

Session-13:

Complex Matrices: Hermitian, Skew-Hermitian, Unitary:

Complex Matrix: If all the elements of a matrix are real, then it is called a real matrix over \mathbb{R} . If at least one element of a matrix is a complex number $x + iy$, where x and y are real, $i = \sqrt{-1}$, then the matrix is called a complex matrix.

Example: Let $A = \begin{bmatrix} 2 + 3i & 5 \\ 7i & 1 - i \end{bmatrix}$.

Then the conjugate of $A = \bar{A} = \begin{bmatrix} 2 - 3i & 5 \\ -7i & 1 + i \end{bmatrix}$.

Here it can be easily shown that $\overline{A^T} = (\bar{A})^T$.

Hermitian Matrix: A square matrix $A = [a_{ij}]$ is said to be Hermitian if

$$\overline{A^T} = A$$

That is, $\overline{a_{ij}} = a_{ji}$

The elements of the leading diagonal of a Hermitian matrix are evidently real, while other element is the complex conjugate of the element in the transposed position.

Example: Let $A = \begin{bmatrix} 2 & 3 + 4i \\ 3 - 4i & 5 \end{bmatrix}$.

Now

$$\bar{A} = \begin{bmatrix} 2 & 3 - 4i \\ 3 + 4i & 5 \end{bmatrix}$$

$$\bar{A}^T = \begin{bmatrix} 2 & 3 + 4i \\ 3 - 4i & 5 \end{bmatrix} = A$$

Thus A is a Hermitian matrix.

Skew-Hermitian Matrix: A square matrix $A = [a_{ij}]$ is said to be Skew-Hermitian if

$$\overline{A^T} = -A$$

That is,

$$\overline{a_{ij}} = -a_{ji}$$

The leading diagonal elements of a Skew-Hermitian matrix are either all zeros or all purely imaginary.

Example: Let $B = \begin{bmatrix} 3i & 4 + 5i \\ -4 + 5i & -i \end{bmatrix}$.

Now

$$\bar{B} = \begin{bmatrix} -3i & 4 - 5i \\ -4 - 5i & i \end{bmatrix}$$

$$\bar{B}^T = \begin{bmatrix} -3i & -4 - 5i \\ 4 - 5i & i \end{bmatrix} = -B$$

Thus B is a Skew-Hermitian matrix.

Unitary Matrix: A square matrix $A = [a_{ij}]$ is said to be Unitary if

$$\bar{A}^T = A^{-1}$$

Example: Let $C = \begin{bmatrix} \frac{1}{2}i & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2}i \end{bmatrix}$.

Now

$$\bar{C} = \begin{bmatrix} -\frac{1}{2}i & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2}i \end{bmatrix}$$

$$\bar{C}^T = \begin{bmatrix} -\frac{1}{2}i & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2}i \end{bmatrix} = C^{-1}$$