

## Session-6

### Cramer's Rule:

Given a system of linear equations, Cramer's Rule is a handy way to solve for just one of the variables without having to solve the whole system of equations. They don't usually teach Cramer's Rule this way, but this is supposed to be the point of the Rule: instead of solving the entire system of equations, you can use Cramer's to solve for just one single variable.

**Solved Example1:** Use Cramer's Rule to determine the solution to the following system of equations.

$$\begin{aligned}2x + y + z &= 3 \\ x - y - z &= 0 \\ x + 2y + z &= 0\end{aligned}$$

We have the left-hand side of the system with the variables (the "coefficient matrix") and the right-hand side with the answer values. Let  $D$  be the determinant of the coefficient matrix of the above system, and let  $D_x$  be the determinant formed by replacing the  $x$ -column values with the answer-column values:

System of equations	Coefficient Matrix's Determinant	Answer Column	$D_x$ : Coefficient Determinant with Answer-column values in $x$ -column
$2x + 1y + 1z = 3$ $1x - 1y - 1z = 0$ $1x + 2y + 1z = 0$	$D = \begin{vmatrix} 2 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & 2 & 1 \end{vmatrix}$	$\begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$	$D_x = \begin{vmatrix} 3 & 1 & 1 \\ 0 & -1 & -1 \\ 0 & 2 & 1 \end{vmatrix}$

Similarly,  $D_y$  and  $D_z$  would then be:

$$D_y = \begin{vmatrix} 2 & 3 & 1 \\ 1 & 0 & -1 \\ 1 & 0 & 1 \end{vmatrix} \quad D_z = \begin{vmatrix} 2 & 1 & 3 \\ 1 & -1 & 0 \\ 1 & 2 & 0 \end{vmatrix}$$

Evaluating each determinant, we get:

$$D = \begin{vmatrix} 2 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & 2 & 1 \end{vmatrix} = (-2) + (-1) + (2) \\ \quad \quad \quad -(-1) - (-4) - (1) = 3$$

$$D_x = \begin{vmatrix} 3 & 1 & 1 \\ 0 & -1 & -1 \\ 0 & 2 & 1 \end{vmatrix} = (-3) + (0) + (0) \\ \quad \quad \quad -(0) - (-6) - (0) = -3 + 6 = 3$$

$$D_y = \begin{vmatrix} 2 & 3 & 1 \\ 1 & 0 & -1 \\ 1 & 0 & 1 \end{vmatrix} = (0) + (-3) + (0) \\ \quad \quad \quad -(0) - (0) - (3) = -3 - 3 = -6$$

$$D_z = \begin{vmatrix} 2 & 1 & 3 \\ 1 & -1 & 0 \\ 1 & 2 & 0 \end{vmatrix} = (0) + (0) + (6) \\ \quad \quad \quad -(-3) - (0) - (0) = 6 + 3 = 9$$

Cramer's Rule says that  $x = D_x \div D$ ,  $y = D_y \div D$ , and  $z = D_z \div D$ . That is:

$$x = 3/3 = 1, y = -6/3 = -2, \text{ and } z = 9/3 = 3.$$

**Solved Example2:** Use Cramer's Rule to determine the solution to the following system of equations.

$$3x_1 - x_2 + 5x_3 = -2$$

$$-4x_1 + x_2 + 7x_3 = 10$$

$$2x_1 + 4x_2 - x_3 = 3$$

**Solution:** The system can be put into matrix form and verify that the coefficient matrix is invertible.

$$\begin{bmatrix} 3 & -1 & 5 \\ -4 & 1 & 7 \\ 2 & 4 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 10 \\ 0 \end{bmatrix}$$

Which is equivalent to  $AX = B$

$\det(A) = -187 \neq 0$  So, the coefficient matrix is invertible and Cramer's Rule can be used on the system. We'll also need  $\det(A)$  in a bit so it's good that we now have it. Let's now write down the formulas for the solution to this system.

$$x_1 = \frac{\det(A_1)}{\det(A)}, x_2 = \frac{\det(A_2)}{\det(A)}, \text{ and } x_3 = \frac{\det(A_3)}{\det(A)}$$

where  $A_1$  is the matrix formed by replacing the 1<sup>st</sup> column of  $A$  with  $B$ ,  $A_2$  is the matrix formed by replacing the 2<sup>nd</sup>

column of  $A$  with  $B$ , and  $A_3$  is the matrix formed by replacing the 3<sup>rd</sup> column of  $A$  with  $B$ .

$$\det(A_1) = \begin{vmatrix} -2 & -1 & 5 \\ 10 & 1 & 7 \\ 3 & 4 & -1 \end{vmatrix} = 212$$

$$\det(A_2) = \begin{vmatrix} 3 & -2 & 5 \\ -4 & 10 & 7 \\ 2 & 3 & -1 \end{vmatrix} = -273$$

$$\det(A_3) = \begin{vmatrix} 3 & -1 & -2 \\ -4 & 1 & 10 \\ 2 & 4 & 3 \end{vmatrix} = -107$$

The solution to the system is

then,  $x_1 = -\frac{212}{187}$        $x_2 = \frac{273}{187}$        $x_3 = \frac{107}{187}$