

## Session-2:

### **Elementary Operations for Equations:**

- Interchange of two equations
- Addition of a constant multiple of one equation to another equation
- Multiplication of an equation by a non-zero constant C

### **Elementary Row Operations for Matrices:**

Elementary *row* operations do not change the solution set of the system of linear equations represented by a matrix, and are used in Gaussian elimination (respectively, Gauss-Jordan elimination) to reduce a matrix to row echelon form (respectively, reduced row echelon form).

The acronym "**ERO**" is commonly used for "elementary row operations".

- **Row switching**

A row within the matrix can be switched with another row.

$$R_i \leftrightarrow R_j$$

- **Row multiplication**

Each element in a row can be multiplied by a non-zero constant.

$$kR_i \rightarrow R_i, \text{ where } k \neq 0$$

- **Row addition**

A row can be replaced by the sum of that row and a multiple of another row.

$$R_i + kR_j \rightarrow R_i, \text{ where } i \neq j$$

The elementary matrix for any row operation is obtained by executing the operation on an identity matrix.

### **Row Echelon Form:**

In linear algebra a matrix is in **row echelon form** if

- All nonzero rows (rows with at least one nonzero element) are above any rows of all zeroes [All zero rows, if any, belong at the bottom of the matrix]
- The leading coefficient (the first nonzero number from the left, also called the pivot) of a nonzero row is always strictly to the right of the leading coefficient of the row above it.
- All entries in a column below a leading entry are zeroes (implied by the first two criteria).

This is an example of  $3 \times 4$  matrix in row echelon form:

$$\begin{bmatrix} 1 & a_1 & a_2 & \vdots & a_3 \\ 0 & 2 & a_4 & \vdots & a_5 \\ 0 & 0 & -1 & \vdots & a_6 \end{bmatrix}$$

A matrix is in **reduced row echelon form** (also called **row canonical form/Row-Echelon Normal Form**) if it satisfies the additional condition:

- Every leading coefficient is 1 and is the only nonzero entry in its column, like in this example:

$$\begin{bmatrix} 1 & 0 & 0 & \vdots & b_1 \\ 0 & 1 & 0 & \vdots & b_2 \\ 0 & 0 & 1 & \vdots & b_3 \end{bmatrix}$$

Note that this does not always mean that the left of the matrix will be an identity matrix. For example, the following matrix is also in reduced row-echelon form:

$$\left[ \begin{array}{cccc|c} 1 & 0 & 1/2 & 0 & b_1 \\ 0 & 1 & -1/3 & 0 & b_2 \\ 0 & 0 & 0 & 1 & b_3 \end{array} \right]$$

## Linear Dependence and Independence:

A finite subset of  $n$  vectors,  $v_1, v_2, v_3, \dots, v_n$  from the vector space  $V$ , is **linearly dependent** if and only if there exists a set of  $n$  scalars,  $\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_n$ , not all zero, such that  $\lambda_1 v_1 + \lambda_2 v_2 + \dots + \lambda_n v_n = 0$

Note that the zero on the right is the zero vectors, not the number zero.

If such scalars do not exist, then the vectors are said to be **linearly independent**.

Alternatively, linear independence can be directly defined as follows: a set of vectors is **linearly independent** if and only if the only representations of the zero vector as linear combinations of its elements are trivial solutions, i.e., whenever  $\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_n$  are

scalars such that  $\lambda_1 v_1 + \lambda_2 v_2 + \dots + \lambda_n v_n = 0$

iff  $\lambda_i = 0$  for  $i = 1, 2, \dots, n$ .

A set of vectors is then said to be **linearly dependent** if it is not linearly independent.

**Solved Example:** Test whether the following sets of vectors are linearly dependent or independent:

$$(1, 1) \text{ and } (-3, 2)$$

**Solution:**

Let  $\lambda_1$  and  $\lambda_2$  be two real numbers such that

$$\lambda_1(1,1) + \lambda_2(-3,2) = (0,0)$$

Taking each coordinate alone, this means

$$\lambda_1 + 2\lambda_2 = 0$$

Solving for  $\lambda_1$  and  $\lambda_2$ , we find that  $\lambda_1 = 0$  and  $\lambda_2 = 0$ .

Thus the vectors  $(1, 1)$  and  $(-3, 2)$  are linearly independent.

### Alternative method using determinants

An alternative method uses the fact that  $n$  vectors in  $\mathbb{R}^n$  are linearly **dependent** if and only if the determinant of the matrix formed by taking the vectors as its columns is zero.

In this case, the matrix formed by the vectors is

$$A = \begin{bmatrix} 1 & -3 \\ 1 & 2 \end{bmatrix}$$

We may write a linear combination of the columns as

$$A\Lambda = \begin{bmatrix} 1 & -3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}.$$

We are interested in whether  $A\Lambda = \mathbf{0}$  for some nonzero vector  $\Lambda$ . This depends on the determinant of  $A$ , which is

$$\det A = 1 \cdot 2 - 1 \cdot (-3) = 5 \neq 0.$$

Since the determinant is non-zero, the vectors  $(1, 1)$  and  $(-3, 2)$  are linearly independent.

**Solved Example:** Test whether the following sets of vectors in  $\mathbf{R}^4$  are linearly dependent or

independent:

$$\begin{bmatrix} 1 \\ 4 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 7 \\ 10 \\ -4 \\ -1 \end{bmatrix} \text{ and } \begin{bmatrix} -2 \\ 1 \\ 5 \\ -4 \end{bmatrix}$$

**Solution:**

We need to find scalars  $\lambda_1$ ,  $\lambda_2$  and  $\lambda_3$  such that

$$\lambda_1 \begin{bmatrix} 1 \\ 4 \\ 2 \\ -3 \end{bmatrix} + \lambda_2 \begin{bmatrix} 7 \\ 10 \\ -4 \\ -1 \end{bmatrix} + \lambda_3 \begin{bmatrix} -2 \\ 1 \\ 5 \\ -4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Forming the simultaneous equations:

$$\lambda_1 + 7\lambda_2 - 2\lambda_3 = 0$$

$$4\lambda_1 + 10\lambda_2 + \lambda_3 = 0$$

$$2\lambda_1 - 4\lambda_2 + 5\lambda_3 = 0$$

$$-3\lambda_1 - \lambda_2 - 4\lambda_3 = 0$$

we can solve (using, for example, Gaussian elimination) to obtain:

$$\lambda_1 = -3\lambda_3/2$$

$$\lambda_2 = \lambda_3/2$$

where  $\lambda_3$  can be chosen arbitrarily.

Since these are nontrivial results, the vectors are linearly dependent.

**Note:**

The rank of a matrix describes its linear independence. It describes the dimensionality of the space. It is defined for any square or rectangular matrix such that

Rank (A) = number of linearly independent rows of **A** = number of linearly independent columns of **A**. The number of linearly independent rows of a matrix is always equal to the number of linearly independent columns.

For **A** of  $n \times p$ , the maximum possible rank is the smaller of  $n$  and  $p$ , where **A** is considered to be of *full rank*. When the rank is smaller than both  $n$  and  $p$ , the matrix is of *deficient rank*. Thus, only a square matrix can be of full rank – rectangular matrices will always have a rank less than either  $n$  or  $p$ .

$0 < \text{rank} \leq \min(n, p)$ . When a matrix is the product of two other matrices,  $\mathbf{C} = \mathbf{AB}$ , its rank cannot be greater than the smaller rank of the two matrices:

Rank **C**  $\leq \min(\text{rank } \mathbf{A}, \text{rank } \mathbf{B})$ .