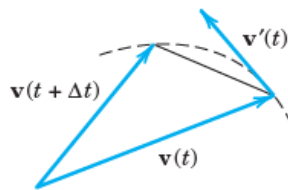


## Session-18

### Derivative of a Vector Function

A vector function  $\mathbf{v}(t)$  is said to be **differentiable** at a point  $t$  if the following limit exists:

$$(9) \quad \mathbf{v}'(t) = \lim_{\Delta t \rightarrow 0} \frac{\mathbf{v}(t + \Delta t) - \mathbf{v}(t)}{\Delta t}.$$



**Fig. 199.** Derivative of a vector function

In components with respect to a given Cartesian coordinate system,

$$(10) \quad \mathbf{v}'(t) = [v_1'(t), v_2'(t), v_3'(t)].$$

$$(c\mathbf{v})' = c\mathbf{v}' \quad (c \text{ constant}),$$

$$(\mathbf{u} + \mathbf{v})' = \mathbf{u}' + \mathbf{v}'$$

and in particular

$$(11) \quad (\mathbf{u} \cdot \mathbf{v})' = \mathbf{u}' \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{v}'$$

$$(12) \quad (\mathbf{u} \times \mathbf{v})' = \mathbf{u}' \times \mathbf{v} + \mathbf{u} \times \mathbf{v}'$$

$$(13) \quad (\mathbf{u} \cdot \mathbf{v} \cdot \mathbf{w})' = (\mathbf{u}' \cdot \mathbf{v} \cdot \mathbf{w}) + (\mathbf{u} \cdot \mathbf{v}' \cdot \mathbf{w}) + (\mathbf{u} \cdot \mathbf{v} \cdot \mathbf{w}').$$

### Partial Derivatives of a Vector Function

Our present discussion shows that partial differentiation of vector functions of two or more variables can be introduced as follows. Suppose that the components of a vector function.

$$\mathbf{v} = [v_1, v_2, v_3] = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$$

are differentiable functions of  $n$  variables  $t_1, \dots, t_n$ . Then the **partial derivative** of  $\mathbf{v}$  with respect to  $t_m$  is denoted by  $\partial\mathbf{v}/\partial t_m$  and is defined as the vector function

$$\frac{\partial\mathbf{v}}{\partial t_m} = \frac{\partial v_1}{\partial t_m}\mathbf{i} + \frac{\partial v_2}{\partial t_m}\mathbf{j} + \frac{\partial v_3}{\partial t_m}\mathbf{k}.$$

Similarly, second partial derivatives are

$$\frac{\partial^2\mathbf{v}}{\partial t_l\partial t_m} = \frac{\partial^2 v_1}{\partial t_l\partial t_m}\mathbf{i} + \frac{\partial^2 v_2}{\partial t_l\partial t_m}\mathbf{j} + \frac{\partial^2 v_3}{\partial t_l\partial t_m}\mathbf{k},$$

and so on.

**Gradient of a Scalar Field. Directional Derivative.**

### Gradient

The setting is that we are given a scalar function  $f(x, y, z)$  that is defined and differentiable in a domain in 3-space with Cartesian coordinates  $x, y, z$ . We denote the **gradient** of that function by  $\text{grad } f$  or  $\nabla f$  (read **nabla**  $f$ ). Then the gradient of  $f(x, y, z)$  is defined as the vector function

$$(1) \quad \text{grad } f = \nabla f = \left[ \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right] = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}.$$

### Directional Derivative

The directional derivative  $D_{\mathbf{b}}f$  or  $df/ds$  of a function  $f(x, y, z)$  at a point  $P$  in the direction of a vector  $\mathbf{b}$  is defined by (see Fig. 215)

$$(2) \quad D_{\mathbf{b}}f = \frac{df}{ds} = \lim_{s \rightarrow 0} \frac{f(Q) - f(P)}{s}.$$

Here  $Q$  is a variable point on the straight line  $L$  in the direction of  $\mathbf{b}$ , and  $|s|$  is the distance between  $P$  and  $Q$ . Also,  $s > 0$  if  $Q$  lies in the direction of  $\mathbf{b}$  (as in Fig. 215),  $s < 0$  if  $Q$  lies in the direction of  $-\mathbf{b}$ , and  $s = 0$  if  $Q = P$ .

$$D_{\mathbf{a}}f = \frac{df}{ds} = \frac{1}{|\mathbf{a}|} \mathbf{a} \cdot \text{grad } f.$$