

ME 101: Engineering Mechanics

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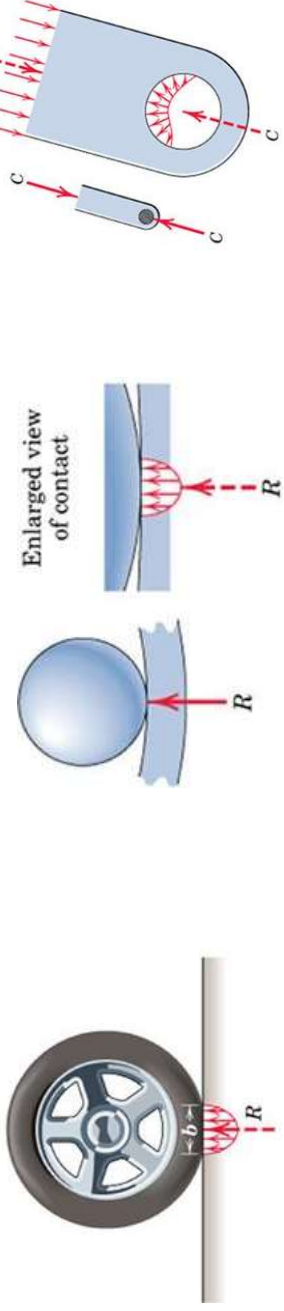
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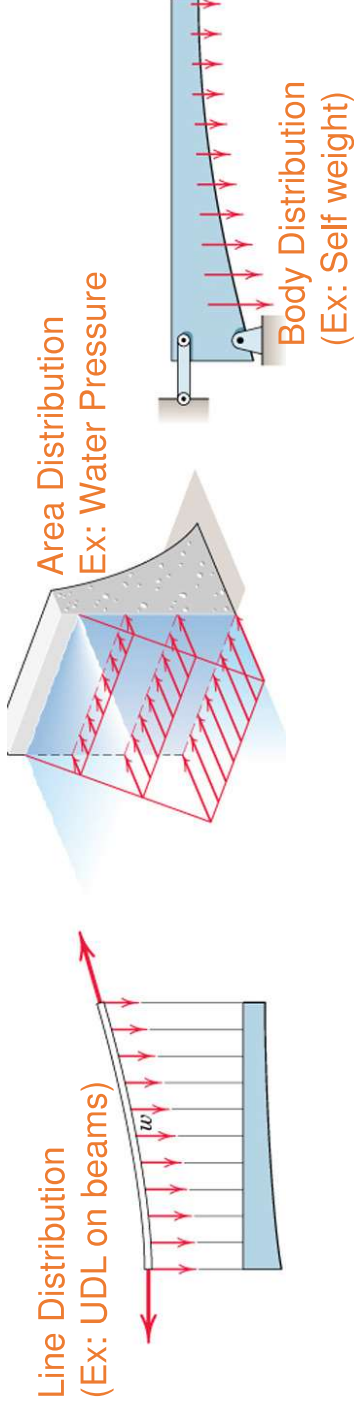
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Center of Mass and Centroids

Concentrated Forces: If dimension of the contact area is negligible compared to other dimensions of the body \rightarrow the contact forces may be treated as Concentrated Forces



Distributed Forces: If forces are applied over a region whose dimension is not negligible compared with other pertinent dimensions \rightarrow proper distribution of contact forces must be accounted for to know intensity of force at any location.

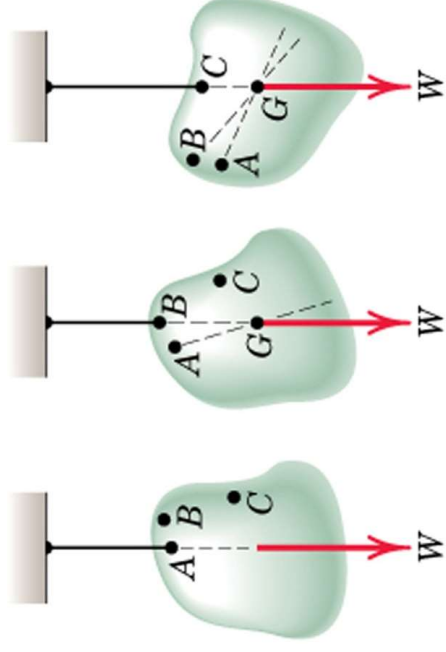


Center of Mass and Centroids

Center of Mass

A body of mass m in equilibrium under the action of tension in the cord, and resultant W of the gravitational forces acting on all particles of the body.

- The resultant is collinear with the cord



Suspend the body from different points on the body

- Dotted lines show lines of action of the resultant force in each case.

- These lines of action will be concurrent at a single point G

As long as dimensions of the body are smaller compared with those of the earth.

- we assume uniform and parallel force field due to the gravitational attraction of the earth.

The unique **Point G** is called the Center of Gravity of the body (CG)

Center of Mass and Centroids

Determination of CG

- Apply Principle of Moments

Moment of resultant gravitational force W about any axis equals sum of the moments about the same axis of the gravitational forces dW acting on all particles treated as infinitesimal elements.

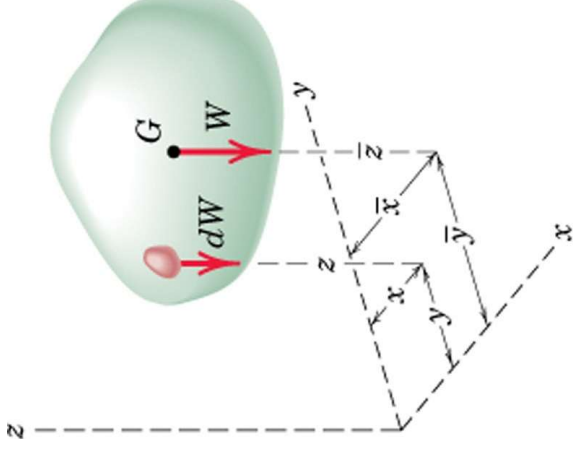
Weight of the body $W = \int dW$

Moment of weight of an element (dW) @ x-axis = ydW

Sum of moments for all elements of body = $\int ydW$

From Principle of Moments: $\int ydW = \bar{y} W$

$$\bar{x} = \frac{\int xdW}{W} \quad \bar{y} = \frac{\int ydW}{W} \quad \bar{z} = \frac{\int zdW}{W}$$



→ Numerator of these expressions represents the **sum of the moments**;
Product of W and corresponding coordinate of G represents the **moment of the sum** → Moment Principle.

Center of Mass and Centroids

Determination of CG

Substituting $W = mg$ and $dW = gdm$

$$\rightarrow \bar{x} = \frac{\int xdm}{m} \quad \bar{y} = \frac{\int ydm}{m} \quad \bar{z} = \frac{\int zdm}{m}$$

In vector notations:

Position vector for elemental mass:

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

Position vector for mass center G:

$$\bar{\mathbf{r}} = \bar{x}\mathbf{i} + \bar{y}\mathbf{j} + \bar{z}\mathbf{k}$$

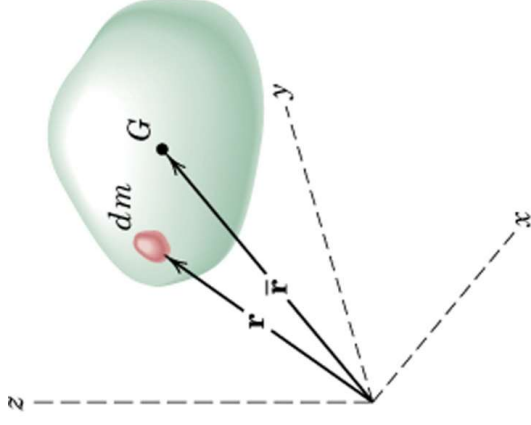
$\rightarrow \bar{\mathbf{r}} = \frac{\int \mathbf{r}dm}{m}$
 The above equations are the components of this single vector equation

Density ρ of a body = **mass per unit volume**

\rightarrow Mass of a differential element of volume $dV \rightarrow dm = \rho dV$
 $\rightarrow \rho$ may not be constant throughout the body

$$\bar{x} = \frac{\int x\rho dV}{\int \rho dV} \quad \bar{y} = \frac{\int y\rho dV}{\int \rho dV} \quad \bar{z} = \frac{\int z\rho dV}{\int \rho dV}$$

$$\bar{x} = \frac{\int x dW}{W} \quad \bar{y} = \frac{\int y dW}{W} \quad \bar{z} = \frac{\int z dW}{W}$$



Center of Mass and Centroids

Center of Mass: Following equations independent of g

$$\bar{x} = \frac{\int x dm}{m} \quad \bar{y} = \frac{\int y dm}{m} \quad \bar{z} = \frac{\int z dm}{m}$$

$$\bar{\mathbf{r}} = \frac{\int \mathbf{r} dm}{m}$$

$$\bar{x} = \frac{\int x \rho dV}{\int \rho dV} \quad \bar{y} = \frac{\int y \rho dV}{\int \rho dV} \quad \bar{z} = \frac{\int z \rho dV}{\int \rho dV}$$

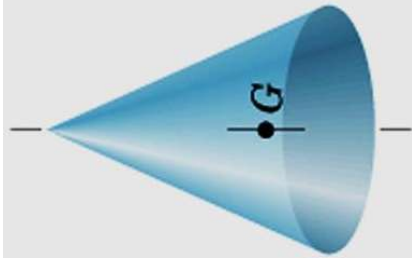
→ They define a unique point, which is a function of distribution of mass

→ This point is **Center of Mass (CM)**

→ **CM coincides with CG as long as gravity field is treated as uniform and parallel**

→ CG or CM may lie outside the body

CM always lie on a line or a plane of symmetry in a homogeneous body



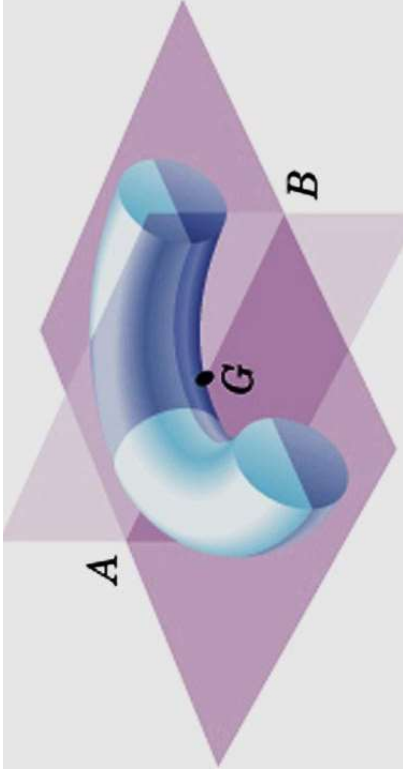
Right Circular Cone

CM on central axis



Half Right Circular Cone

CM on vertical plane of symmetry



Half Ring

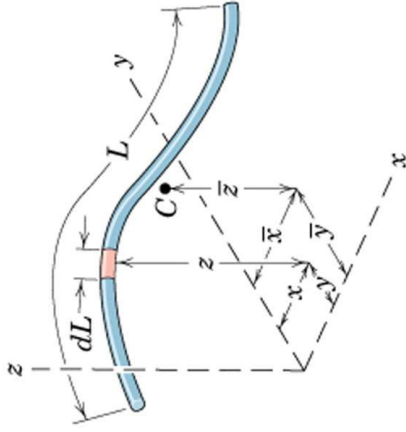
CM on intersection of two planes of symmetry (line AB)

Center of Mass and Centroids

Centroids of Lines, Areas, and Volumes

Centroid is a geometrical property of a body

→ When density of a body is uniform throughout, centroid and CM coincide



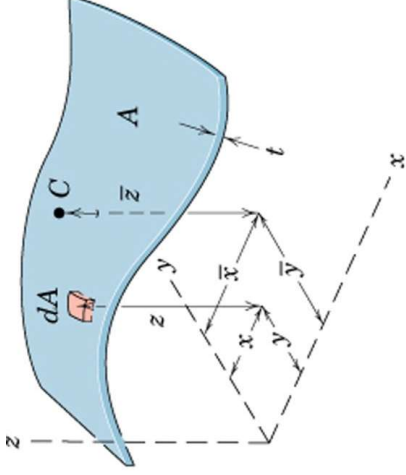
Lines: Slender rod, Wire

Cross-sectional area = A

ρ and A are constant over L

$dm = \rho A dL$; Centroid = CM

$$\bar{x} = \frac{\int x dL}{L} \quad \bar{y} = \frac{\int y dL}{L} \quad \bar{z} = \frac{\int z dL}{L}$$



Areas: Body with small but constant thickness t

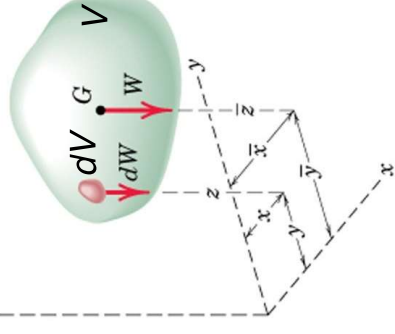
Cross-sectional area = A

ρ and A are constant over A

$dm = \rho t dA$; Centroid = CM

$$\bar{x} = \frac{\int x dA}{A} \quad \bar{y} = \frac{\int y dA}{A} \quad \bar{z} = \frac{\int z dA}{A}$$

Numerator = First moments of Area



Volumes: Body with volume V

ρ constant over V

$dm = \rho dV$ Centroid = CM

$$\bar{x} = \frac{\int x dV}{V} \quad \bar{y} = \frac{\int y dV}{V} \quad \bar{z} = \frac{\int z dV}{V}$$

$$\bar{x} = \frac{\int x dm}{m} \quad \bar{y} = \frac{\int y dm}{m} \quad \bar{z} = \frac{\int z dm}{m}$$

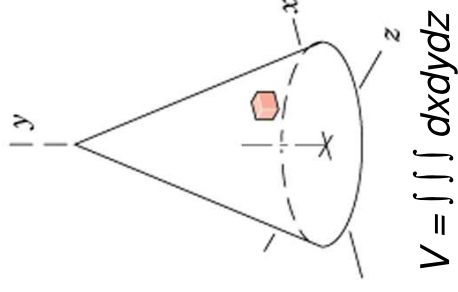
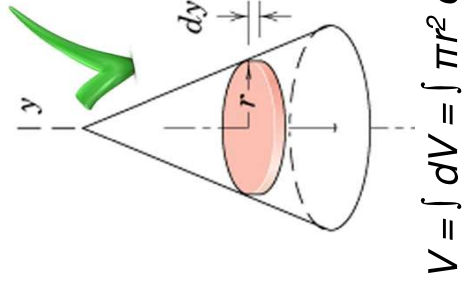
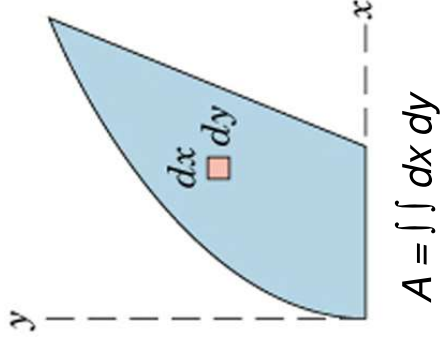
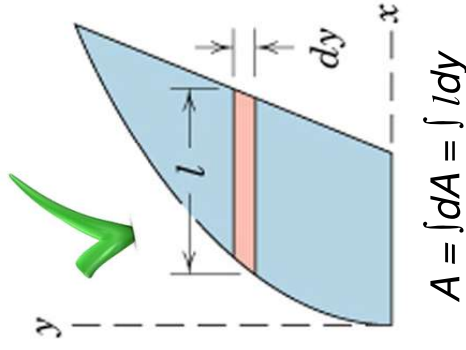
Center of Mass and Centroids

Centroids of Lines, Areas, and Volumes

Guidelines for Choice of Elements for Integration

- **Order of Element Selected for Integration**

A first order differential element should be selected in preference to a higher order element → only one integration should cover the entire figure

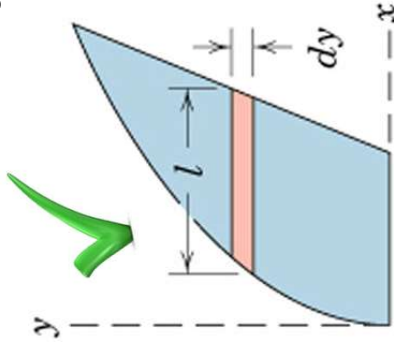


Center of Mass and Centroids

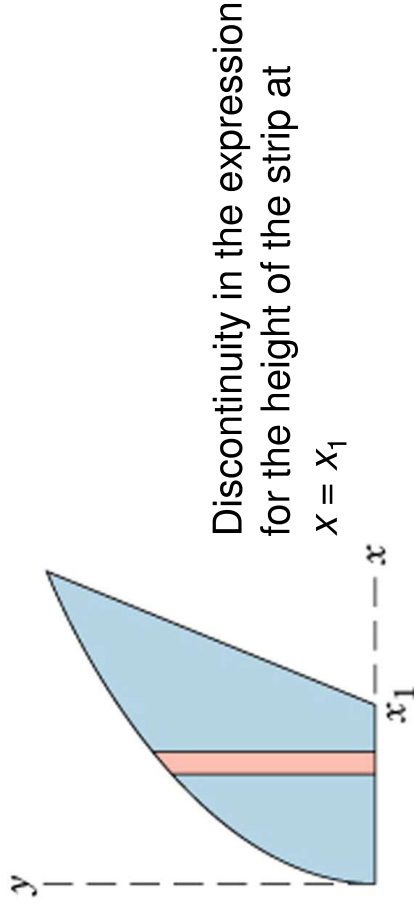
Centroids of Lines, Areas, and Volumes Guidelines for Choice of Elements for Integration

- **Continuity**

Choose an element that can be integrated in one continuous operation to cover the entire figure → the function representing the body should be continuous
→ only one integral will cover the entire figure



Continuity in the expression for the width of the strip

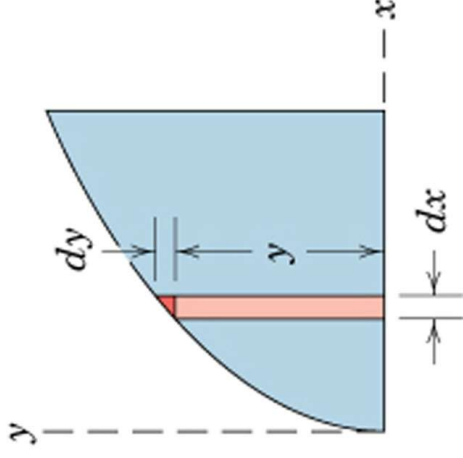


Center of Mass and Centroids

Centroids of Lines, Areas, and Volumes Guidelines for Choice of Elements for Integration

- **Discarding Higher Order Terms**
Higher order terms may always be dropped compared with lower order terms

Vertical strip of area under the curve is given by the first order term $\rightarrow dA = ydx$
The second order triangular area $0.5dx dy$ may be discarded



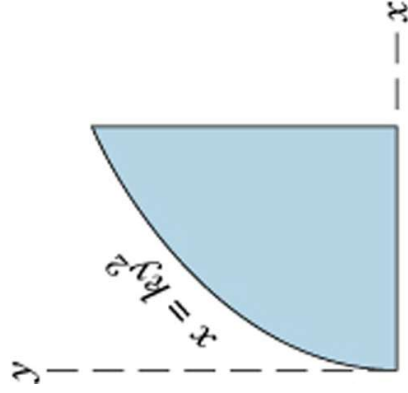
Center of Mass and Centroids

Centroids of Lines, Areas, and Volumes Guidelines for Choice of Elements for Integration

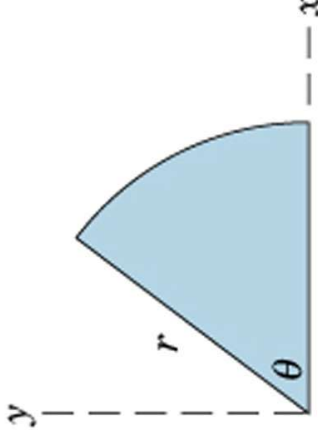
- **Choice of Coordinates**

Coordinate system should best match the boundaries of the figure

→ easiest coordinate system that satisfies boundary conditions should be chosen



Boundaries of this area (not circular) can be easily described in rectangular coordinates



Boundaries of this circular sector are best suited to polar coordinates

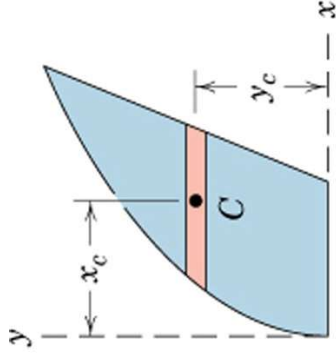
Center of Mass and Centroids

Centroids of Lines, Areas, and Volumes

Guidelines for Choice of Elements for Integration

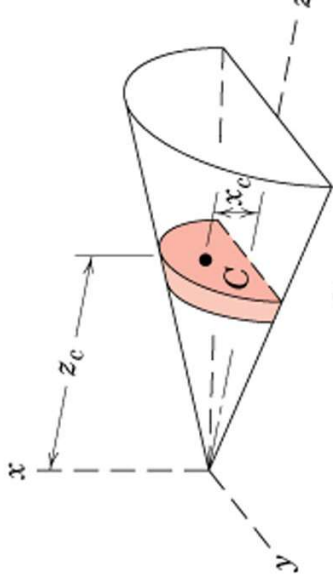
- **Centroidal Coordinate of Differential Elements**

While expressing moment of differential elements, take coordinates of the centroid of the differential element as lever arm (not the coordinate describing the boundary of the area)



Modified Equations

$$\bar{x} = \frac{\int x_c dA}{A} \quad \bar{y} = \frac{\int y_c dA}{A} \quad \bar{z} = \frac{\int z_c dA}{A}$$



$$\bar{x} = \frac{\int x_c dV}{V} \quad \bar{y} = \frac{\int y_c dV}{V} \quad \bar{z} = \frac{\int z_c dV}{V}$$

Center of Mass and Centroids

Centroids of Lines, Areas, and Volumes

Guidelines for Choice of Elements for Integration

1. Order of Element Selected for Integration
2. Continuity
3. Discarding Higher Order Terms
4. Choice of Coordinates
5. Centroidal Coordinate of Differential Elements

$$\bar{x} = \frac{\int x dL}{L} \quad \bar{y} = \frac{\int y dL}{L} \quad \bar{z} = \frac{\int z dL}{L}$$

$$\bar{x} = \frac{\int x_c dA}{A} \quad \bar{y} = \frac{\int y_c dA}{A} \quad \bar{z} = \frac{\int z_c dA}{A}$$

$$\bar{x} = \frac{\int x_c dV}{V} \quad \bar{y} = \frac{\int y_c dV}{V} \quad \bar{z} = \frac{\int z_c dV}{V}$$

Center of Mass and Centroids

Examples: Centroids

Locate the centroid of the circular arc

Solution: Polar coordinate system is better

Since the figure is symmetric: centroid lies on the x axis

Differential element of arc has length $dL = r d\theta$

Total length of arc: $L = 2\alpha r$

x-coordinate of the centroid of differential element: $x = r \cos \theta$

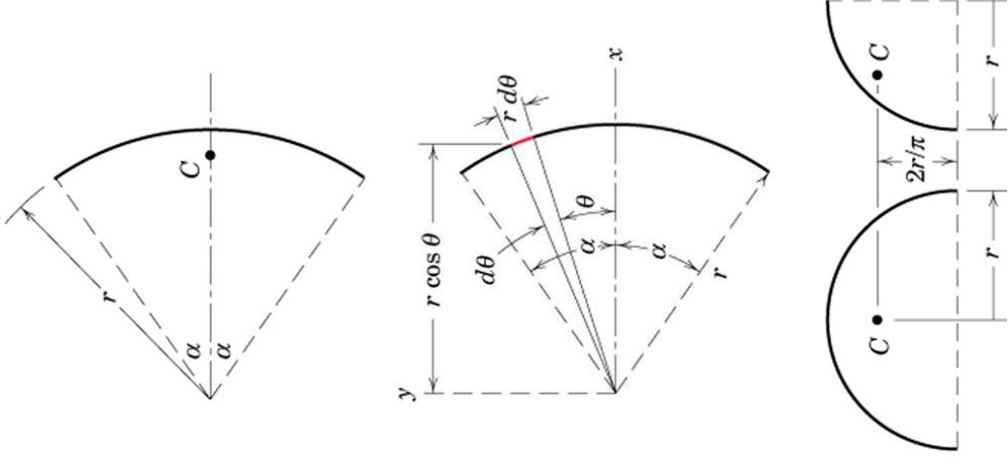
$$\bar{x} = \frac{\int x dL}{L} \quad \bar{y} = \frac{\int y dL}{L} \quad \bar{z} = \frac{\int z dL}{L}$$

$$L\bar{x} = \int x dL \quad 2\alpha r \bar{x} = \int_{-\alpha}^{\alpha} (r \cos \theta) r d\theta$$

$$2\alpha r \bar{x} = 2r^2 \sin \alpha$$

$$\bar{x} = \frac{r \sin \alpha}{\alpha}$$

For a semi-circular arc: $2\alpha = \pi \rightarrow$ centroid lies at $2r/\pi$



Center of Mass and Centroids

Examples: Centroids

Locate the centroid of the triangle along h from the base

Solution:

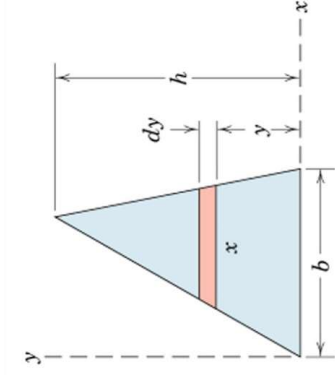
$$dA = xdy \quad \frac{x}{(h-y)} = \frac{b}{h}$$

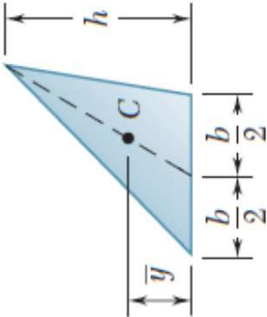
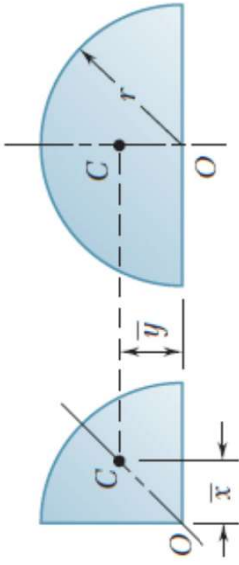
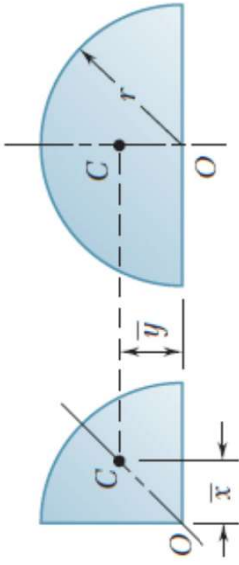
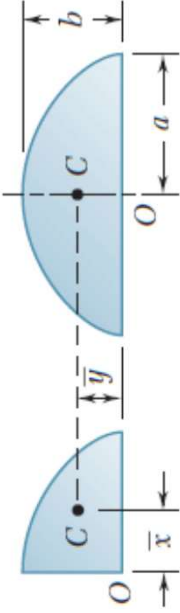
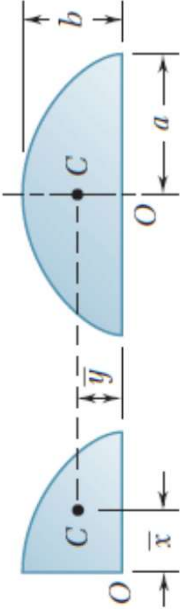
$$\text{Total Area } A = \frac{1}{2}bh \quad y = y_c$$

$$\bar{x} = \frac{\int x_c dA}{A} \quad \bar{y} = \frac{\int y_c dA}{A} \quad \bar{z} = \frac{\int z_c dA}{A}$$

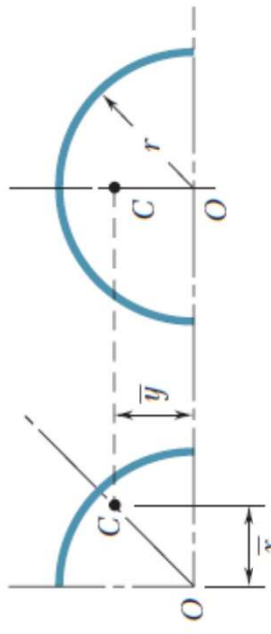
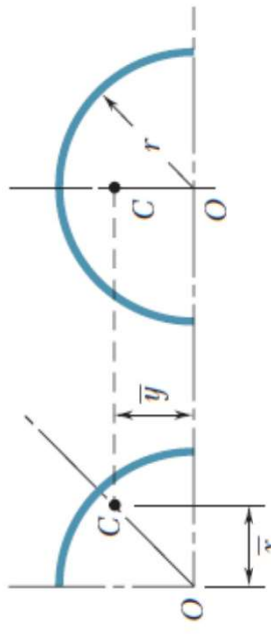
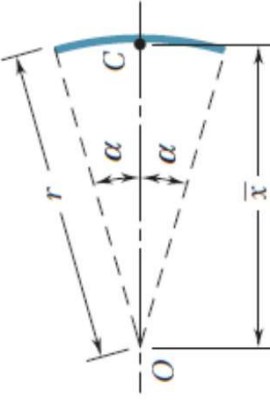
$$A\bar{y} = \int y_c dA \Rightarrow \frac{bh}{2}\bar{y} = \int_0^h y \frac{b(h-y)}{y} dy = \frac{bh^2}{6}$$

$$\bar{y} = \frac{h}{3}$$



Shape		\bar{x}	\bar{y}	Area
Triangular area			$\frac{h}{3}$	$\frac{bh}{2}$
Quarter-circular area		$\frac{4r}{3\pi}$	$\frac{4r}{3\pi}$	$\frac{\pi r^2}{4}$
Semicircular area		0	$\frac{4r}{3\pi}$	$\frac{\pi r^2}{2}$
Quarter-elliptical area		$\frac{4a}{3\pi}$	$\frac{4b}{3\pi}$	$\frac{\pi ab}{4}$
Semielliptical area		0	$\frac{4b}{3\pi}$	$\frac{\pi ab}{2}$

Semiparabolic area		$\frac{3a}{8}$	$\frac{3h}{5}$	$\frac{2ah}{3}$
		0	$\frac{3h}{5}$	$\frac{4ah}{3}$
Parabolic spandrel		$\frac{3a}{4}$	$\frac{3h}{10}$	$\frac{ah}{3}$
General spandrel		$\frac{n+1}{n+2}a$	$\frac{n+1}{4n+2}h$	$\frac{ah}{n+1}$
Circular sector		$\frac{2r \sin \alpha}{3\alpha}$	0	αr^2

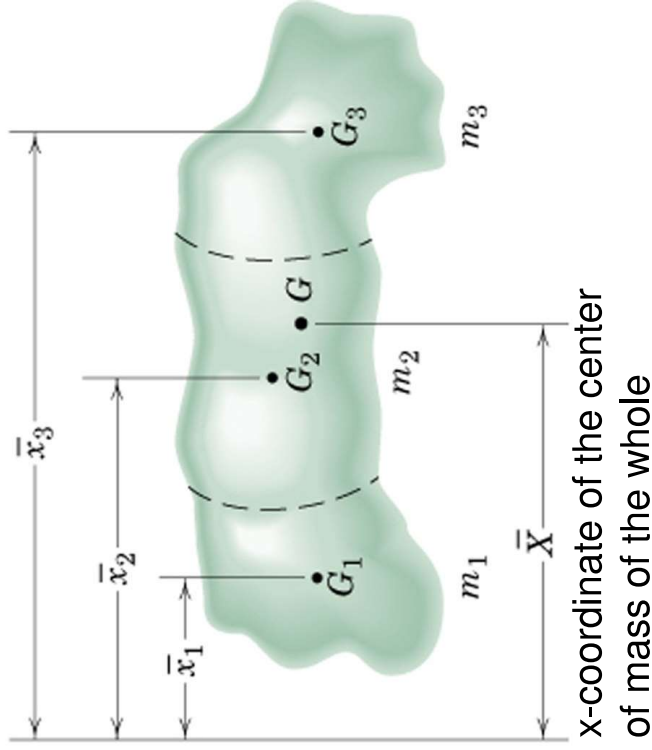
Shape	Diagram	\bar{x}	\bar{y}	Length
Quarter-circular arc		$\frac{2r}{\pi}$	$\frac{2r}{\pi}$	$\frac{\pi r}{2}$
Semicircular arc		0	$\frac{2r}{\pi}$	πr
Arc of circle		$\frac{r \sin \alpha}{\alpha}$	0	$2\alpha r$

Center of Mass and Centroids

Composite Bodies and Figures

Divide bodies or figures into several parts such that their mass centers can be conveniently determined

→ Use Principle of Moment for all finite elements of the body



$$(m_1 + m_2 + m_3) \bar{X} = m_1 \bar{x}_1 + m_2 \bar{x}_2 + m_3 \bar{x}_3$$

Mass Center Coordinates can be written as:

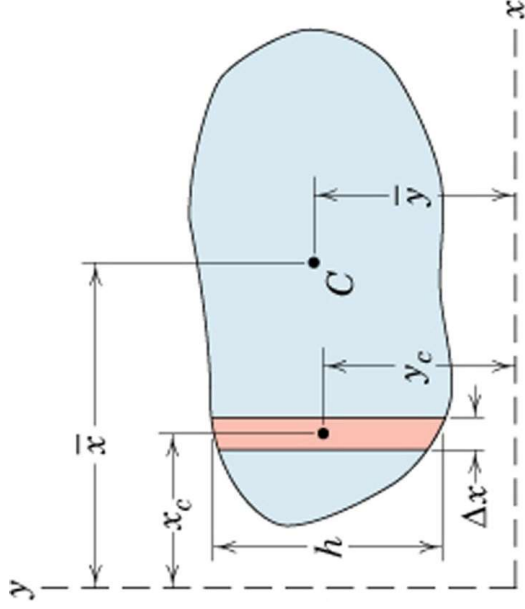
$$\bar{X} = \frac{\sum m\bar{x}}{\sum m} \quad \bar{Y} = \frac{\sum m\bar{y}}{\sum m} \quad \bar{Z} = \frac{\sum m\bar{z}}{\sum m}$$

m 's can be replaced by L 's, A 's, and V 's for lines, areas, and volumes

Center of Mass and Centroids: Composite Bodies and Figures

Integration vs Appx Summation: Irregular Area

In some cases, the boundaries of an area or volume might not be expressible mathematically or in terms of simple geometrical shapes → Appx Summation may be used instead of integration



Divide the area into several strips

Area of each strip = $h\Delta x$

Moment of this area about x- and y-axis

= $(h\Delta x)y_c$ and $(h\Delta x)x_c$

→ Sum of moments for all strips

divided by the total area will give

corresponding coordinate of the centroid

$$\bar{x} = \frac{\sum Ax_c}{\sum A} \quad \bar{y} = \frac{\sum Ay_c}{\sum A}$$

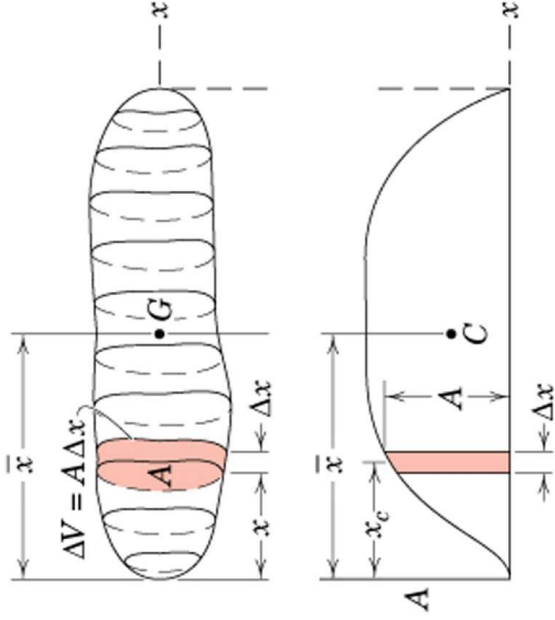
Accuracy may be improved by reducing the width of the strip

Center of Mass and Centroids: Composite Bodies and Figures

Integration vs Appx Summation: Irregular Volume

Reduce the problem to one of locating the centroid of area

→ Appx Summation may be used instead of integration



Divide the area into several strips

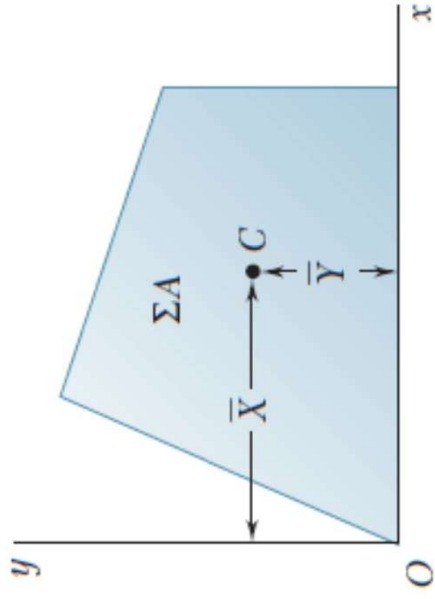
Volume of each strip = $A\Delta x$

Plot all such A against x.

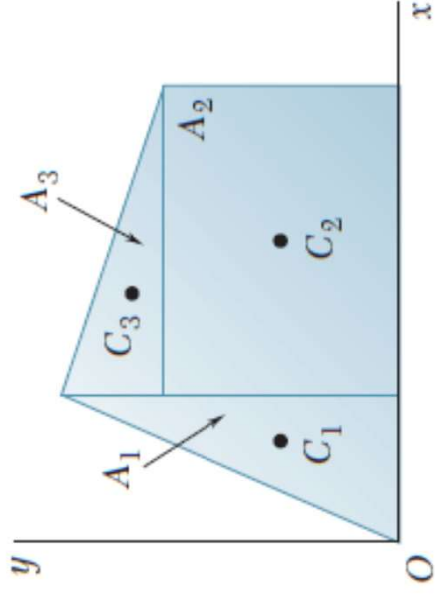
→ Area under the plotted curve represents volume of whole body and the x-coordinate of the centroid of the area under the curve is given by:

$$\bar{x} = \frac{\sum (A\Delta x)x_c}{\sum A\Delta x} \Rightarrow \bar{x} = \frac{\sum Vx_c}{\sum V}$$

Accuracy may be improved by reducing the width of the strip



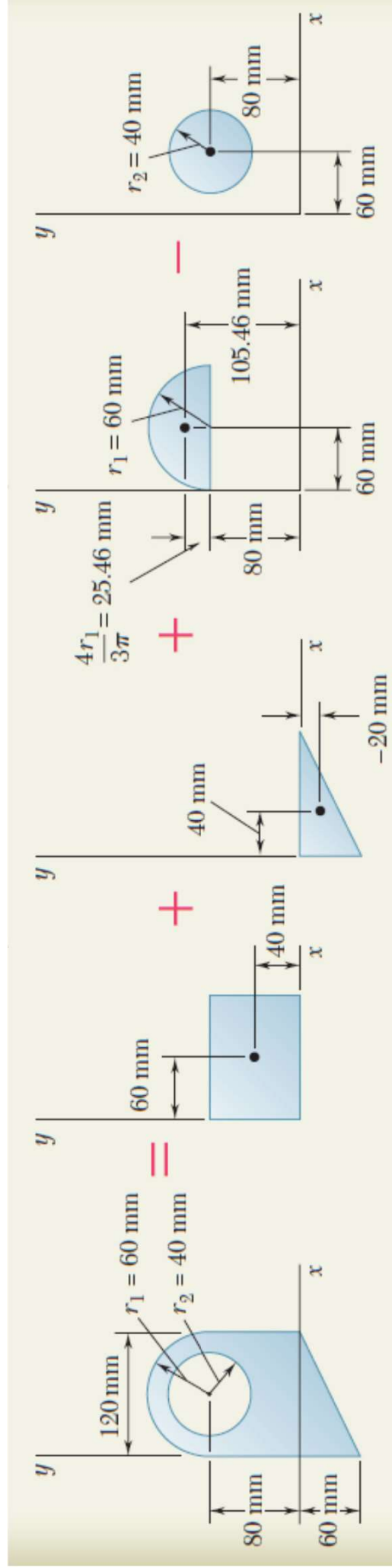
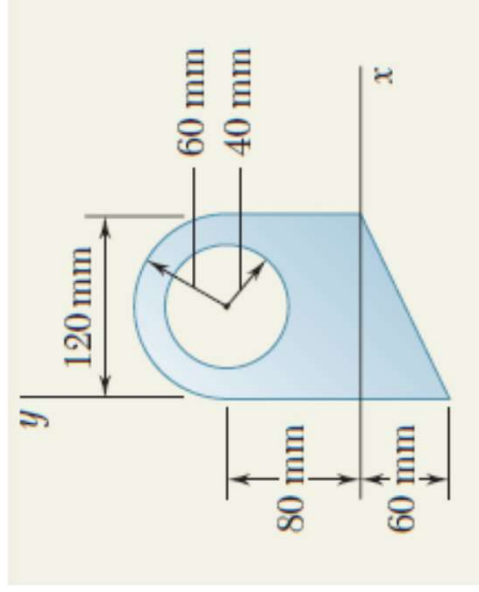
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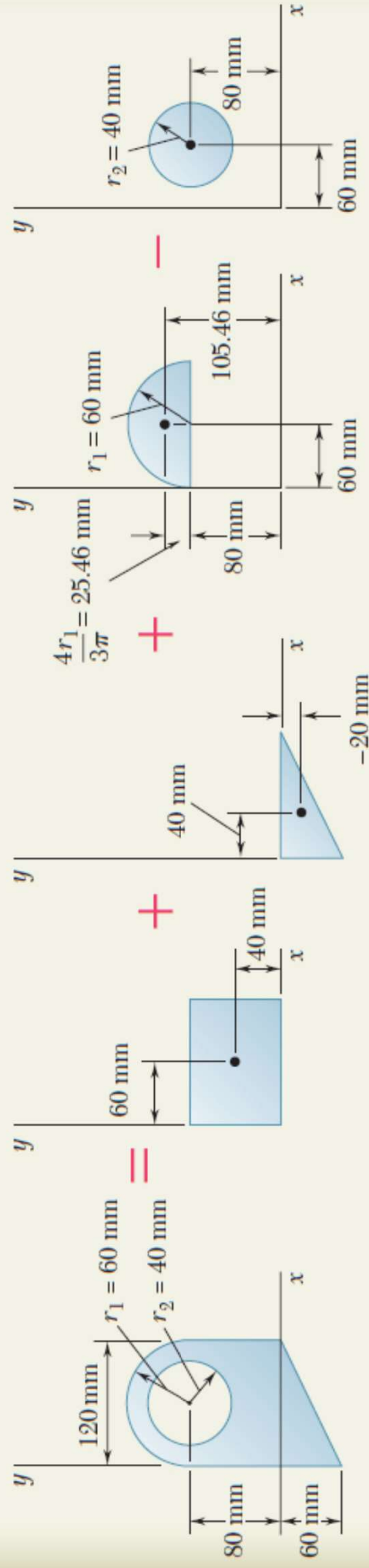


$$Q_y = \bar{X} \Sigma A = \Sigma \bar{x} A$$

$$Q_x = \bar{Y} \Sigma A = \Sigma \bar{y} A$$

For the plane area shown, determine (a) the first moments with respect to the x and y axes, (b) the location of the centroid.





Component	A, mm^2	\bar{x}, mm	\bar{y}, mm	$\bar{x}A, \text{mm}^3$	$\bar{y}A, \text{mm}^3$
Rectangle	$(120)(80) = 9.6 \times 10^3$	60	40	$+576 \times 10^3$	$+384 \times 10^3$
Triangle	$\frac{1}{2}(120)(60) = 3.6 \times 10^3$	40	-20	$+144 \times 10^3$	-72×10^3
Semicircle	$\frac{1}{2}\pi(60)^2 = 5.655 \times 10^3$	60	105.46	$+339.3 \times 10^3$	$+596.4 \times 10^3$
Circle	$-\pi(40)^2 = -5.027 \times 10^3$	60	80	-301.6×10^3	-402.2×10^3
	$\Sigma A = 13.828 \times 10^3$			$\Sigma \bar{x}A = +757.7 \times 10^3$	$\Sigma \bar{y}A = +506.2 \times 10^3$

First Moments of the Area.

$$Q_x = \Sigma \bar{y}A = 506.2 \times 10^3 \text{ mm}^3$$

$$Q_y = \Sigma \bar{x}A = 757.7 \times 10^3 \text{ mm}^3$$

Location of Centroid.

$$\bar{X} \Sigma A = \Sigma \bar{x}A: \quad \bar{X}(13.828 \times 10^3 \text{ mm}^2) = 757.7 \times 10^3 \text{ mm}^3$$

$$\bar{X} = 54.8 \text{ mm}$$

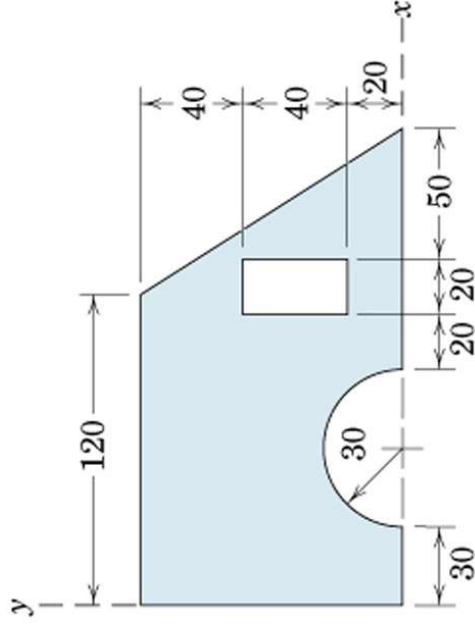
$$\bar{Y} \Sigma A = \Sigma \bar{y}A: \quad \bar{Y}(13.828 \times 10^3 \text{ mm}^2) = 506.2 \times 10^3 \text{ mm}^3$$

$$\bar{Y} = 36.6 \text{ mm}$$

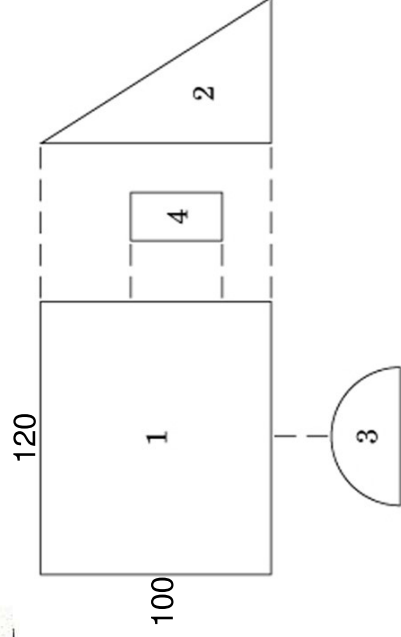
Center of Mass and Centroids: Composite Bodies and Figures

Example:
 Locate the centroid of the shaded area

Solution: Divide the area into four elementary shapes: Total Area = $A_1 + A_2 - A_3 - A_4$



Dimensions in millimeters



PART	A mm ²	\bar{x} mm	\bar{y} mm	$\bar{x}A$ mm ³	$\bar{y}A$ mm ³
1	12 000	60	50	720 000	600 000
2	3000	140	100/3	420 000	100 000
3	-1414	60	12.73	-84 800	-18 000
4	-800	120	40	-96 000	-32 000
TOTALS	12 790			959 000	650 000

$$\left[\bar{X} = \frac{\sum A\bar{x}}{\sum A} \right]$$

$$\left[\bar{Y} = \frac{\sum A\bar{y}}{\sum A} \right]$$

$$\bar{X} = \frac{959\,000}{12\,790} = 75.0 \text{ mm}$$

$$\bar{Y} = \frac{650\,000}{12\,790} = 50.8 \text{ mm}$$

A uniform semicircular rod of weight W and radius r is attached to a pin at A and rests against a frictionless surface at B . Determine the reactions at A and B .

$$+\uparrow \Sigma M_A = 0: \quad B(2r) - W\left(\frac{2r}{\pi}\right) = 0$$

$$B = +\frac{W}{\pi}$$

$$+\rightarrow \Sigma F_x = 0: \quad A_x + B = 0$$

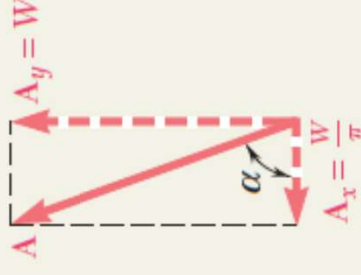
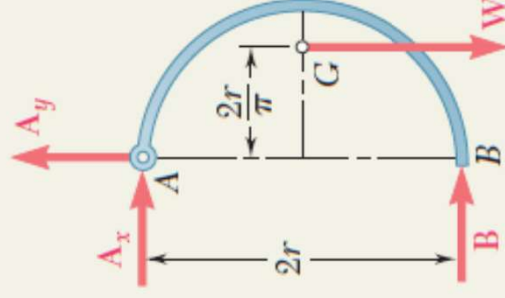
$$A_x = -B = -\frac{W}{\pi} \quad A_x = \frac{W}{\pi} \leftarrow$$

$$+\uparrow \Sigma F_y = 0: \quad A_y - W = 0 \quad A_y = W \uparrow$$

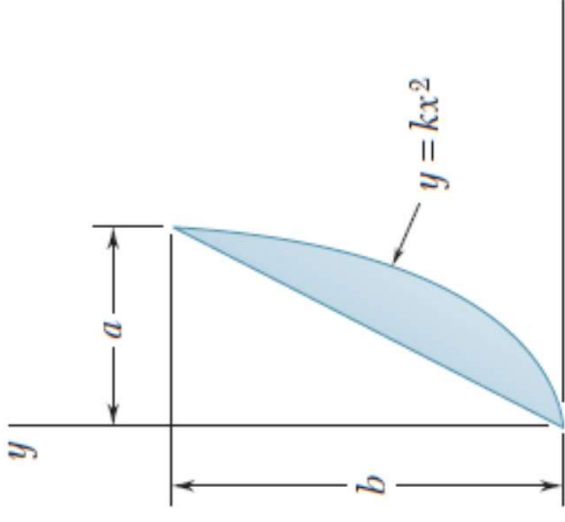
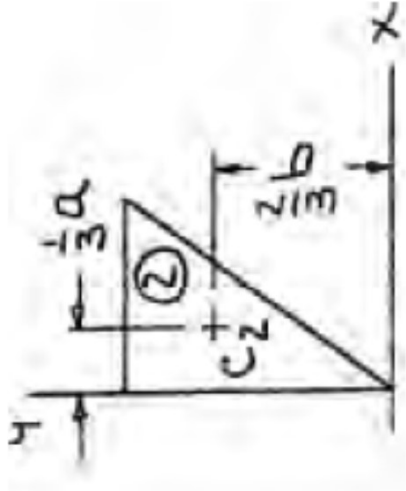
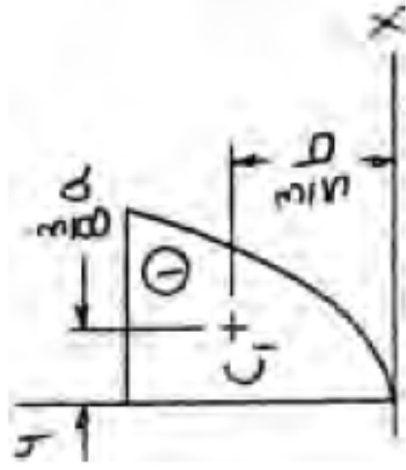
$$A = \left[W^2 + \left(\frac{W}{\pi} \right)^2 \right]^{1/2}$$

$$\tan \alpha = \frac{W}{W/\pi} = \pi$$

$$A = 1.049W \quad \sphericalangle 72.3^\circ \quad B = 0.318W \rightarrow$$



For the area shown, determine the ratio a/b for which $\bar{x} = \bar{y}$



	A	\bar{x}	\bar{y}	$\bar{x}A$	$\bar{y}A$
1	$\frac{2}{3}ab$	$\frac{3}{8}a$	$\frac{3}{5}b$	$\frac{a^2b}{4}$	$\frac{2ab^2}{5}$
2	$-\frac{1}{2}ab$	$\frac{1}{3}a$	$\frac{2}{3}b$	$-\frac{a^2b}{6}$	$-\frac{ab^2}{3}$
Σ	$\frac{1}{6}ab$			$\frac{a^2b}{12}$	$\frac{ab^2}{15}$

Then

$$\bar{X} \Sigma A = \Sigma \bar{x} A$$

$$\bar{X} \left(\frac{1}{6} ab \right) = \frac{a^2 b}{12}$$

or

$$\bar{X} = \frac{1}{2} a$$

$$\bar{Y} \Sigma A = \Sigma \bar{y} A$$

$$\bar{Y} \left(\frac{1}{6} ab \right) = \frac{ab^2}{15}$$

or

$$\bar{Y} = \frac{2}{5} b$$

Now

$$\bar{X} = \bar{Y} \Rightarrow$$

$$\frac{1}{2} a = \frac{2}{5} b$$

$$\text{or } \frac{a}{b} = \frac{4}{5} \blacktriangleleft$$