

Session-17

Inner Product (Dot Product) of Vectors

The **inner product** or **dot product** $\mathbf{a} \cdot \mathbf{b}$ (read “**a dot b**”) of two vectors \mathbf{a} and \mathbf{b} is the product of their lengths times the cosine of their angle (see Fig. 178),

$$(1) \quad \begin{array}{ll} \mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| |\mathbf{b}| \cos \gamma & \text{if } \mathbf{a} \neq \mathbf{0}, \mathbf{b} \neq \mathbf{0} \\ \mathbf{a} \cdot \mathbf{b} = 0 & \text{if } \mathbf{a} = \mathbf{0} \text{ or } \mathbf{b} = \mathbf{0}. \end{array}$$

The angle γ , $0 \leq \gamma \leq \pi$, between \mathbf{a} and \mathbf{b} is measured when the initial points of the vectors coincide, as in Fig. 178. In components, $\mathbf{a} = [a_1, a_2, a_3]$, $\mathbf{b} = [b_1, b_2, b_3]$, and

$$(2) \quad \mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3.$$

Orthogonality: A vector \mathbf{a} is called **orthogonal** to a vector \mathbf{b} $\mathbf{a} \cdot \mathbf{b} = 0$ if . Then \mathbf{b} is also orthogonal to \mathbf{a} , and we call \mathbf{a} and \mathbf{b} **orthogonal vectors**. Clearly, this happens for nonzero vectors if and only if $\cos \gamma = 0$; thus $\gamma = \frac{\pi}{2} (90^\circ)$.

Note: (Orthogonality Criterion)

The inner product of two nonzero vectors is 0 if and only if these vectors are perpendicular.

The second line in (1) is needed because γ is undefined when $\mathbf{a} = \mathbf{0}$ or $\mathbf{b} = \mathbf{0}$. The derivation of (2) from (1) is shown below.

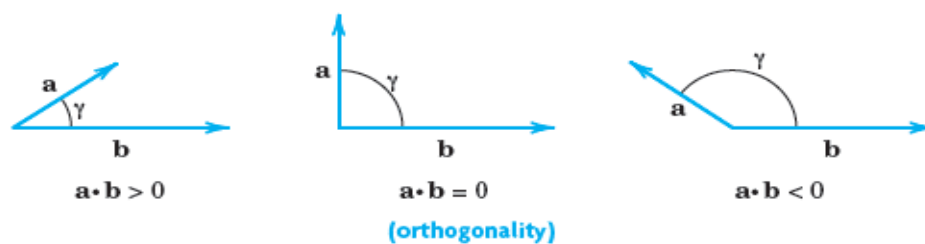


Fig. 178. Angle between vectors and value of inner product

Length and Angle:

From equation (1) with $\mathbf{a} = \mathbf{b}$ gives $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$. Hence $|\mathbf{a}| = \sqrt{\mathbf{a} \cdot \mathbf{a}}$ (3) represents length of a vector.

From (3) and (1) we obtain for the angle γ between two nonzero vectors

$$\cos \gamma = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}| |\mathbf{b}|} = \frac{\mathbf{a} \cdot \mathbf{b}}{\sqrt{\mathbf{a} \cdot \mathbf{a}} \sqrt{\mathbf{b} \cdot \mathbf{b}}} \quad (4)$$

Properties of inner product:

Let us consider any three vectors a, b, c and scalars k_1, k_2 such the the following properties exist

- (a) $(k_1 a + k_2) \cdot c = k_1 a \cdot c + k_2 b \cdot c$ (Linearity)
- (b) $a \cdot b = b \cdot a$ (commutative and symmetry)
- (c) $a \cdot a \geq 0$ and $a \cdot a = 0$ iff $a = 0$ (Positive definiteness)
- (d) $(a + b) \cdot c = a \cdot c + b \cdot c$ (Distributivity)
- (e) $|a \cdot b| \leq |a||b|$ since from (1) $|\cos \gamma| \leq 1$ (Cauchy-schwarz inequality)
- (f) $|a + b| \leq |a| + |b|$ (Triangle inequality)
- (g) $|a + b|^2 + |a - b|^2 = 2(|a|^2 + |b|^2)$ (Parallelogram equality)

Vector Product (Cross Product, Outer Product) of Vectors

The **vector product** or **cross product** of two vectors \mathbf{a} and \mathbf{b} is the vector \mathbf{v} denoted as $\mathbf{a} \times \mathbf{b}$ (read “ \mathbf{a} cross \mathbf{b} ”) and is defined by $\mathbf{V} = \mathbf{a} \times \mathbf{b}$.

Case(I): If $\mathbf{a} = \mathbf{0}$ or $\mathbf{b} = \mathbf{0}$, then we define $\mathbf{V} = \mathbf{a} \times \mathbf{b} = \mathbf{0}$.

Case(II) If both vectors are nonzero vectors, then vector \mathbf{v} has the length

$$|V| = |\mathbf{a} \times \mathbf{b}| = |a||b|\cos\gamma, \text{ where } \gamma \text{ is the angle between } \mathbf{a} \text{ and } \mathbf{b}$$

Furthermore, by design, \mathbf{a} and \mathbf{b} form the sides of a parallelogram on a plane in space. Hence

$$|V| = |\mathbf{a} \times \mathbf{b}| \text{ is equal to the area of that parallelogram.}$$

Case (III). If \mathbf{a} and \mathbf{b} lie in the same straight line, i.e., \mathbf{a} and \mathbf{b} have the same or opposite directions, then γ is

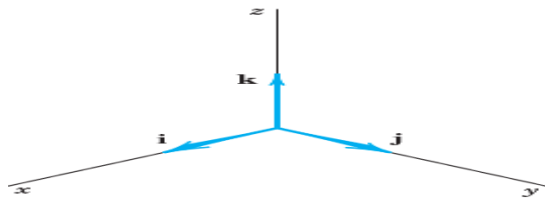
$$0^\circ \text{ or } 180^\circ \text{ or so that } \sin\gamma = 0. \text{ In that case } |V| = 0 \text{ so that } |V| = |\mathbf{a} \times \mathbf{b}| = 0$$

Case(IV). If cases I and III do not occur, then \mathbf{v} is a nonzero vector. The direction of

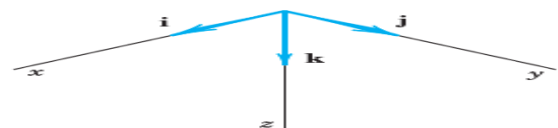
$\mathbf{V} = \mathbf{a} \times \mathbf{b}$ is perpendicular to both \mathbf{a} and \mathbf{b} such that $\mathbf{a}, \mathbf{b}, \mathbf{v}$ precisely in this order.

Right-Handed Cartesian Coordinate System.

The system is called **right-handed** if the corresponding unit vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ in the positive directions of the axes (see the below fig.) form a right-handed triple. The system is called **left-handed** if the sense of \mathbf{k} is reversed,



(a) Right-handed



(b) Left-handed

Example 3 is typical of applications that deal with the **component** or **projection** of a vector \mathbf{a} in the direction of a vector $\mathbf{b} (\neq \mathbf{0})$. If we denote by p the length of the orthogonal projection of \mathbf{a} on a straight line l parallel to \mathbf{b} as shown in Fig. 181, then

$$(10) \quad p = |\mathbf{a}| \cos \gamma.$$

Here p is taken with the plus sign if $p\mathbf{b}$ has the direction of \mathbf{b} and with the minus sign if $p\mathbf{b}$ has the direction opposite to \mathbf{b} .

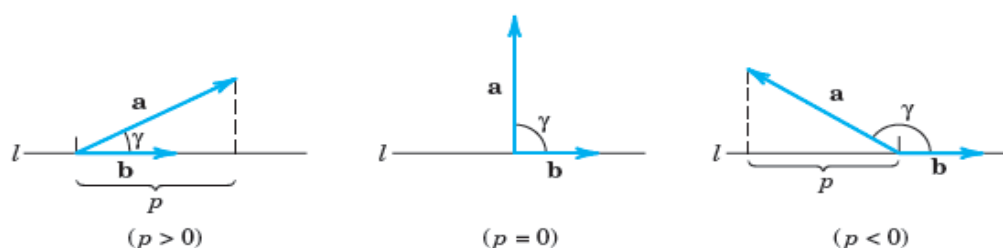


Fig. 181. Component of a vector \mathbf{a} in the direction of a vector \mathbf{b}

Multiplying (10) by $|\mathbf{b}|/|\mathbf{b}| = 1$, we have $\mathbf{a} \cdot \mathbf{b}$ in the numerator and thus

$$(11) \quad p = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|} \quad (\mathbf{b} \neq \mathbf{0}).$$

If \mathbf{b} is a unit vector, as it is often used for fixing a direction, then (11) simply gives

$$(12) \quad p = \mathbf{a} \cdot \mathbf{b} \quad (|\mathbf{b}| = 1).$$

THEOREM 1

General Properties of Vector Products

(a) For every scalar l ,

$$(4) \quad (l\mathbf{a}) \times \mathbf{b} = l(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (l\mathbf{b}).$$

(b) Cross multiplication is distributive with respect to vector addition; that is,

$$(5) \quad (\alpha) \quad \mathbf{a} \times (\mathbf{b} + \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) + (\mathbf{a} \times \mathbf{c}),$$

$$(\beta) \quad (\mathbf{a} + \mathbf{b}) \times \mathbf{c} = (\mathbf{a} \times \mathbf{c}) + (\mathbf{b} \times \mathbf{c}).$$

(c) Cross multiplication is **not commutative** but **anticommutative**; that is,

$$(6) \quad \mathbf{b} \times \mathbf{a} = -(\mathbf{a} \times \mathbf{b}) \quad (\text{Fig. 189}).$$

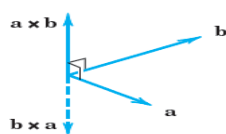


Fig. 189. Anticommutativity of cross multiplication

(d) Cross multiplication is **not associative**; that is, in general,

$$(7) \quad \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \neq (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$$

so that the parentheses cannot be omitted.

Scalar Triple Product

Certain products of vectors, having three or more factors, occur in applications. The most important of these products is the scalar triple product or mixed product of three vectors \mathbf{a} , \mathbf{b} , \mathbf{c} .

$$(10^*) \quad (\mathbf{a} \ \mathbf{b} \ \mathbf{c}) = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}).$$

The scalar triple product is indeed a scalar since (10*) involves a dot product, which in turn is a scalar. We want to express the scalar triple product in components and as a third-order determinant. To this end, let $\mathbf{a} = [a_1, a_2, a_3]$, $\mathbf{b} = [b_1, b_2, b_3]$, and $\mathbf{c} = [c_1, c_2, c_3]$.

Properties and Applications of Scalar Triple Products

(a) *In (10) the dot and cross can be interchanged:*

$$(11) \quad (\mathbf{a} \ \mathbf{b} \ \mathbf{c}) = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}.$$

(b) **Geometric interpretation.** *The absolute value $|(\mathbf{a} \ \mathbf{b} \ \mathbf{c})|$ of (10) is the volume of the parallelepiped (oblique box) with \mathbf{a} , \mathbf{b} , \mathbf{c} as edge vectors (Fig. 193).*

(c) **Linear independence.** *Three vectors in R^3 are linearly independent if and only if their scalar triple product is not zero.*