

Session-2:

Gauss Elimination Method:

Solution by Gauss Elimination: The Gauss Elimination is a standard method for solving linear equations. It is a systematic elimination. It is a reduction to triangular form, from which we shall obtain the values of the unknowns by substitution. This method will be very clear by taking the following problems.

Solved Example: Solve the following system by Gauss Elimination:

$$-x_1 + x_2 - x_3 = 0$$

$$10x_2 + 25x_3 = 90$$

$$20x_1 + 10x_2 = 80$$

Solution: We can write the system and its augmented matrix side by side:

$$-x_1 + x_2 - x_3 = 0$$

$$10x_2 + 25x_3 = 90$$

$$20x_1 + 10x_2 = 80$$

$$K = \begin{bmatrix} 1 & -1 & 1 & 0 \\ -1 & 1 & -1 & 0 \\ 0 & 10 & 25 & 90 \\ 20 & 10 & 0 & 80 \end{bmatrix}$$

Step -1: Elimination of x_1

The first equation is called pivot equation and the coefficient 1 of its x_1 -term the pivot. Using pivot equation, we can eliminate x_1 from other equation. For this, we have to do:

Adding 1 times pivot equation to the 2nd and -20 times pivot equation to the 4th equation, the result is:

$$0 = 0$$

$$10x_2 + 25x_3 = 90$$

$$30x_2 - 20x_3 = 80$$

$$K = \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 10 & 2 & 90 \\ 0 & 30 & -20 & 80 \end{bmatrix}$$

Step -2: Elimination of x_2

The 1st equation which has just served as the pivot equation remains unchanged. The 2nd equation is the next pivot equation, but for this problem 2nd equation contains no x_2 -term. Now it is required to change the order of the equation to get a non-zero pivot. We can put 2nd

equation at the end and the 3rd and 4th equations one place up. This is called **partial pivoting**. We get:

$$\begin{aligned} x_1 - x_2 + x_3 &= 0 \\ 10x_2 + 25x_3 &= 90 \\ 30x_2 - 20x_3 &= 80 \\ 0 &= 0 \end{aligned} \quad K = \begin{bmatrix} 1 & -1 & 1 & \text{R}0 \\ 0 & 10 & 25 & \text{R}90 \\ 0 & 30 & -20 & \text{R}80 \\ 0 & 0 & 0 & \text{R}0 \end{bmatrix}.$$

Adding 3 times pivot equation to the 3rd equation, the result is:

$$\begin{aligned} 10x_2 + 25x_3 &= 90 \\ -95x_3 &= -190 \\ 0 &= 0 \end{aligned} \quad K = \begin{bmatrix} 1 & -1 & 1 & \text{R}0 \\ 0 & 10 & 25 & \text{R}90 \\ 0 & 0 & -95 & \text{R}-190 \\ 0 & 0 & 0 & \text{R}0 \end{bmatrix}.$$

Back Substitution: Determination of x_3 , x_2 , and x_1 (in this order)

$$-95x_3 = -190 \Rightarrow x_3 = 2$$

$$10x_2 + 25x_3 = 90 \Rightarrow x_2 = 4$$

$$x_1 - x_2 + x_3 = 0 \Rightarrow x_1 = 2$$

The solution of the system is:

$$x_1 = 2, x_2 = 4, \text{ and } x_3 = 2.$$

Elementary Operations for Equations:

- Interchange of two equations
- Addition of a constant multiple of one equation to another equation
- Multiplication of an equation by a non-zero constant C

Elementary Row Operations for Matrices:

Elementary *row* operations do not change the solution set of the system of linear equations represented by a matrix, and are used in Gaussian elimination (respectively, Gauss-Jordan elimination) to reduce a matrix to row echelon form (respectively, reduced row echelon form).

The acronym "**ERO**" is commonly used for "elementary row operations".

➤ Row switching

A row within the matrix can be switched with another row.

$$R_i \leftrightarrow R_j$$

➤ **Row multiplication**

Each element in a row can be multiplied by a non-zero constant.

$$kR_i \rightarrow R_i, \text{ where } k \neq 0$$

➤ **Row addition**

A row can be replaced by the sum of that row and a multiple of another row.

$$R_i + kR_j \rightarrow R_i, \text{ where } i \neq j$$

The elementary matrix for any row operation is obtained by executing the operation on an identity matrix.

Row Echelon Form:

In linear algebra a matrix is in **row echelon form** if

- All nonzero rows (rows with at least one nonzero element) are above any rows of all zeroes [All zero rows, if any, belong at the bottom of the matrix]
- The leading coefficient (the first nonzero number from the left, also called the pivot) of a nonzero row is always strictly to the right of the leading coefficient of the row above it.
- All entries in a column below a leading entry are zeroes (implied by the first two criteria).

This is an example of 3×4 matrix in row echelon form:

$$\begin{bmatrix} 1 & a_1 & a_2 & : & a_3 \\ 0 & 2 & a_4 & : & a_5 \\ 0 & 0 & -1 & : & a_6 \end{bmatrix}$$

A matrix is in **reduced row echelon form** (also called **row canonical form/Row-Echelon Normal Form**) if it satisfies the additional condition:

- Every leading coefficient is 1 and is the only nonzero entry in its column, like in this example:

$$\begin{bmatrix} 1 & 0 & 0 & : & b_1 \\ 0 & 1 & 0 & : & b_2 \\ 0 & 0 & 1 & : & b_3 \end{bmatrix}$$

Note that this does not always mean that the left of the matrix will be an identity matrix. For example, the following matrix is also in reduced row-echelon form:

$$\left[\begin{array}{cccc|c} 1 & 0 & 1/2 & 0 & b_1 \\ 0 & 1 & -1/3 & 0 & b_2 \\ 0 & 0 & 0 & 1 & b_3 \end{array} \right]$$