
Continuum Mechanics

Lecture 7

Theory of 2D potential flows

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Outline

- velocity potential and stream function
- complex potential
- elementary solutions
- flow past a cylinder
- lift force: Blasius formulae
- Joukowski transform: flow past a wing
- Kutta condition
- Kutta-Joukowski theorem

2D potential flows

From the Helmholtz decomposition, we have $\vec{v} = \vec{\nabla}\phi + \vec{\nabla} \times \vec{A}$

2D flows are defined by $\partial_z(\cdot) = 0$ and $v_z = 0$.

We have therefore $\vec{A} = \psi \vec{e}_z$

We consider in this chapter incompressible and irrotational flows.

$$\begin{array}{lll} \vec{\nabla} \cdot \vec{v} = 0 & \Delta\phi = 0 & + \text{B. C.} \\ \vec{\nabla} \times \vec{v} = 0 & \Delta\psi = 0 & \vec{v} \cdot \vec{n} = 0 \end{array}$$

We have two alternative but equivalent approaches.

$v_x = \partial_x\phi$ $v_y = \partial_y\phi$ where the velocity potential satisfies the Laplace equation.

$v_x = \partial_y\psi$ $v_y = -\partial_x\psi$ where the stream function satisfies the Laplace equation.

In the potential case, the irrotational condition is satisfied automatically.

In the stream function approach, this is the divergence free condition.

Since both conditions are satisfied, both velocity fields are equal.

Isopotential curves and stream lines

The velocity field is defined equivalently by two scalar fields

$$v_x = \partial_x \phi = \partial_y \psi$$

$$v_y = \partial_y \phi = -\partial_x \psi$$

They are conjugate functions that satisfy the Cauchy-Riemann relations.

They are also harmonic functions (Laplace equation), with however different B. C.

$$\Delta \phi = 0 \quad \text{in } S \quad \text{with} \quad \vec{\nabla} \phi \cdot \vec{n} = 0 \quad \text{on the boundary } L$$

$$\Delta \psi = 0 \quad \text{in } S \quad \text{with} \quad \vec{\nabla} \psi \cdot \vec{t} = 0 \quad \text{on the boundary } L \quad \text{or} \quad \psi = \text{constant} \quad \text{along } L$$

Isopotential curves are defined by $d\phi = dx\partial_x\phi + dy\partial_y\phi = 0$ or $\frac{dy}{dx} = -\frac{v_x}{v_y}$

Stream lines are defined by $d\psi = dx\partial_x\psi + dy\partial_y\psi = 0$ or $\frac{dy}{dx} = \frac{v_y}{v_x}$

Isopotential curves and stream lines are orthogonal to each other.

$$\partial_x \psi \partial_x \phi + \partial_y \psi \partial_y \phi = (-v_y)v_x + (+v_x)v_y = 0$$

Complex potential and complex derivative

We define the complex potential $F(z) = \phi(x, y) + i\psi(x, y)$

where $z = x + iy$ and $i^2 = -1$.

From complex derivation theory, we know that any complex function F is differentiable if and only if the two functions ϕ and ψ satisfy the Cauchy-Riemann relations. Such complex functions are called *analytic*.

Luckily, since the velocity potential and the stream function are conjugate, the complex velocity potential is differentiable.

We define the complex velocity $w(z) = \frac{dF}{dz}$
where the complex derivative is defined as $\frac{dF}{dz} = \frac{\partial F}{\partial x} = \frac{1}{i} \frac{\partial F}{\partial y}$

We obtain $w(z) = \partial_x \phi + i\partial_x \psi = \partial_y \psi - i\partial_y \phi = v_x - iv_y$

In cylindrical coordinates, we have $v_r = \partial_r \phi = \frac{1}{r} \partial_\theta \psi$ $v_\theta = \frac{1}{r} \partial_\theta \phi = -\partial_r \psi$

and the complex velocity writes $w(z) = v_x - iv_y = (v_r - iv_\theta) \exp -i\theta$

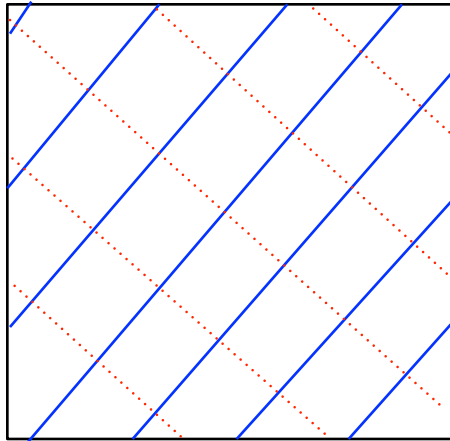
Uniform flow

Complex potential

$$F(z) = U \exp^{-i\alpha} z$$

Complex velocity

$$w(z) = U \exp^{-i\alpha}$$



Velocity field

$$v_x = U \cos(\alpha)$$

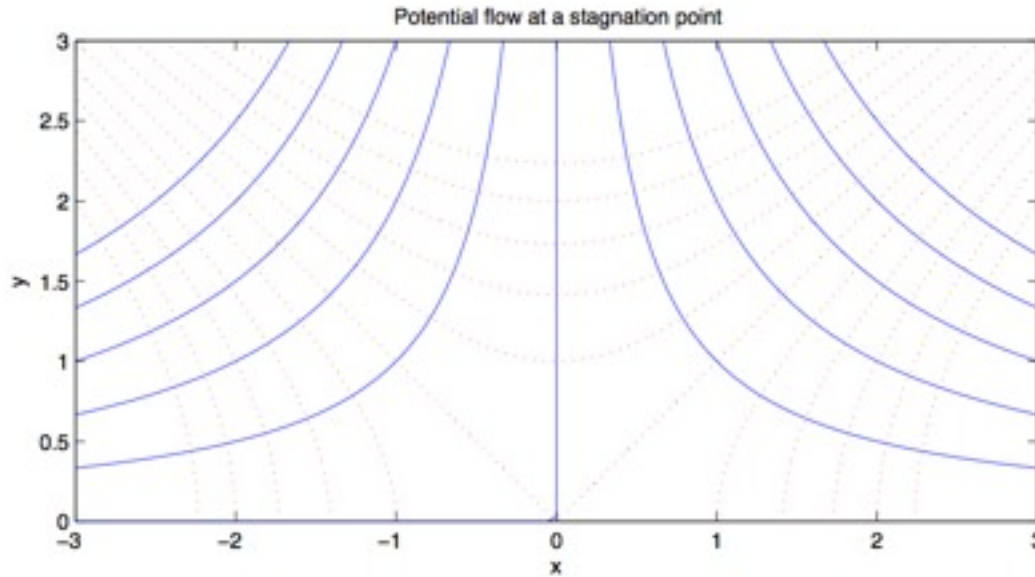
$$v_y = U \sin(\alpha)$$

..... Potential lines
———— Streamlines

Velocity potential $\phi = U \cos(\alpha)x + U \sin(\alpha)y$

Stream function $\psi = U \cos(\alpha)y - U \sin(\alpha)x$

Stagnation flow



$$F(z) = Cz^2$$

Streamlines are hyperbolae.

$$\phi = C(x^2 - y^2)$$

$$v_x = 2Cx$$

$$\psi = 2Cxy$$

$$v_y = -2Cy$$

In polar coordinates

$$\phi = Cr^2 \cos(2\theta)$$

$$\psi = Cr^2 \sin(2\theta)$$

This potential can also be used to describe a flow past a corner.

Flow past an edge

Complex potential

$$F(z) = c\sqrt{z} = cr^{1/2} \exp^{i\theta/2}$$

Velocity potential

$$\phi = c\sqrt{r} \cos \frac{\theta}{2}$$

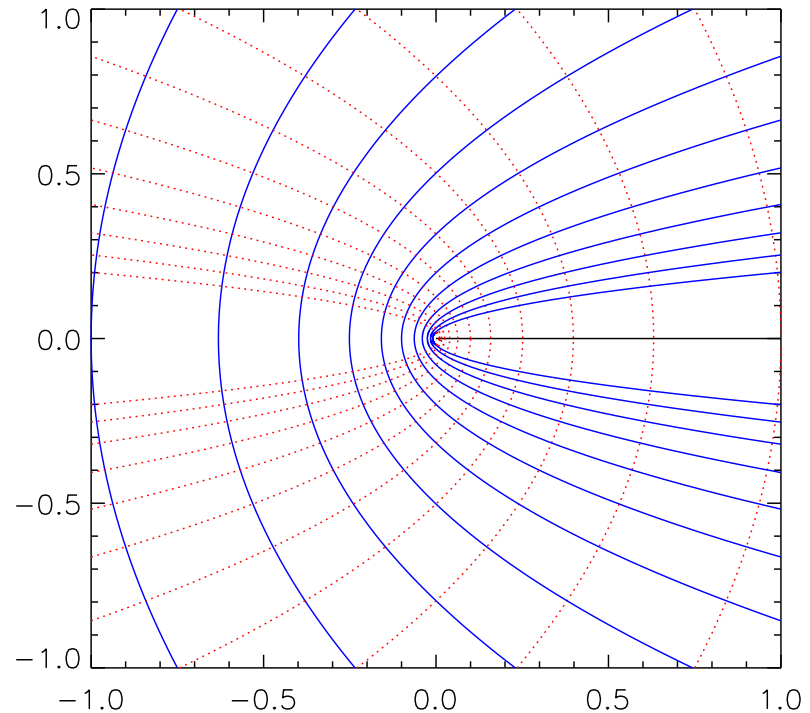
Stream function

$$\psi = c\sqrt{r} \sin \frac{\theta}{2}$$

Velocity field

$$v_r = \frac{c}{2} \frac{1}{\sqrt{r}} \cos \frac{\theta}{2}$$

$$v_\theta = \frac{c}{2} \frac{1}{\sqrt{r}} \sin \frac{\theta}{2}$$



The velocity field becomes infinite at the tip of the edge.

Flow around a source or a sink

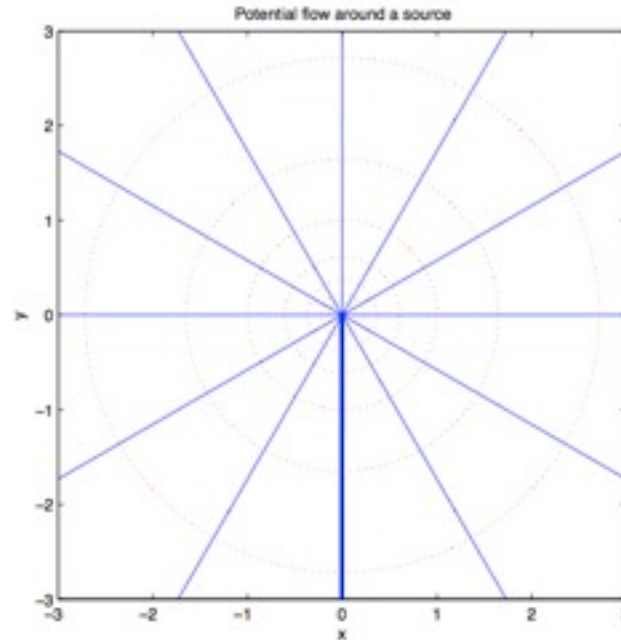
Complex potential

$$F(z) = \frac{m}{2\pi} \log(z - z_0)$$

Complex velocity

$$w(z) = \frac{m}{2\pi} \frac{1}{z - z_0}$$

$$\phi = \frac{m}{2\pi} \log r \quad \psi = \frac{m\theta}{2\pi}$$



Velocity field

$$v_x = \frac{m}{2\pi} \frac{x - x_0}{r^2}$$

$$v_y = \frac{m}{2\pi} \frac{y - y_0}{r^2}$$

In polar coordinates

$$v_r = \frac{m}{2\pi r}$$

The velocity divergence is zero everywhere $\vec{\nabla} \cdot \vec{v} = \partial_r v_r + \frac{v_r}{r} = 0$ for $r > 0$.

We apply the divergence theorem to a circle centered on the singularity:

$$\int_L \vec{v} \cdot \vec{n} dl = v_r 2\pi r = m$$

The kinetic energy in the flow is
$$\int_S \frac{1}{2} v^2 dx dy = \int_{R_{\min}}^{R_{\max}} \frac{m^2}{4\pi} \frac{dr}{r} = \frac{m^2}{4\pi} \log \left(\frac{R_{\max}}{R_{\min}} \right)$$

In a real flow, the singularity is usually embedded inside the boundary condition.

Flow around a point vortex

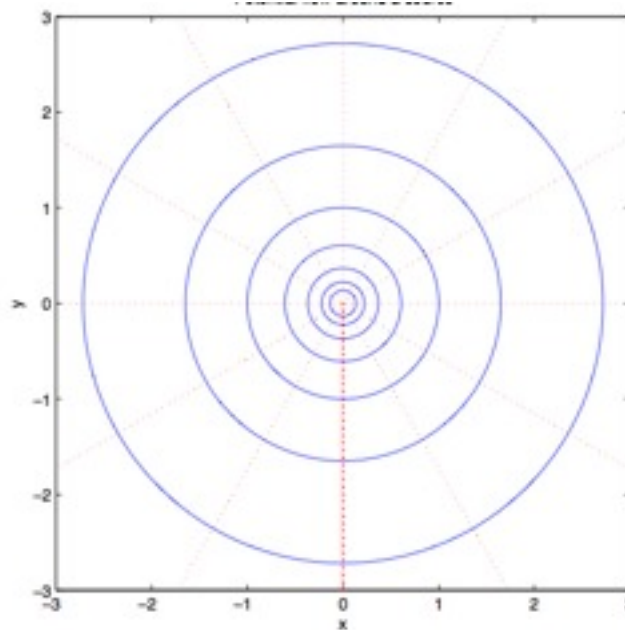
Complex potential

$$F(z) = -i \frac{\Omega}{2\pi} \log(z - z_0)$$

Complex velocity

$$w(z) = -i \frac{\Omega}{2\pi} \frac{1}{z - z_0}$$

$$\phi = \frac{\Omega\theta}{2\pi} \quad \psi = -\frac{\Omega}{2\pi} \log r$$



Velocity field

$$v_x = -\frac{\Omega}{2\pi} \frac{y - y_0}{r^2}$$

$$v_y = \frac{\Omega}{2\pi} \frac{x - x_0}{r^2}$$

In polar coordinates

$$v_\theta = \frac{\Omega}{2\pi r}$$

The velocity curl is zero everywhere $\vec{\nabla} \times \vec{v} = \left(\partial_r v_\theta + \frac{v_\theta}{r} \right) \vec{e}_z = 0$ for $r > 0$.

We apply the curl theorem on a circle centered on the singularity:

$$\int_L \vec{v} \cdot \vec{t} dl = v_\theta 2\pi r = \Omega$$

The kinetic energy in the flow is $\int_S \frac{1}{2} v^2 dx dy = \int_{R_{\min}}^{R_{\max}} \frac{m^2}{4\pi} \frac{dr}{r} = \frac{m^2}{4\pi} \log \left(\frac{R_{\max}}{R_{\min}} \right)$

There is a direct analogy with the energy of dislocations in a solid.

Superposition principle and boundary conditions

Like for the Navier equation for thermoelastic equilibrium problems, the Laplace equation for the potential and/or the stream function is a linear boundary value problem.

When proper boundary conditions are imposed (no vorticity), the solution always exists and is unique. Two different solutions can be added linearly and the sum represent also a solution with the corresponding boundary conditions.

The previous elementary solutions form a library that you can combine to build up more complex curl-free and divergence-free flows.

Streamlines are perpendicular to potential curves. The velocity component normal to a streamline is always zero. Therefore, each streamline can be used to define *a posteriori* the boundary condition.

You can therefore add up randomly complex potential to get any kind of analytical complex function. Then, you compute the streamlines. Then, you define the embedded body by picking any streamline. You finally get yourself a valid potential flow !

Flow around a doublet

We superpose a source and a sink

$$F(z) = \frac{m}{2\pi} (\log(z - z_0) - \log(z + z_0))$$

very close to each other $z_0 = \epsilon \exp^{i\alpha}$

Taylor expanding, we find

$$F(z) \simeq -\frac{m\epsilon \exp^{i\alpha}}{\pi z} = -\frac{\mu \exp^{i\alpha}}{z}$$

Parameter μ is called the doublet strength.

For $\alpha = 0$, we find

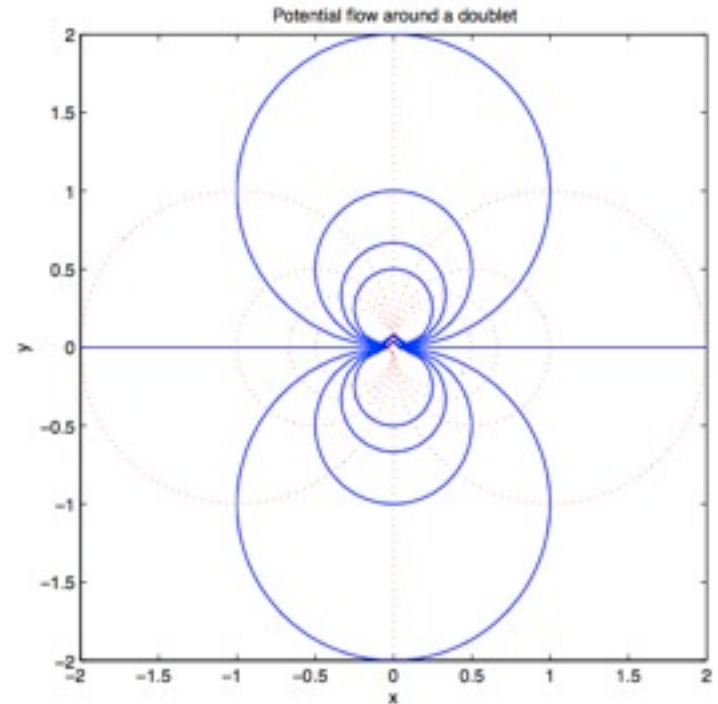
$$\phi = -\frac{\mu x}{r^2} = -\frac{\mu \cos \theta}{r}$$

$$\psi = \frac{\mu y}{r^2} = \frac{\mu \sin \theta}{r}$$

The velocity field is given by $v_r = \partial_r \phi = \frac{\mu \cos \theta}{r^2}$ and $v_\theta = \frac{1}{r} \partial_\theta \phi = \frac{\mu \sin \theta}{r^2}$

It is a dipole field $\vec{v} = \mu \frac{\vec{r}}{r^3}$

The general form around an arbitrary center z_0 is: $F(z) = -\mu \frac{\exp^{i\alpha}}{z - z_0}$



Potential and streamlines are circles.

Flow past a cylinder

We superpose a uniform flow and a doublet.

$$F(z) = U_\infty z + \frac{\mu}{z}$$

We find $\phi = U_\infty x + \mu \frac{x}{r^2}$ $\psi = U_\infty y - \mu \frac{y}{r^2}$

The streamline $\psi = 0$ is the circle $r = \sqrt{\frac{\mu}{U_\infty}}$

We reverse engineer the process.

For a cylinder a radius a , if we define $\mu = U_\infty a^2$

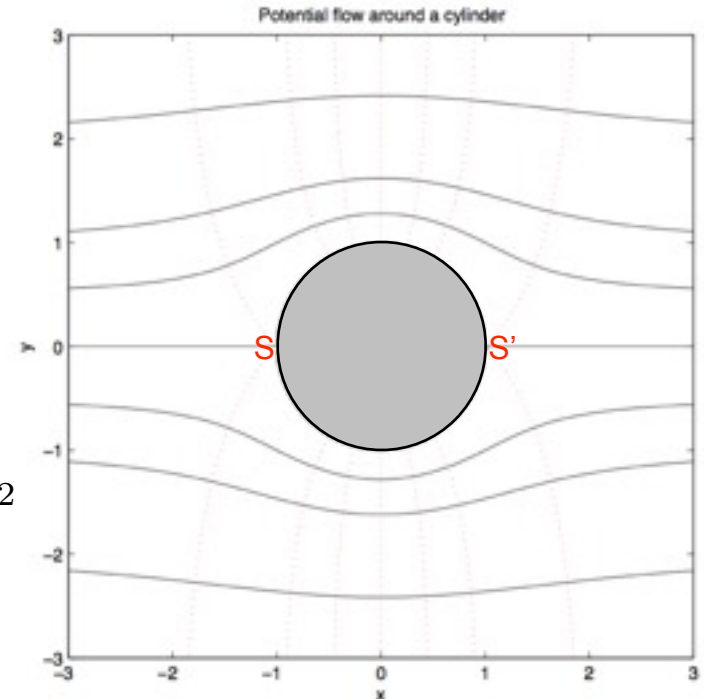
then the potential flow around the cylinder is

$$F(z) = U_\infty \left(z + \frac{a^2}{z} \right)$$

The velocity field is given by $v_r = U_\infty \left(1 - \frac{a^2}{r^2} \right) \cos \theta$ $v_\theta = -U_\infty \left(1 + \frac{a^2}{r^2} \right) \sin \theta$

The flow has 2 stagnation points S and S' given by $r=a$ and $\theta=0$ and π .

The doublet is inside the embedded body, so there is no singularity in the flow.



Force acting on the cylinder

Using the second Bernoulli theorem (curl-free, incompressible, no gravity),

we know that the quantity $H = \frac{P}{\rho} + \frac{1}{2}v^2$ is uniform.

We thus have $p + \frac{1}{2}\rho v^2 = p_\infty + \frac{1}{2}\rho U_\infty^2$

The force acting on the cylinder is given by $\vec{F} = - \int_L p \vec{n} dl$

On the cylinder, we have $v_r = 0$ and $v_\theta = -2U_\infty \sin \theta$

The pressure field on the cylinder is thus $p = p_\infty + \frac{1}{2}\rho U_\infty^2 (1 - 4 \sin^2 \theta)$

Using $\vec{n} = (\cos \theta, \sin \theta)$ we find:

$$F_x = -\frac{1}{2}\rho U_\infty^2 a \int_0^{2\pi} (1 - 4 \sin^2 \theta) \cos \theta d\theta \propto \left[\sin \theta - \frac{4}{3} \sin^3 \theta \right]_0^{2\pi} = 0$$

$$F_y = -\frac{1}{2}\rho U_\infty^2 a \int_0^{2\pi} (1 - 4 \sin^2 \theta) \sin \theta d\theta \propto \left[3 \cos \theta - \frac{4}{3} \cos^3 \theta \right]_0^{2\pi} = 0$$

Exercise: compute the torque on the cylinder (use the cylinder axis). It is also zero !

Flow past a cylinder with vorticity

We superpose a uniform flow, a doublet and a vortex.

$$F(z) = U_\infty \left(z + \frac{a^2}{z} \right) - \frac{i\Omega}{2\pi} \log \left(\frac{z}{a} \right)$$

Streamlines are given by

$$\psi = U_\infty \left(1 - \frac{a^2}{r^2} \right) y - \frac{\Omega}{2\pi} \log \left(\frac{r}{a} \right)$$

The cylinder $r=a$ is still a proper boundary condition.

$$v_r = U_\infty \left(1 - \frac{a^2}{r^2} \right) \cos \theta$$

$$v_\theta = -U_\infty \left(1 + \frac{a^2}{r^2} \right) \sin \theta + \frac{\Omega}{2\pi r}$$

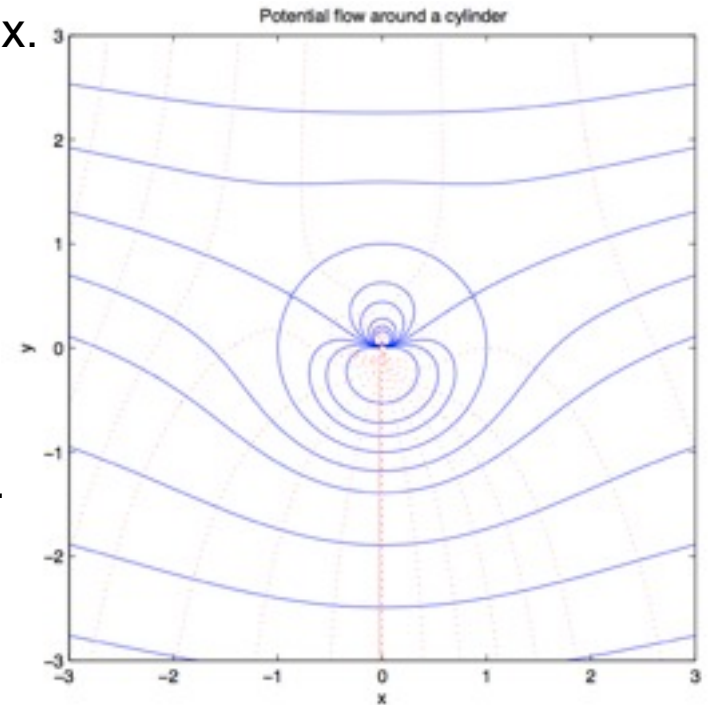
On the cylinder, we have to stagnation point given by $\sin \theta_s = \frac{\Omega}{4\pi U_\infty a}$

or one stagnation point away from the cylinder if $\Omega < -4\pi U_\infty a$

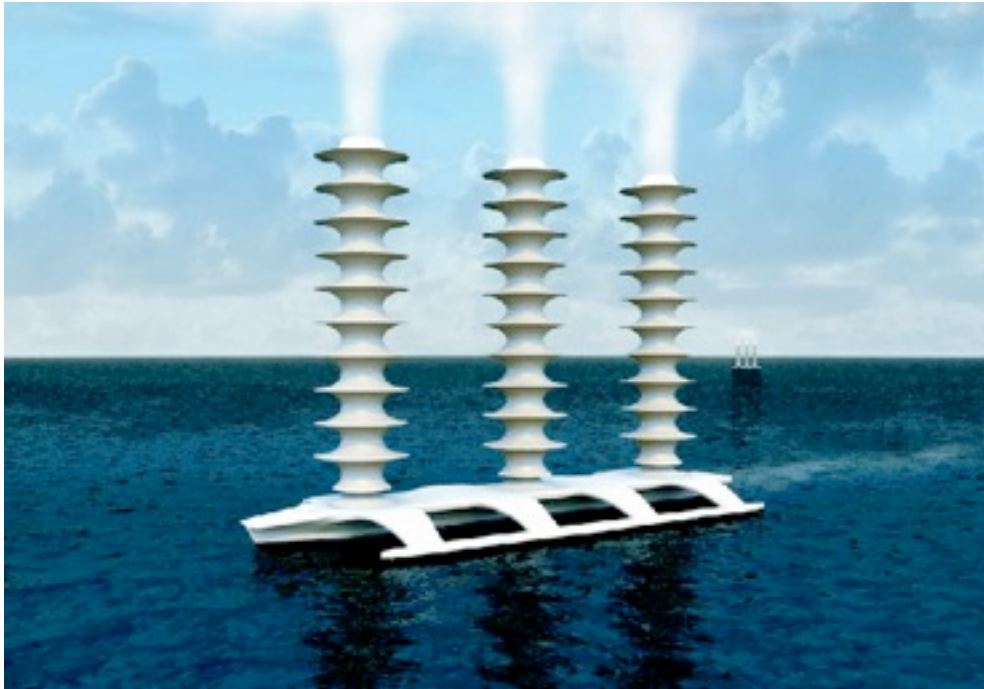
At the boundary, we have $v_\theta = -2U_\infty (\sin \theta - \sin \theta_s)$

Using the Bernoulli theorem and integrating the pressure field on the boundary,

we can compute the force on the cylinder (exercise) $F_x = 0$ $F_y = -\rho U_\infty \Omega$



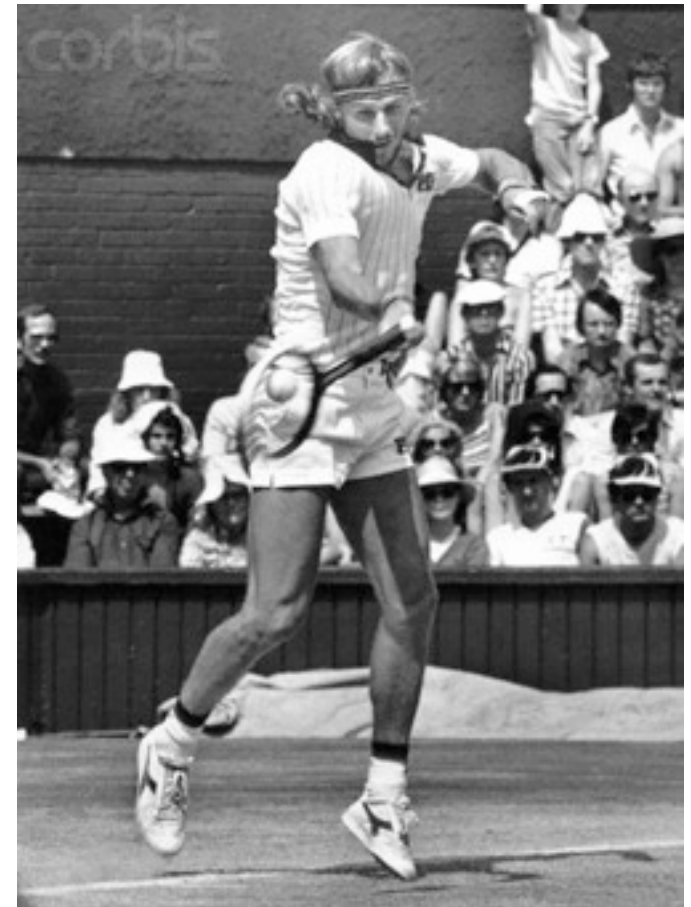
The Magnus effect



Rotating pipes induce a force perpendicular to the wind direction

Warning: viscosity effects can't be ignored !

Topspin tennis ball trajectory curves down.



The complex force: Blasius formulae

$$\vec{F} = - \int_L p \vec{n} dl$$

We use curvilinear coordinates along the body

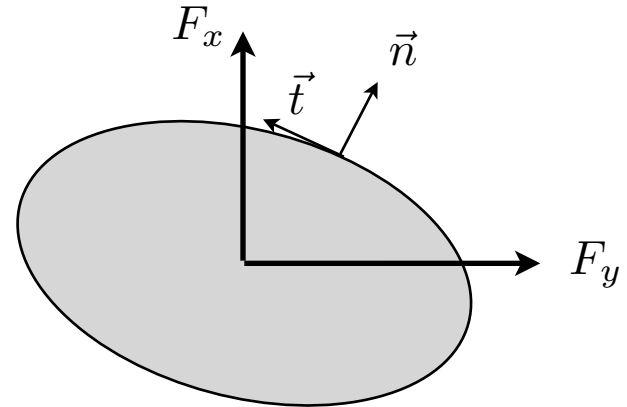
$$\vec{t} = (\cos \theta, \sin \theta) \quad \vec{n} = (\sin \theta, -\cos \theta)$$

The force components are

$$F_x = - \int_L p \sin \theta dl \quad F_y = \int_L p \cos \theta dl$$

In Cartesian coordinates, we have

$$dx = \cos \theta dl \quad dy = \sin \theta dl$$



The complex force is defined as

$$F_x - iF_y = - \int_L p (dy + idx) = -i \int_L p dz^*$$

Bernoulli theorem: $p = p_\infty + \frac{1}{2}\rho U_\infty^2 - \frac{1}{2}\rho v^2$ with $v^2 = w(z)w(z)^*$

Boundary condition: $\vec{v} \cdot \vec{n} = 0 \longrightarrow v_x dy - v_y dx = 0$ and $w^* dz^* = w dz$

We finally get the force for an arbitrary shaped body:

$$F_x - iF_y = \frac{i}{2}\rho \int_L w^2(z) dz$$

The complex circulation

We consider an arbitrary closed contour in the complex plane.

We define the complex circulation as $C = \int_L w(z) dz$

Using the same definition as before along the contour, we have

$$C = \int_L (v_x dx + v_y dy) + i \int_L (v_x dy - v_y dx)$$

where the Cartesian coordinates are related to the curvilinear ones by

$$dx = dl \cos \theta \quad dy = dl \sin \theta$$

We finally get $C = \int_L \vec{v} \cdot \vec{t} dl + i \int_L \vec{v} \cdot \vec{n} dl = \Gamma + iQ$

Γ is the physical circulation and Q is the physical mass flux.

On the contour defining the body shape, the mass flux is zero and we have

$$C = \Gamma = \int_L \vec{v} \cdot \vec{t} dl$$

Conformal mapping

We need to build more complex profile than just a cylinder. We use for that a mathematical trick called *conformal mapping*.

A conformal mapping is a differentiable complex function M that maps the complex plane z into another complex plane Z .

We have $Z = M(z)$ and $z = m(Z)$ with $m = M^{-1}$

If a flow is defined by a potential function $f(z)$ in the z plane, then the function

$$F(Z) = f(m(Z))$$

is also analytic (it satisfies the Cauchy-Riemann relations). It is therefore a valid vector potential.

The new streamlines and potential curves are the transform of the old one.

The new complex velocity writes $W(Z) = \frac{dF}{dZ} = \frac{df}{dz} \frac{dm}{dZ} = w(z)m'(Z)$

The complex circulation is conserved by conformal mapping

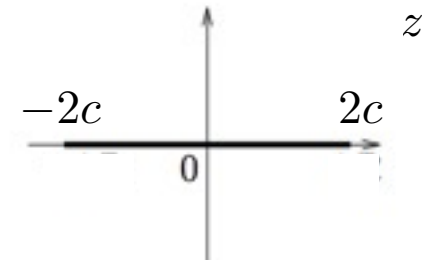
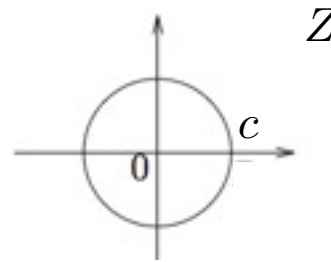
$$C = \int_L W(Z)dZ = \int_L w(z)m'(Z)dZ = \int_l w(z)dz$$

The Joukowski transform

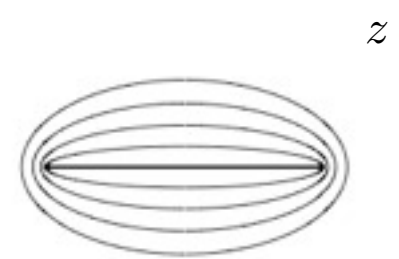
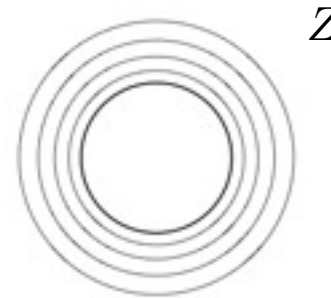
Definition: $z = Z + \frac{c^2}{Z}$

The circle of radius c becomes the line segment $[-2c, 2c]$

$$Z = c \exp^{i\theta} \longrightarrow z = 2c \cos \theta$$



The circle of radius $a > c$ becomes an ellipse.



$$Z = a \exp^{i\theta}$$

$$\longrightarrow z = \left(a + \frac{c^2}{a}\right) \cos \theta + i \left(a - \frac{c^2}{a}\right) \sin \theta$$

The inverse transform is $Z = \frac{z}{2} + \sqrt{\left(\frac{z}{2}\right)^2 - c^2}$

The derivative $M'(z) = \frac{1}{2} + \frac{z}{2\sqrt{\left(\frac{z}{2}\right)^2 - c^2}}$ has 2 singular points at $z = \pm 2c$

Acyclic flow past an ellipse

We use the Joukowski transform from a flow past a circular cylinder.

The flow is acyclic: no circulation and no vortex component.

We assume that the flow at infinity is at an angle with the x-axis.

The complex potential and velocity of the original flow are

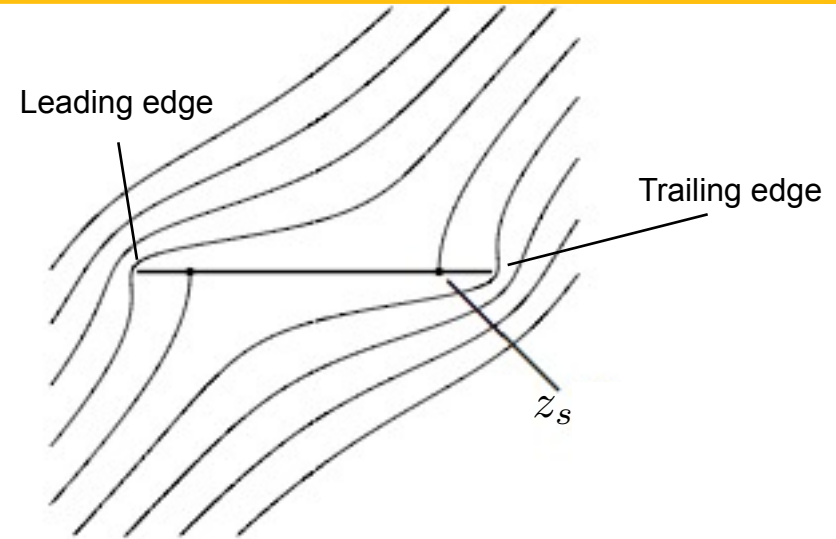
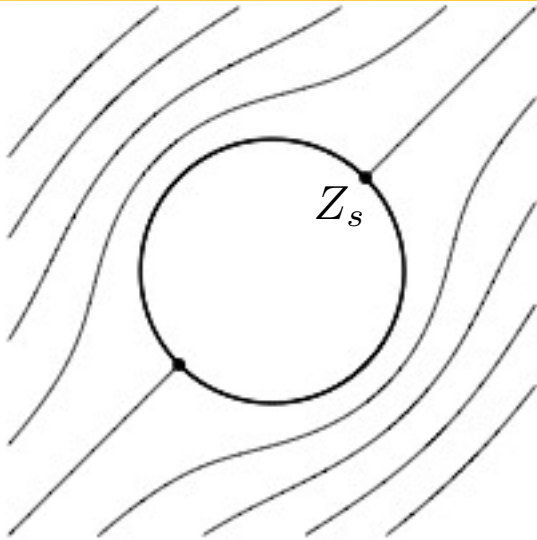
$$F(Z) = U_{\infty} \left(Z \exp^{-i\alpha} + a^2 \frac{\exp^{i\alpha}}{Z} \right) \quad W(Z) = U_{\infty} \exp^{-i\alpha} \left(1 - \frac{a^2}{Z^2} \exp^{i2\alpha} \right)$$

Using the Joukowski mapping $Z = M(z)$ with $a > c$, we get the potential around an ellipsoidal cylinder.

$$\text{Using } \frac{c^2}{Z} = z - Z, \text{ we get } f(z) = U_{\infty} \left[z \frac{a^2}{c^2} \exp^{i\alpha} + M(z) \left(\exp^{-i\alpha} - \frac{a^2}{c^2} \exp^{i\alpha} \right) \right]$$

The original stagnation points $Z_s = \pm a \exp^{i\alpha}$ become $z_s = \pm \left(a \exp^{i\alpha} + \frac{c^2}{a} \exp^{-i\alpha} \right)$

Acyclic flow past a plate



We use the previous results, taking $c \rightarrow a$

$$f(z) = U_\infty \left[z \exp^{i\alpha} - 2i \sin \alpha \left(\frac{z}{2} + \sqrt{\left(\frac{z}{2}\right)^2 - a^2} \right) \right]$$

The stagnation points are on the x-axis $z_s = \pm 2a \cos \alpha$

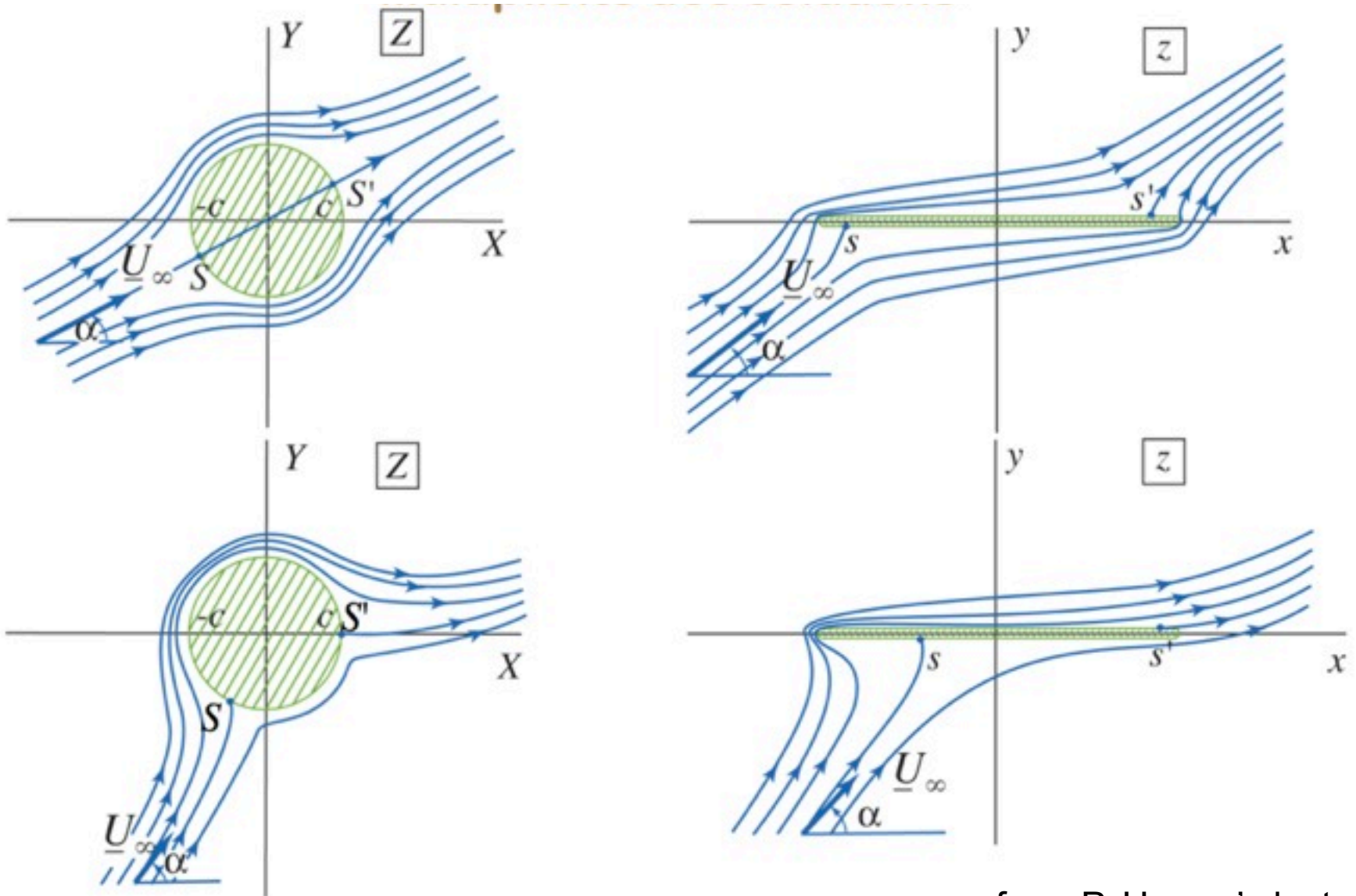
The complex velocity is given by $w(z) = W(Z)M'(z)$

The velocity at the leading and trailing edges is:

$$Z = \pm a \quad W(\pm a) = -2iU_\infty \sin \alpha \quad w(\pm 2a) \rightarrow \infty$$

(see flow past an edge). This is unphysical !

Flow past a plate with circulation



from P. Huerre's lectures

Flow past a plate with circulation

On the original circular cylinder, we have:

$$F(Z) = U_{\infty} \left(Z \exp^{-i\alpha} + a^2 \frac{\exp^{i\alpha}}{Z} \right) - \frac{i\Omega}{2\pi} \log \left(\frac{Z}{a} \right)$$
$$W(Z) = U_{\infty} \exp^{-i\alpha} \left(1 - \frac{a^2}{Z^2} \exp^{i2\alpha} \right) - \frac{i\Omega}{2\pi Z}$$

The stagnation points are now defined by $\sin(\theta_s - \alpha) = \frac{\Omega}{4\pi U_{\infty} a}$

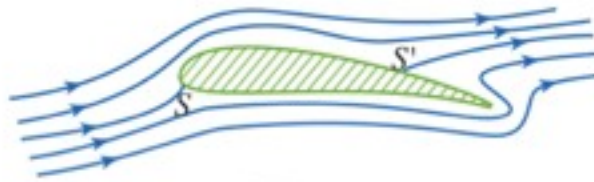
We still have an infinite number of solutions, depending on the value of the point vorticity.

For a particular value of the circulation, the stagnation point will coincide with the trailing edge, therefore removing the singularity.

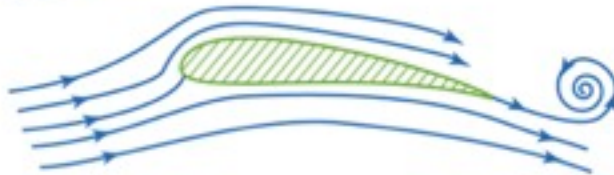
$$\Omega_c = -\sin \alpha 4\pi U_{\infty} a$$

For a given body shape, we always choose the critical circulation as defining the unique *physical* solution.

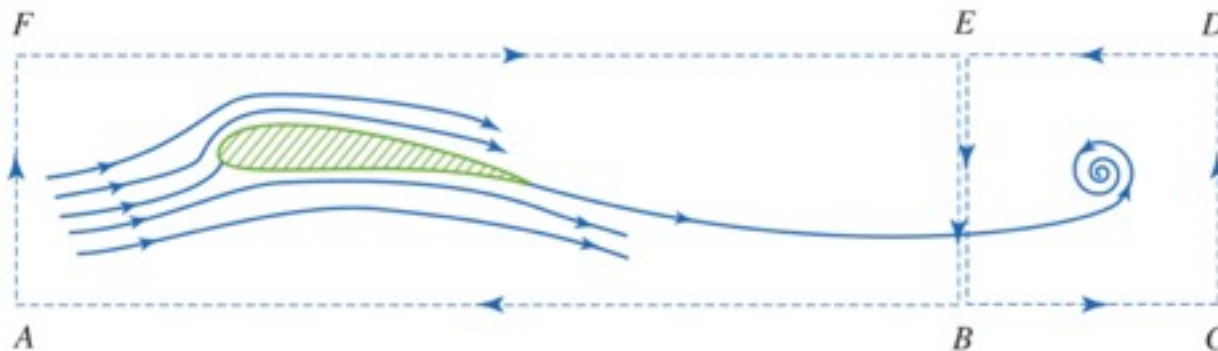
The Kutta condition



Initially, we have zero circulation



Starting vortex produces vorticity



Kelvin's theorem

«A body with a sharp trailing edge which is moving through a fluid will create about itself a circulation of sufficient strength to hold the rear stagnation point at the trailing edge.»

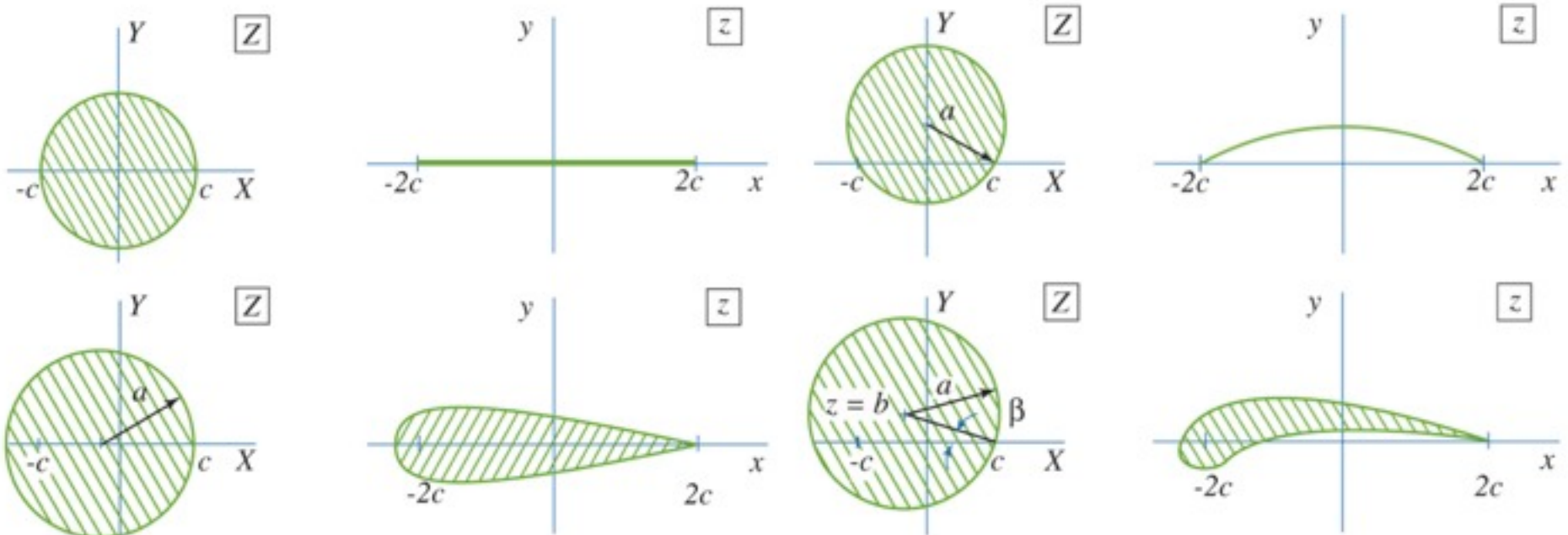
The Joukowski profiles

We consider now the more general case of a circular cylinder for which the center has been offset from the origin.

$$F(Z) = U_\infty \left((Z - b) \exp^{-i\alpha} + a^2 \frac{\exp^{i\alpha}}{Z - b} \right) - \frac{i\Omega}{2\pi} \log \left(\frac{Z - b}{a} \right)$$

$$W(Z) = U_\infty \exp^{-i\alpha} \left(1 - \frac{a^2}{(Z - b)^2} \exp^{i2\alpha} \right) - \frac{i\Omega}{2\pi(Z - b)}$$

Recipe: using the Kutta condition, we impose the singular trailing edge to be a stagnation point. By adjusting b , we remove the singularity at the leading edge.



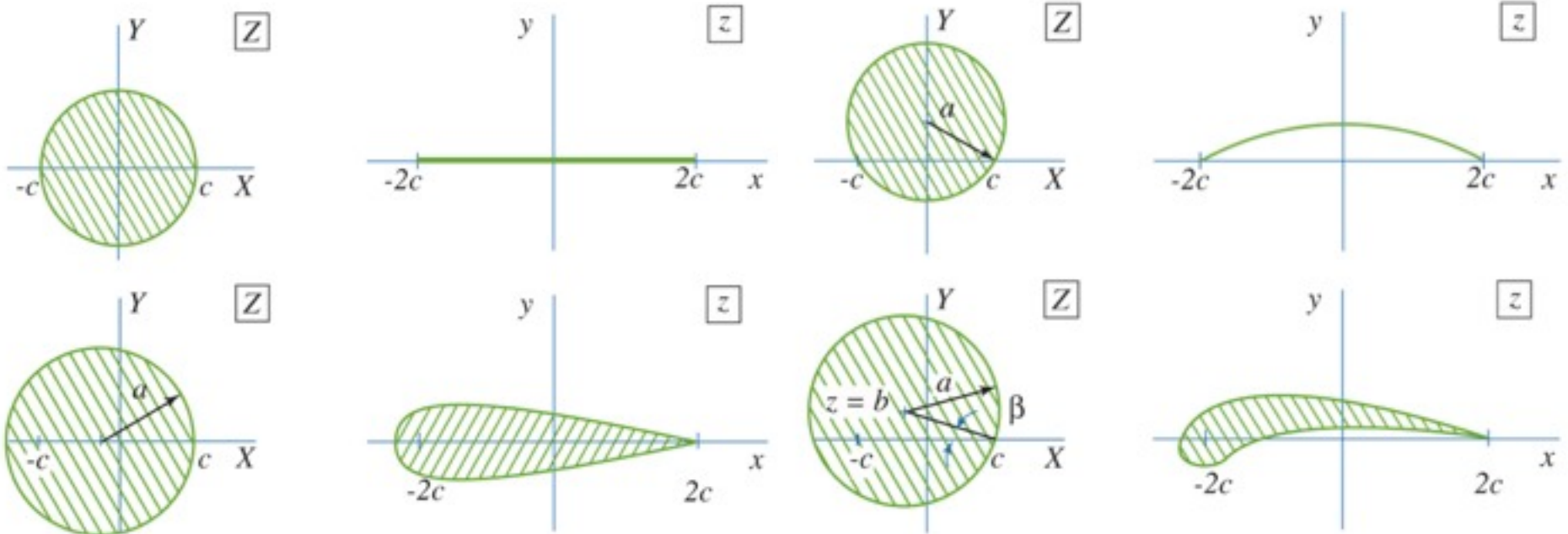
Critical circulation for Joukowski profiles

The trailing edge $Z = +c$ is imposed to be a stagnation point.

$$U_\infty \exp^{-i\alpha} \left(1 - \frac{a^2}{(c-b)^2} \exp^{i2\alpha} \right) - \frac{i\Omega}{2\pi(c-b)} = 0$$

Since b is the center of the cylinder, we can define the angle $c - b = a \exp^{-i\beta}$

$$\Omega_c = -4\pi U_\infty a \sin(\alpha + \beta)$$



Flow past an arbitrarily shaped cylinder

We now consider the inverse problem: we know the shape of the cylinder and we would like to find the conformal mapping to a circular cylinder.

Any analytic complex function can be expanded in its Laurent series around the origin. We restrict ourselves to mapping for which points at infinity are invariants.

$$M(z) = z + \sum_{n=0}^{\infty} \frac{a_n}{z^n} \quad \text{where} \quad a_n = \frac{1}{2\pi i} \int_l M(z) z^{n+1} dz$$

The general flow around the circular cylinder is given by the potential

$$F(Z) = U_{\infty} \left(Z \exp^{-i\alpha} + a^2 \frac{\exp^{i\alpha}}{Z} \right) - \frac{i\Omega}{2\pi} \log \left(\frac{Z}{a} \right)$$

Injecting the mapping for Z and Taylor expanding around infinity, we get:

$$f(z) = U_{\infty} z \exp^{-i\alpha} - \frac{i\Omega}{2\pi} \log \left(\frac{z}{a} \right) + \sum_{n=0}^{\infty} \frac{b_n}{z^n} \quad \text{and}$$

$$w(z) = U_{\infty} \exp^{-i\alpha} - \frac{i\Omega}{2\pi z} - \sum_{n=1}^{\infty} \frac{nb_n}{z^{n+1}}$$

The general flow is uniform to leading order, then a vortex flow to next order, then a doublet flow to higher order, and so on...

The circulation on the new body is $C = \int_l w(z) dz = \int_L W(Z) dZ = \Omega$

The Kutta-Joukowski theorem

We now compute the force acting on the arbitrarily shaped body.

We have the Clausius formula $F_x - iF_y = \frac{i}{2}\rho \int_l w^2(z) dz$

The kinetic energy is expanded as $w^2(z) = U_\infty^2 \exp^{-2i\alpha} - U_\infty \exp^{-i\alpha} \frac{i\Omega}{\pi z} + \sum_{n=2}^{\infty} \frac{c_n}{z^n}$

We have (residue theorem) $\int_L \frac{dz}{z} = 2i\pi$ and $\int_L \frac{dz}{z^n} = 0$ for $n \geq 2$

The force is for any profile $F_x - iF_y = i\rho U_\infty \exp^{-i\alpha} \Omega$

We recover the force acting on the circular cylinder.

General results:

- no drag $F_x = 0$

- without circulation $F_y = 0$ (d'Alembert's paradox).

The force on a general Joukowski profile is $F_y = 4\pi\rho U_\infty^2 a \sin(\alpha + \beta)$

Lift coefficient

The lift coefficient is a dimensionless number that measures the performance of a wing profile (L is the length of the wing section).

$$C_y = \frac{F_y}{\frac{1}{2}\rho U_\infty^2 L}$$

For a Joukowski profile with small attack angle and small bending angle,

$$C_y = 8\pi \frac{a}{L} \sin(\alpha + \beta) \simeq 2\pi(\alpha + \beta)$$

The theory disagrees more and more with the experiment: we have neglected viscous effects.

It breaks down completely above 10 degrees. This is because the zero streamline is detaching from the wing.

