{x_b} = 0.$$

$$\text{Bandwidth } \omega_b = x \omega_n$$

$$\text{Bandwidth } B.\omega = \omega_n \left\{ 1 - 2\xi^2 + \sqrt{2 - 4\xi^2 + 4\xi^4} \right\}^{1/2}$$

$$\Rightarrow B.\omega = \omega_n \sqrt{1 - 2\xi^2 + \sqrt{2 - 4\xi^2 + 4\xi^4}}$$



### \* Cut-off rate:-

The slope of the log magnitude curve near the cut-off frequency is called the cut-off rate. The cut-off rate is the ability of the system to distinguish the signal from noise.

### \* Gain Margin:-

The gain Margin ( $K_g$ ) is defined as the reciprocal of the magnitude of open loop transfer function at phase cross over frequency. The frequency at which the phase of open loop transfer function is  $180^\circ$  is called the phase cross over frequency,  $\omega_{pc}$ .

$$\text{Gain Margin } K_g = \frac{1}{|G(j\omega_{pc})|}$$

$$\text{Gain Margin in dB} = 20 \log K_g = 20 \log \frac{1}{|G(j\omega_{pc})|} = -20 \log |G(j\omega_{pc})|$$

So the gain margin in dB is defined as the negative of the db magnitude of  $G(j\omega)$  at phase cross over frequency. The gain margin indicates the amount by which gain of the system can be increased without affecting the stability of the system.





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### \* Phase Margin:-

The phase margin is that amount of additional phase lag at the gain cross over frequency required to bring the system to the verge of instability.

The gain cross over frequency  $\omega_{gc}$  is the frequency at which the magnitude of the open loop transfer function is unity (or it is the frequency at which the db magnitude is zero).

The phase margin ( $\gamma$ ) is obtained by adding  $180^\circ$  to the phase angle  $\phi$  of the open loop transfer function at the gain cross over frequency.

$$\text{phase margin } \gamma = 180^\circ + \phi_{gc} \quad \text{where } \phi_{gc} = \angle G(j\omega_c)$$

### Derivation:-

The open loop transfer function of a 2nd order system is given by

$$G(s) = \frac{\omega_n^2}{s(s + 2\zeta\omega_n)}$$

The sinusoidal transfer function  $G(j\omega)$  is obtained by putting  $s = j\omega$

$$\begin{aligned} G(j\omega) &= \frac{\omega_n^2}{j\omega(j\omega + 2\zeta\omega_n)} = \frac{\omega_n^2}{-\omega^2 + j2\zeta\omega\omega_n} \\ &= \frac{\omega_n^2}{\omega_n^2 \left( -\frac{\omega^2}{\omega_n^2} + j2\zeta \frac{\omega}{\omega_n} \right)} \end{aligned}$$

$$\text{take } x = \frac{\omega}{\omega_n}$$

$$\text{Now } G(j\omega) = \frac{1}{-x^2 + j2\zeta x}$$





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Magnitude of  $G(j\omega) = |G(j\omega)| = \frac{1}{\sqrt{x^4 + 4\epsilon^2 x^2}}$

phase of  $G(j\omega) = \angle G(j\omega) = -\tan^{-1} \frac{2\epsilon x}{-x^2} = -\tan^{-1} \frac{2\epsilon}{-x} = \tan^{-1} \frac{2\epsilon}{x}$

At the gain crossover frequency  $\omega_{gc}$ , the magnitude of  $G(j\omega)$  is unity.

Take  $x_{gc} = \frac{\omega_{gc}}{\omega_n}$

$\therefore$  Now at  $x = x_{gc}$   $|G(j\omega)| = 1 \Rightarrow \frac{1}{\sqrt{x_{gc}^4 + 4\epsilon^2 x_{gc}^2}} = 1$   $x_{gc}^2 = \frac{-4\epsilon^2 \pm \sqrt{16\epsilon^4 + 4}}{2}$

$\Rightarrow x_{gc}^4 + 4\epsilon^2 x_{gc}^2 = 1 \Rightarrow x_{gc}^4 + 4\epsilon^2 x_{gc}^2 - 1 = 0$   $= -2\epsilon^2 \pm \sqrt{4\epsilon^4 + 1}$

$x_{gc} = \left[ -2\epsilon^2 + \sqrt{4\epsilon^4 + 1} \right]^{1/2}$  (By taking the +ve sign only)

The phase margin  $\gamma = 180^\circ + \angle G(j\omega)|_{\omega=\omega_{gc}, x=x_{gc}}$

$\gamma = 180^\circ + \tan^{-1} \frac{2\epsilon}{x_{gc}} = 180^\circ + \tan^{-1} \frac{2\epsilon}{(-2\epsilon^2 + \sqrt{4\epsilon^4 + 1})^{1/2}}$

(Note:- The gain margin of 2nd order system is infinite)