

## Session-12:

**Theorem:** Show that every square matrix  $A$  can be uniquely expressed as a sum of a symmetric matrix and a skew-symmetric matrix.

**Proof:** Let  $A$  be a square matrix. Then  $A$  can be written as:

$$A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$$

Let

$$B = \frac{1}{2}(A + A^T)$$

And

$$C = \frac{1}{2}(A - A^T)$$

Now

$$B^T = \frac{1}{2}(A + A^T)^T = \frac{1}{2}(A^T + (A^T)^T) = \frac{1}{2}(A + A^T) = B$$

Thus  $B$  is symmetric.

Again

$$C^T = \frac{1}{2}(A - A^T)^T = \frac{1}{2}(A^T - (A^T)^T) = \frac{1}{2}(A^T - A) = -\frac{1}{2}(A - A^T) = -C$$

Thus  $C$  is skew-symmetric.

Hence  $A$  can be expressed as a sum of a symmetric matrix and a skew-symmetric matrix.

Uniqueness of  $A$ :

If possible let another form of

$$A = P + Q \dots \dots \dots (1)$$

Where

$P$  is a symmetric matrix and  $Q$  is a skew-symmetric matrix.

Thus

$$P^T = P \text{ and } Q^T = -Q$$

Now

$$A^T = (P + Q)^T = P^T + Q^T = P - Q \dots \dots \dots (2)$$

From (1) and (2) we have

$$A + A^T = 2P \Rightarrow P = \frac{1}{2}(A + A^T) = B$$

$$A - A^T = 2Q \Rightarrow Q = \frac{1}{2}(A - A^T) = C$$

And

Thus uniqueness is proved.

**Solved Example:** Express the matrix

$$A = \begin{bmatrix} 4 & 2 & -3 \\ 1 & 3 & -6 \\ -5 & 0 & -7 \end{bmatrix}$$

as a sum of a symmetric matrix and a skew-symmetric matrix.

**Solution:** Here

$$\text{Now } A^T = \begin{bmatrix} 4 & 1 & -5 \\ 2 & 3 & 0 \\ -3 & -6 & -7 \end{bmatrix}$$

$$\text{So } A + A^T = \begin{bmatrix} 8 & 3 & -8 \\ 3 & 9 & -6 \\ -8 & -6 & -14 \end{bmatrix} \text{ and}$$

$$\frac{1}{2}(A + A^T) = \begin{bmatrix} 4 & 3/2 & -4 \\ 3/2 & 9/2 & -3 \\ -4 & -3 & -7 \end{bmatrix}$$

$$\text{Again } A - A^T = \begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & -6 \\ -2 & 6 & 0 \end{bmatrix} \text{ and}$$

$$\frac{1}{2}(A - A^T) = \begin{bmatrix} 0 & 1/2 & 1 \\ -1/2 & 0 & -3 \\ -1 & 3 & 0 \end{bmatrix}$$

Thus

$$A = \begin{bmatrix} 4 & 3/2 & -4 \\ 3/2 & 9/2 & -3 \\ -4 & -3 & -7 \end{bmatrix} + \begin{bmatrix} 0 & 1/2 & 1 \\ -1/2 & 0 & -3 \\ -1 & 3 & 0 \end{bmatrix}.$$

### **Theorem :( Eigenvalues)**

- The eigenvalues of a symmetric matrix are real.
- The eigenvalues of a skew-symmetric matrix are pure imaginary or zero.

**Proof:** a) Let  $\lambda$  be an eigenvalue of a symmetric matrix  $A$  and  $X$  be the corresponding eigenvector. So  $(A)^T = A$ .

$$\text{Now } (A - \lambda)X = 0$$

$$\Rightarrow AX = \lambda X \dots \dots \dots (1)$$

$$\Rightarrow \bar{A}\bar{X} = \bar{\lambda}\bar{X}$$

$$\Rightarrow (\bar{A}\bar{X})^T = (\bar{\lambda}\bar{X})^T$$

$$\Rightarrow (\bar{X})^T(\bar{A})^T = \bar{\lambda}(\bar{X})^T$$

$$\Rightarrow (\bar{X})^T(A)^T = \bar{\lambda}(\bar{X})^T, \text{ as } \bar{A} = A$$

$$\Rightarrow (\bar{X})^T A = \bar{\lambda}(\bar{X})^T, \text{ as } (A)^T = A$$

$$\Rightarrow (\bar{X})^T AX = \bar{\lambda}(\bar{X})^T X \dots \dots (2), \text{ (Post - multiplying by } X)$$

Pre-multiplying (1) by  $(\bar{X})^T$ , we get

$$(\bar{X})^T AX = \lambda(\bar{X})^T X \dots \dots \dots (3)$$

Since  $(\bar{X})^T X \neq 0$ , using (2) and (3) we obtain

$$\lambda = \bar{\lambda} \Rightarrow \lambda \text{ is a real number.}$$

b) Let  $\lambda$  be an eigenvalue of a skew-symmetric matrix  $A$  and  $X$  be the corresponding eigenvector. So  $(A)^T = -A$ .

Now  $(A - \lambda)X = 0$

$$\Rightarrow AX = \lambda X \dots \dots \dots (1)$$

$$\Rightarrow \bar{A}\bar{X} = \bar{\lambda}\bar{X}$$

$$\Rightarrow (\bar{A}\bar{X})^T = (\bar{\lambda}\bar{X})^T$$

$$\Rightarrow (\bar{X})^T(\bar{A})^T = \bar{\lambda}(\bar{X})^T$$

$$\Rightarrow (\bar{X})^T(A)^T = \bar{\lambda}(\bar{X})^T, \text{ as } \bar{A}=A$$

$$\Rightarrow (\bar{X})^T(-A) = \bar{\lambda}(\bar{X})^T, \text{ as } (A)^T = -A$$

$$\Rightarrow (\bar{X})^T - AX = \bar{\lambda}(\bar{X})^T X$$

$$\Rightarrow (\bar{X})^T(-A)X = \bar{\lambda}(\bar{X})^T X \dots \dots (2), \text{ (Post - multiplying by } X)$$

$$\Rightarrow (\bar{X})^T(-AX) = \bar{\lambda}(\bar{X})^T X$$

$$\Rightarrow (\bar{X})^T(-\lambda X) = \bar{\lambda}(\bar{X})^T X$$

$$\Rightarrow -\lambda((\bar{X})^T X) = \bar{\lambda}((\bar{X})^T X)$$

Since  $(\bar{X})^T X \neq 0$ , we obtain

$$-\lambda = \bar{\lambda} \Rightarrow \lambda \text{ is pure imaginary or zero.}$$

## Orthogonal Transformation & Matrices:

Orthogonal Transformations are transformations

$$Y = AX$$

with  $A$  an orthogonal matrix.

**Theorem1:** (Invariance of Inner Product) An orthogonal transformation preserves the value of inner product

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{a}^T \mathbf{b}$$

( $\mathbf{a}$  and  $\mathbf{b}$  are column vectors).

Hence it preserves also the length or norm of a vector in  $R^n$  given by

$$\|\mathbf{a}\| = \sqrt{\mathbf{a} \cdot \mathbf{a}} = \sqrt{\mathbf{a}^T \mathbf{a}}$$

**Proof:** Refer page no:-383 of book-Advanced Engineering Mathematics by Erwin Kreyszig (8<sup>th</sup> Edition)

**Theorem2 :**( Orthonormality of column and row vectors) A real square matrix is orthogonal if and only if its column vectors  $a_1, a_2, a_3, \dots \dots \dots a_n$  (and also its row vectors) form an Orthonormal system, that is,

$$a_j \cdot a_k = a_j^T a_k$$

$$= \begin{cases} 0 & \text{if } j \neq k \\ 1 & \text{if } j = k \end{cases}$$

**Proof:** Refer page no:-383 of book-Advanced Engineering Mathematics by Erwin Kreyszig (8<sup>th</sup> Edition)

**Theorem3 :( Determinant of an orthogonal matrix)** The determinant of an orthogonal matrix has the value +1 or -1.

**Proof:** We that

$$\det (AB) = (\det A) (\det B)$$

and

$$\begin{aligned} \det (A^T) &= \det A \\ 1 = \det I &= \det(AA^{-1}) \\ \Rightarrow 1 &= \det(AA^T) \end{aligned}$$

$$\begin{aligned} (\text{Since } A \text{ is orthogonal } (A^T \bar{A} = A^{-1})) \\ \Rightarrow \det(A) (\det \bar{A}) &= (\det A^{-1}) \\ \Rightarrow \det A &= \pm 1. \end{aligned}$$

**Theorem4 :( Eigenvalues of an orthogonal matrix)**

The eigenvalues of an orthogonal matrix  $A$  are real or complex conjugates in pairs and have absolute value 1.

**Proof:** Let  $\lambda$  be an eigenvalue of an orthogonal matrix  $A$  and  $X$  be the corresponding eigenvector .So  $(A)^T = A^{-1}$ .

$$\text{Now } (A - \lambda)X = 0$$

$$\Rightarrow AX = \lambda X \dots \dots \dots (1)$$

$$\Rightarrow \bar{A}\bar{X} = \bar{\lambda}\bar{X}$$

$$\Rightarrow (\bar{A}\bar{X})^T = (\bar{\lambda}\bar{X})^T$$

$$\Rightarrow (\bar{X})^T(\bar{A})^T = \bar{\lambda}(\bar{X})^T$$

$$\Rightarrow (\bar{X})^T(A)^T = \bar{\lambda}(\bar{X})^T, \text{ as } \bar{A} = A$$

$$\Rightarrow (\bar{X})^T A^{-1} = \bar{\lambda}(\bar{X})^T, \text{ as } (A)^T = A^{-1}$$

$$\Rightarrow (\bar{X})^T A^{-1} X = \bar{\lambda}(\bar{X})^T X$$

$$\Rightarrow (\bar{X})^T (A^{-1})AX = \bar{\lambda}(\bar{X})^T AX \dots \dots (2), \text{ (Post - multiplying by } AX)$$

$$\Rightarrow (\bar{X})^T (A^{-1}A)X = \bar{\lambda}(\bar{X})^T AX$$

$$\Rightarrow (\bar{X})^T (IX) = \bar{\lambda}(\bar{X})^T \lambda X$$

$$\Rightarrow ((\bar{X})^T X) = |\lambda|^2 ((\bar{X})^T X)$$

Since  $(\bar{X})^T X \neq 0$ , we obtain

$$|\lambda|^2 = 1 \Rightarrow |\lambda| = 1.$$