

Linear Algebra

Session-1:

Matrix: A matrix is an array of numbers. A set of mn numbers (real or complex) arranged in a rectangular array having m rows (horizontal lines) and n columns (vertical lines), the numbers /entries being enclosed by brackets $[]$ or $()$ is called an $m \times n$ matrix (read as “ m by n matrix”). In what follows, a matrix is denoted by an upper-case alphabet in boldface (e.g., \mathbf{A}), and its (i, j) th element (the element at the i th row and j th column) is denoted by the corresponding lower-case alphabet with subscripts ij (e.g. a_{ij}). Specifically, an $m \times n$ matrix \mathbf{A} contains m rows and n columns and can be expressed as

$$\mathbf{A} = \begin{bmatrix} a^{11} & a^{12} \dots & a^{1n} \\ \vdots & \ddots & \vdots \\ a^{m1} & a^{m2} \dots & a^{mn} \end{bmatrix}$$

Basic Concepts:

1) Matrix Addition: Let $\mathbf{A} = [a^{ij}]$ $\mathbf{B} = [b^{ij}]$ be two matrices of same order, then their sum, written

$\mathbf{A} + \mathbf{B}$ is obtained by adding the corresponding entries. Matrices of different orders cannot be added.

Example: If $\mathbf{A} = \begin{bmatrix} 1 & 3 & -5 \\ 5 & 0 & 9 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 3 & 5 & 8 \\ -8 & 2 & 7 \end{bmatrix}$ then $\mathbf{A} + \mathbf{B} = \begin{bmatrix} 4 & 8 & 3 \\ -3 & 2 & 16 \end{bmatrix}$

2) Scalar Multiplication: The product of any $m \times n$ matrix $\mathbf{A} = [a^{ij}]$ and any scalar λ , written

$\lambda \mathbf{A}$, is the $m \times n$ matrix $\lambda \mathbf{A} = [\lambda a^{ij}]$ obtained by multiplying each entry in \mathbf{A} by λ .

Example: If $\mathbf{A} = \begin{bmatrix} 1 & 3 & -5 \\ 5 & 0 & 9 \end{bmatrix}$ and $\lambda = -5$ then $\lambda \mathbf{A} = \begin{bmatrix} -5 & -15 & 25 \\ -25 & 0 & -45 \end{bmatrix}$

3) Properties of Matrix Addition:

a) Matrix Addition is commutative. That is $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$.

b) Matrix Addition is associative. That is $(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$.

c) For any matrix \mathbf{A} , there exists a null matrix $\mathbf{0}$ of same order such that $\mathbf{A} + \mathbf{0} = \mathbf{0} + \mathbf{A}$.

d) For any matrix \mathbf{A} , there exists a matrix $-\mathbf{A}$ of same order such that $\mathbf{A} + (-\mathbf{A}) = (-\mathbf{A}) + \mathbf{A} = \mathbf{0}$.

Equality of Matrices: Two matrices **A** and **B** are said to be **equal** iff they have the same order and their corresponding elements are equal. Thus, if $A = [a^{ij}]$ and $B = [b^{ij}]$ then $A=B$ iff $a^{ij} = b^{ij}$,

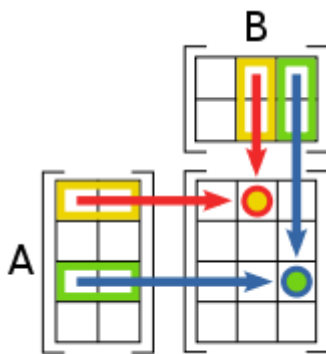
for all i and j.

Matrix Multiplication: Two matrices **A** and **B** are said to be conformable for the product (in this very order of A and B) if the number of columns in **A** (called the prefactor) is equal to the number of rows in **B** (called the post factor).

Here (i, j) th element of **AB**

=Sum of the products of the elements of ith row of **A**

with the corresponding jth column of **B**



Matrix Multiplication

Solved Example: If $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ and $B = \begin{bmatrix} k & l \\ m & n \\ o & p \end{bmatrix}$,

$$\text{then } \mathbf{AB} = \begin{bmatrix} ak + bm + co & al + bn + cp \\ dk + em + fo & dl + en + fp \\ gk + hm + io & gl + hn + ip \end{bmatrix}$$

Properties of Matrix Multiplication:

- Matrix Multiplication is not commutative: That is $\mathbf{AB} \neq \mathbf{BA}$.
- Matrix Multiplication is associative: That is $(\mathbf{AB}) \mathbf{C} = \mathbf{A} (\mathbf{BC})$.
- Matrix Multiplication is distributive with respect to Matrix Addition : That is $\mathbf{A} (\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}$.
- If **A** and **I** are square matrices of same order then $\mathbf{AI} = \mathbf{IA} = \mathbf{A}$.

Example: $A = \begin{bmatrix} .5 & \sqrt{3} & 0 \\ -6 & 11 & \sqrt[3]{7} \end{bmatrix}$.

c) Square Matrix: A matrix in which the number of rows is equal to the number of columns is called a **square matrix**, otherwise, it is said to be a **rectangular matrix**.

Example: $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$, the elements a, e, i are called its diagonal elements and the diagonal along which these elements lie is called the **principal diagonal** or **leading diagonal**.

d) Trace of a (square) Matrix: The sum of the diagonal elements of a square matrix **A** is called the **trace of A**. Thus, the trace of the n-rowed square matrix $A = [a^{ij}]$ is

$$a_{11} + a_{22} + a_{33} + \dots + a_{nn} = \sum_{i=1}^n a_{ii}$$

e) Diagonal Matrix: A square matrix in which all non-diagonal elements are zero is called a **diagonal matrix**.

Example: $A = \begin{bmatrix} -3 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 7 \end{bmatrix}$.

f) Scalar Matrix: A diagonal matrix in which all the diagonal elements are equal is called a **scalar matrix**.

Example: $A = \begin{bmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{bmatrix}$.

g) Unit Matrix/Identity Matrix: A scalar matrix, in which each diagonal element is unit, is called a **unit matrix/identity matrix**.

Example: $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

h) Null Matrix/Zero Matrix: A matrix, in which each element is zero, is called a **null/zero matrix**.

Example: If $A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Special Matrices (Continued):

i) Triangular Matrix: A square matrix in which all the elements either below or above the principal diagonal are zero is called a triangular matrix. Thus a triangular matrix is either upper triangular or lower triangular.

1) Upper Triangular Matrix: A square matrix in which all the elements below the principal

diagonal are zero is called an **upper triangular matrix**.

$$\text{Example: } \mathbf{A} = \begin{bmatrix} -5 & 11 & 2 \\ 0 & 10 & 8 \\ 0 & 0 & 27 \end{bmatrix}.$$

2) Lower Triangular Matrix: A square matrix in which all the elements above the principal

diagonal are zero is called a **lower triangular matrix**.

$$\text{Example: } \mathbf{A} = \begin{bmatrix} -8 & 0 & 0 \\ -3 & 20 & 0 \\ 9 & 2 & 7 \end{bmatrix}.$$

j) Symmetric Matrix: A square matrix $\mathbf{A} = [a_{ij}]$ is said to be **symmetric** when $a_{ij} = a_{ji}$

for all i and j . That is, $\mathbf{A}^T = \mathbf{A}$.

$$\text{Example: } \mathbf{A} = \begin{bmatrix} a & e & f \\ e & b & g \\ f & g & c \end{bmatrix}.$$

k) Skew-Symmetric Matrix (or Anti-Symmetric Matrix): A square matrix $\mathbf{A} = [a_{ij}]$ is said to be **skew-symmetric**

when 1) $a_{ij} = -a_{ji}$ for all i and j

and 2) All diagonal elements $a_{ii} = 0$.

That is, $\mathbf{A}^T = -\mathbf{A}$.

$$\text{Example: } \mathbf{A} = \begin{bmatrix} 0 & e & f \\ -e & 0 & -g \\ -f & g & 0 \end{bmatrix}.$$

1) Inverse of a Square Matrix: The inverse of an $n \times n$ square matrix $\mathbf{A}=[a_{ij}]$ is denoted by \mathbf{A}^{-1} and is an $n \times n$ matrix such that $\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$, where \mathbf{I} is the $n \times n$ unit matrix.

If \mathbf{A} has an inverse, then \mathbf{A} is called a **non-singular matrix** (That is, $\text{Det}(\mathbf{A}) \neq 0$).

If \mathbf{A} has no inverse, then \mathbf{A} is called a **singular matrix** (That is, $\text{Det}(\mathbf{A}) = 0$).

Inner Product of Vectors: If \mathbf{a} is a row vector and \mathbf{b} is a column vector, both with n components then the matrix multiplication gives a 1×1 matrix, which is a single entry. This product is called the inner product or dot product of \mathbf{a} and \mathbf{b} and is denoted by $\mathbf{a} \cdot \mathbf{b}$.

$$\text{Let } \mathbf{a} = [a^1 \quad a^2 \quad \dots \quad a^{n-1} \quad a^n] \text{ and } \mathbf{b} = \begin{bmatrix} b^1 \\ b^2 \\ \vdots \\ b^{n-1} \\ b^n \end{bmatrix}$$

$$\text{Then } \mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + \dots + a_n b_n = \sum_{i=1}^n a_i b_i$$

$$\text{Example: } \mathbf{a} = [2 \quad -3 \quad 11], \mathbf{b} = \begin{bmatrix} 2 \\ -6 \\ 5 \end{bmatrix}, \text{ then } \mathbf{a} \cdot \mathbf{b} = (2 \times 2) + (-3) \times (-6) + (11 \times 5) = 77$$

Matrix Product in Terms of Row and Column Vectors: Let \mathbf{A} be a matrix of order $m \times n$ and \mathbf{B} be a matrix of order $n \times p$ represented as:

$$\mathbf{A} = \begin{bmatrix} a^{11} & a^{12} & \dots & a^{1n} \\ a^{21} & a^{22} & \dots & a^{2n} \\ \dots & \dots & \dots & \dots \\ a^{m1} & a^{m2} & \dots & a^{mn} \end{bmatrix} = \begin{bmatrix} a^1 \\ a^2 \\ \dots \\ a^{m-1} \\ a^m \end{bmatrix}, \text{ where } a^1, a^2, \dots, a^m \text{ are row vectors of order}$$

$1 \times n$.

$$\text{and } \mathbf{B} = \begin{bmatrix} b_{11} & b_{12} & \Lambda & b^{1p} \\ b^{21} & b^{22} & \Lambda & b^{2p} \\ \Lambda & \Lambda & \Lambda & \Lambda \\ b^{n1} & b^{n2} & \Lambda & b^{np} \end{bmatrix} = \begin{bmatrix} b^1 & b^2 & \Lambda & \Lambda & b^{p-1} & b^p \end{bmatrix}, \text{ where } b_1, b_2, \dots, b_p$$

are column vectors of order $n \times 1$.

$$\text{Then } \mathbf{AB} = \begin{bmatrix} a^1 \cdot b^1 & a^1 \cdot b^2 & \Lambda & a^1 \cdot b^p \\ a^2 \cdot b^1 & a^2 \cdot b^2 & \Lambda & a^2 \cdot b^p \\ \Lambda & \Lambda & \Lambda & \Lambda \\ a^m \cdot b^1 & a^m \cdot b^2 & \Lambda & a^m \cdot b^p \end{bmatrix} \text{ is in terms of row and column vectors.}$$

Solved Example: Express the product of the matrices in terms of row and column vectors where

$$A = \begin{bmatrix} 4 & 3 \\ 7 & 2 \\ 9 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & 5 \\ 1 & 6 \end{bmatrix}.$$

$$\text{Solution: } A = \begin{bmatrix} 4 & 3 \\ 7 & 2 \\ 9 & 0 \end{bmatrix} = \begin{bmatrix} a^1 \\ a^2 \\ a^3 \end{bmatrix}$$

$$\text{where } a_1 = [4 \ 3], a_2 = [7 \ 2], \text{ and } a_3 = [9 \ 0]$$

$$\text{and } B = \begin{bmatrix} 2 & 5 \\ 1 & 6 \end{bmatrix} = [b^1 \ b^2]$$

$$\text{where } b_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \text{ and } b_2 = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

$$\text{Then } AB = \begin{bmatrix} a^1 \cdot b^1 & a^1 \cdot b^2 \\ a^2 \cdot b^1 & a^2 \cdot b^2 \\ a^3 \cdot b^1 & a^3 \cdot b^2 \end{bmatrix}$$

$$= \begin{bmatrix} (4 \times 2 + 3 \times 1) & (4 \times 5 + 3 \times 6) \\ (7 \times 2 + 2 \times 1) & (7 \times 5 + 2 \times 6) \\ (9 \times 2 + 0 \times 1) & (9 \times 5 + 0 \times 6) \end{bmatrix} = \begin{bmatrix} 11 & 38 \\ 16 & 47 \\ 18 & 45 \end{bmatrix}.$$

System of Linear Equations:

The concept of a matrix was historically introduced to simplify the solution of linear systems although they today have much greater and broad-reaching applications. Let's see how a linear system can be represented using a matrix.

Consider a general system of m linear equations with n unknowns:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2n}x_n &= b_2 \\ \dots & \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \dots + a_{mn}x_n &= b_m \end{aligned}$$

This system is equivalent to a matrix equation of the form

$$AX = B \dots \dots \dots (1)$$

Where A is an $m \times n$ matrix, X is a column matrix with n entries, and B is a column matrix with m entries.

$$A = \begin{bmatrix} a_{11} & a_{12} \dots & a_{1n} \\ a_{21} & a_{22} \dots & a_{2n} \\ \dots & \dots \dots & \dots \\ a_{m1} & a_{m2} \dots & a_{mn} \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix}, \text{ and } B = \begin{bmatrix} b_1 \\ b_2 \\ \dots \\ b_m \end{bmatrix}$$

Clearly our manner of defining matrix multiplication is used in representing the linear system in this fashion because now the product of the matrix A (called the **coefficient matrix**) and the matrix x gives us precisely the matrix b (called the **right hand side matrix**).

$$K = \begin{bmatrix} a_{11} & a_{12} \dots & a_{1n} & \vdots & b_1 \\ a_{21} & a_{22} \dots & a_{2n} & \vdots & b_2 \\ \dots & \dots \dots & \dots & \vdots & \dots \\ a_{m1} & a_{m2} \dots & a_{mn} & \vdots & b_m \end{bmatrix} \text{ is called the } \mathbf{Augmented Matrix}.$$

If the elements in B are not all zero, then (i) is called **Non-homogeneous System**, otherwise it is called **Homogeneous System**. A solution of (1) is defined by a set of values of the variables $x_1, x_2, x_3, \dots, x_n$ which satisfy simultaneously all the equations of (1). If the

system given by (1) has a solution, it is called a **consistent system**, otherwise the system is called **inconsistent**. In fact, a consistent system has either unique solution or infinitely many solutions. Obviously any homogeneous system of equations is always a consistent system, since $x_i=0$ for all i is a solution which is known as trivial solution.

Gauss Elimination Method:

Solution by Gauss Elimination: The Gauss Elimination is a standard method for solving linear equations. It is a systematic elimination. It is a reduction to triangular form, from which we shall obtain the values of the unknowns by substitution. This method will be very clear by taking the following problems.

Solved Example: Solve the following system by Gauss Elimination:

$$-x_1 + x_2 - x_3 = 0$$

$$10x_2 + 25x_3 = 90$$

$$20x_1 + 10x_2 = 80$$

Solution: We can write the system and its augmented matrix side by side:

$$-x_1 + x_2 - x_3 = 0$$

$$10x_2 + 25x_3 = 90$$

$$20x_1 + 10x_2 = 80$$

$$K = \begin{bmatrix} 1 & -1 & 1 & 0 \\ -1 & 1 & -1 & 0 \\ 0 & 10 & 25 & 90 \\ 20 & 10 & 0 & 80 \end{bmatrix}$$

Step -1: Elimination of x_1

The first equation is called pivot equation and the coefficient 1 of its x_1 -term the pivot. Using pivot equation, we can eliminate x_1 from other equation. For this, we have to do:

Adding 1 times pivot equation to the 2nd and -20 times pivot equation to the 4th equation, the result is:

$$0 = 0$$

$$10x_2 + 25x_3 = 90$$

$$30x_2 - 20x_3 = 80$$

$$K = \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 10 & 25 & 90 \\ 0 & 30 & -20 & 80 \end{bmatrix}$$

Step -2: Elimination of x_2

The 1st equation which has just served as the pivot equation remains unchanged. The 2nd equation is the next pivot equation, but for this problem 2nd equation contains no x_2 -term.

Now it is required to change the order of the equation to get a non-zero pivot. We can put 2nd equation at the end and the 3rd and 4th equations one place up. This is called **partial pivoting**. We get:

$$\begin{aligned} x_1 - x_2 + x_3 &= 0 \\ 10x_2 + 25x_3 &= 90 \\ 30x_2 - 20x_3 &= 80 \\ 0 &= 0 \end{aligned} \quad K = \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 10 & 25 & 90 \\ 0 & 30 & -20 & 80 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Adding 3 times pivot equation to the 3rd equation, the result is:

$$\begin{aligned} 10x_2 + 25x_3 &= 90 \\ -95x_3 &= -190 \\ 0 &= 0 \end{aligned} \quad K = \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 10 & 25 & 90 \\ 0 & 0 & -95 & -190 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Back Substitution: Determination of x_3 , x_2 and x_1 (in this order)

$$-95x_3 = -190 \Rightarrow x_3 = 2$$

$$10x_2 + 25x_3 = 90 \Rightarrow x_2 = 4$$

$$x_1 - x_2 + x_3 = 0 \Rightarrow x_1 = 2$$

The solution of the system is:

$$x_1 = 2, x_2 = 4, \text{ and } x_3 = 2.$$