

Session-3:

Rank of a Matrix: The maximum number of linearly independent rows in a matrix A is called the **row rank** of A , and the maximum number of linearly independent columns in A is called the **column rank** of A . If A is an m by n matrix, that is, if A has m rows and n columns, then it is obvious that

$$\text{Row rank of } A \leq m$$

$$\text{Column rank of } A \leq n$$

For any matrix A ,

- the row rank of A = the column rank of A
- If A is $m \times n$, it follows from the above inequalities that rank of $A \leq \min(m, n)$

Solved Example 1: Find the rank of the matrix

$$B = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \\ 1 & 1 & 4 \end{bmatrix}$$

Solution: First, because the matrix is 4×3 , its rank can be no greater than 3. Therefore, at least one of the four rows will become a row of zeros. Perform the following row operations:

$$\begin{bmatrix} 2 & -1 & 3 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \\ 1 & 1 & 4 \end{bmatrix}$$

$$\xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 0 & 1 \\ 2 & -1 & 3 \\ 0 & 2 & -1 \\ 1 & 1 & 4 \end{bmatrix}$$

$$\xrightarrow{R_2 \rightarrow R_2 - 2R_1 \text{ and } R_4 \rightarrow R_4 - R_1} \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 2 & -1 \\ 0 & 1 & 3 \end{bmatrix}$$

$$\xrightarrow{R_3 \rightarrow R_3 + 2R_2 \text{ and } R_4 \rightarrow R_4 + R_2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 4 \end{bmatrix}$$

$$\xrightarrow{R_2 \rightarrow (-1)R_2 \text{ and } R_4 \rightarrow R_4 - 4R_3} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Since there are 3 nonzero rows remaining in this echelon form of B , Rank of $B=3$.

Solved Example2: Determine the rank of the 4 by 4 checkerboard matrix

$$C = \begin{bmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{bmatrix}$$

Solution: Since $r_2 = r_4 = -r_1$ and $r_3 = r_1$, all rows but the first vanish upon row-reduction:

$$\begin{bmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{bmatrix} \xrightarrow{\substack{r_1 \text{ added to } r_2 \\ -r_1 \text{ added to } r_3 \\ r_1 \text{ added to } r_4}} \begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since only 1 nonzero row remains, rank of $C = 1$.

Properties (Related to Rank of Matrices):

We assume that A is an $m \times n$ matrix over either the real numbers or the complex numbers:

- Only a zero matrix has rank zero.
- rank of $A \leq \min(m, n)$
- If A is a square matrix (i.e., $m = n$), then A is invertible if and only if A has rank n (that is, A has full rank).
- If B is any $n \times k$ matrix, then

$$\text{rank}(AB) \leq \min(\text{rank } A, \text{rank } B).$$

- If B is an $n \times k$ matrix with rank n , then $\text{rank}(AB) = \text{rank}(A)$.

- If C is an $l \times m$ matrix with rank m , then

$$\text{rank}(CA) = \text{rank}(A).$$

- The rank of A is equal to r if and only if there exists an invertible $m \times m$ matrix X and an invertible $n \times n$ matrix Y such that

$$XAY = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix},$$

where I_r denotes the $r \times r$ identity matrix.

- The rank of a matrix plus the nullity of the matrix equals the number of columns of the matrix. (This is the rank–nullity theorem.)
- $\text{rank}(A^T A) = \text{rank}(A A^T) = \text{rank}(A) = \text{rank}(A^T)$.
- If A^* denotes the conjugate transpose of A (i.e., the adjoint of A), then

$$\text{rank}(A) = \text{rank}(\overline{A}) = \text{rank}(A^T) = \text{rank}(A^*) = \text{rank}(A^* A).$$

- Let A be an $n \times n$ matrix:

A is nonsingular if and only if $\text{rank}(A) = n$.

$\text{rank}(A) = n \Leftrightarrow A$ is nonsingular $\Leftrightarrow \det(A) \neq 0$

$\Leftrightarrow Ax=b$ has a unique solution.

$\text{rank}(A) < n \Leftrightarrow Ax=0$ has a nontrivial solution