

Session-20

Divergence of a Vector Field

Let $V(x, y, z)$ be a differentiable vector function, where x, y, z are Cartesian coordinates, and let v_1, v_2, v_3 be the components of \mathbf{v} . Then the function $\text{div } V = \nabla \cdot V = \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k \right) \cdot (v_1 i + v_2 j + v_3 k)$

$$\text{div } V = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}$$

is called the **divergence** of \mathbf{v} or the *divergence of the vector field defined by \mathbf{v}* .

Some important Results:

(1) $V = \text{grad } f = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k$ then $\text{div } V = \text{div}(\text{grad } f) = \nabla^2 f$ called *Laplacian* form of a vectors.

(2) If $\text{div } V = 0$ then V is called *Solenoidal* vector function or *incompressible* vector function.

Curl of a Vector Field

Let $V(x, y, z)$ be a differentiable vector function, where x, y, z are Cartesian coordinates, and let $V = [v_1, v_2, v_3]$ be the components of \mathbf{v} . Then the function $\text{curl } V = \nabla \times V = \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k \right) \times (v_1 i + v_2 j + v_3 k)$

Or

$$\begin{aligned} \text{curl } V = \nabla \times V &= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix} \\ &= \left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z} \right) i + \left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x} \right) j + \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} \right) k \end{aligned}$$

This is the formula when x, y, z are *right-handed*. If they are *left-handed*, the determinant has a minus sign in front.

Some important Results:(Relation between Grad, Div, Curl)

(1) **Gradient fields** are *irrotational*. That is, if a continuously differentiable vector function is the gradient of a scalar function, then its curl is the zero vector, i.e. $\text{curl}(\text{grad } f) = 0$.

(2) The divergence of the curl of a twice continuously differentiable vector function \mathbf{v} is zero, i.e. $\text{div}(\text{curl } V) = 0$.

(3) If $\text{curl } V = 0$ then V is called *irrotational* vector field.

(4) If $\text{curl } V \neq 0$ then V is called *rotational* vector field.