

Session-14:

Note:

1. If a Hermitian matrix is real, then $\bar{A}^T = A^T = A$. Hence a real Hermitian matrix is a symmetric matrix.

2. If a Skew-Hermitian matrix is real, then $\bar{A}^T = A^T = -A$.
Hence a real Skew-Hermitian matrix is a skew-symmetric matrix.

3. If a Unitary matrix is real, then $\bar{A}^T = A^T = A^{-1}$. Hence a real Unitary matrix is an Orthogonal matrix.

Hence Hermitian, Skew-Hermitian and Unitary matrices generalize Symmetric, Skew-symmetric and Orthogonal matrices, respectively.

Theorem :(Eigenvalues)

a) The eigenvalues of a Hermitian matrix are real.

b) The eigenvalues of a skew- Hermitian matrix are pure imaginary or zero.

c) The eigenvalues of a Unitary matrix have absolute value 1.

Proof: a) Let λ be an eigenvalue of a Hermitian matrix A and X be the corresponding eigenvector .So $(\bar{A})^T = A$.

$$\begin{aligned}\text{Now } (A - \lambda)X &= 0 \\ \Rightarrow AX &= \lambda X \dots \dots \dots (1) \\ \Rightarrow \bar{A}\bar{X} &= \bar{\lambda}\bar{X} \\ \Rightarrow (\bar{A}\bar{X})^T &= (\bar{\lambda}\bar{X})^T \\ \Rightarrow (\bar{X})^T(\bar{A})^T &= \bar{\lambda}(\bar{X})^T\end{aligned}$$

$$\begin{aligned}\Rightarrow (\bar{X})^T(A^T)^T &= \bar{\lambda}(\bar{X})^T, \text{ as } (\bar{A})^T = A \\ \Rightarrow (\bar{X})^T A &= \bar{\lambda}(\bar{X})^T \\ \Rightarrow (\bar{X})^T AX &= \bar{\lambda}(\bar{X})^T X \dots \dots (2), \text{ (Post - multiplying by } X)\end{aligned}$$

Pre-multiplying (1) by $(\bar{X})^T$, we get

$$(\bar{X})^T AX = \lambda(\bar{X})^T X \dots \dots \dots (3)$$

Since $(\bar{X})^T X \neq 0$, using (2) and (3) we obtain

$$\lambda = \bar{\lambda} \Rightarrow \lambda \text{ is a real number.}$$

b) Let λ be an eigenvalue of a skew-Hermitian matrix A and X be the corresponding eigenvector .So $(\bar{A})^T = -A$.

$$\begin{aligned}\text{Now } (A - \lambda)X &= 0 \\ \Rightarrow AX &= \lambda X \dots \dots \dots (1) \\ \Rightarrow \bar{A}\bar{X} &= \bar{\lambda}\bar{X}\end{aligned}$$

$$\begin{aligned}
&\Rightarrow (\bar{A}\bar{X})^T = (\bar{\lambda}\bar{X})^T \\
&\Rightarrow (\bar{X})^T(\bar{A})^T = \bar{\lambda}(\bar{X})^T \\
&\Rightarrow (\bar{X})^T(-A) = \bar{\lambda}(\bar{X})^T, \text{ as } (\bar{A})^T = -A \\
&\Rightarrow (\bar{X})^T - AX = \bar{\lambda}(\bar{X})^T X \\
&\Rightarrow (\bar{X})^T(-A)X = \bar{\lambda}(\bar{X})^T X \dots \dots (2), \text{ (Post - multiplying by } X) \\
&\Rightarrow (\bar{X})^T(-AX) = \bar{\lambda}(\bar{X})^T X \\
&\Rightarrow (\bar{X})^T(-\lambda X) = \bar{\lambda}(\bar{X})^T X \\
&\Rightarrow -\lambda((\bar{X})^T X) = \bar{\lambda}((\bar{X})^T X)
\end{aligned}$$

Since $(\bar{X})^T X \neq 0$, we obtain
 $-\lambda = \bar{\lambda} \Rightarrow \lambda$ is pure imaginary or zero.

c) Let λ be an eigenvalue of a Unitary matrix A and X be the corresponding eigenvector
 .So $(\bar{A})^T = A^{-1}$.

Now $(A - \lambda)X = 0$

$$\begin{aligned}
&\Rightarrow AX = \lambda X \dots \dots \dots (1) \\
&\Rightarrow \bar{A}\bar{X} = \bar{\lambda}\bar{X} \\
&\Rightarrow (\bar{A}\bar{X})^T = (\bar{\lambda}\bar{X})^T \\
&\Rightarrow (\bar{X})^T(\bar{A})^T = \bar{\lambda}(\bar{X})^T \\
&\Rightarrow (\bar{X})^T A^{-1} = \bar{\lambda}(\bar{X})^T, \text{ as } (\bar{A})^T = A^{-1} \\
&\Rightarrow (\bar{X})^T A^{-1} X = \bar{\lambda}(\bar{X})^T X \\
&\Rightarrow (\bar{X})^T (A^{-1})AX = \bar{\lambda}(\bar{X})^T AX \dots \dots (2), \text{ (Post - multiplying by } AX) \\
&\Rightarrow (\bar{X})^T (A^{-1}A)X = \bar{\lambda}(\bar{X})^T AX \\
&\Rightarrow (\bar{X})^T (IX) = \bar{\lambda}(\bar{X})^T \lambda X \\
&\Rightarrow ((\bar{X})^T X) = |\lambda|^2 ((\bar{X})^T X)
\end{aligned}$$

Since $(\bar{X})^T X \neq 0$, we obtain
 $|\lambda|^2 = 1 \Rightarrow |\lambda| = 1$.