

## Session-8:

### **Spectrum & Spectral Radius:**

Let  $A = [a_{ij}]$  be a given  $n \times n$  matrix and we can consider the vector equation

$$AX = \lambda X \dots \dots \dots (1)$$

Here  $X$  is an unknown vector and  $\lambda$  is an unknown scalar and we want to determine both.

The zero vector  $X = 0$  is a solution of (1) for any value of  $\lambda$ . A value of  $\lambda$  for which (1) has a solution  $X \neq 0$  is called an eigenvalue or characteristic value or latent root of the matrix  $A$ . The corresponding solutions  $X \neq 0$  of (1) are called eigenvectors or characteristic vectors of  $A$  corresponding to that eigenvalue  $\lambda$ . The set of eigenvalues is called the spectrum of  $A$ . The largest of the absolute values of the eigenvalues of  $A$  is called the spectral radius of  $A$ .

### **Eigenvalues:**

#### **Characteristic Equation**

If  $A$  is any square matrix of order  $n$ , we can form the matrix  $A - \lambda I$ , where  $I$  is the  $n \times n$  identity matrix. The determinant of this matrix equated to zero, that is,

$$|A - \lambda I| = \begin{vmatrix} a_{11} - \lambda & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} - \lambda & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} - \lambda \end{vmatrix} = 0$$
 is called the *characteristic equation* (or, less

often, the secular equation) of  $A$ .

On expanding the determinant, the characteristic equation takes the form

$(-1)^n \lambda^n + k_1 \lambda^{n-1} + k_2 \lambda^{n-2} + \dots + k_n = 0$ , where  $k$ 's, are expressible in terms of the elements  $a_{ij}$ . The roots of the equation are called the characteristic roots/latent roots/eigenvalues of the matrix  $A$ .

We can write  $C(\lambda) = |A - \lambda I|$  = The characteristic polynomial of  $A$ ; it is a polynomial of degree  $n$  and it has  $n$  roots, some or all of which may be repeated, or some or all of which may be complex.

### **Eigenvectors:**

If

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \text{ and } A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$

then the linear transformation  $Y = AX$  carries the column vector  $X$  into the column vector  $Y$  by means of the square matrix  $A$ .

Let  $X$  be such a vector which transforms into  $\lambda X$  by means of the square matrix  $A$  i.e.

$$\lambda X = AX$$

or,  $AX - \lambda X = 0$

or,  $(A - \lambda I)X = 0$

This matrix equation represents  $n$  homogeneous linear equations

$$(a_{11} - \lambda)x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n = 0$$

$$a_{21}x_1 + (a_{22} - \lambda)x_2 + a_{23}x_3 + \dots + a_{2n}x_n = 0$$

$$\dots$$

$$a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \dots + (a_{nn} - \lambda)x_n = 0$$

Which will have a nontrivial solution iff the coefficient matrix is singular, i. e. if

$$|A - \lambda I| = 0 \quad (\text{as } X \neq 0)$$

The scalar  $\lambda$  represents the eigenvalues of  $A$ , and the corresponding non-zero vector  $X$  are the eigenvectors / latent vectors / characteristic vectors of  $A$ . The set of eigenvalues is called the spectrum of  $A$ . The largest of the absolute values of the eigenvalues of  $A$  is called the spectral radius of  $A$ .