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System Modeling & Control

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Lesson - 4

Mathematical Modeling of Control Systems

INTRODUCTION

A mathematical model of a dynamic system is defined as a set of equations that represents the dynamics of the system accurately, or at least fairly well. Note that a mathematical model is not unique to a given system. A system may be represented in many different ways and, therefore, may have many mathematical models, depending on one's perspective. The dynamics of many systems, whether they are mechanical, electrical, thermal, economic, biological, and so on, may be described in terms of differential equations. Such differential equations may be obtained by using physical laws governing a particular system—for example, Newton's laws for mechanical systems and Kirchhoff's laws for electrical systems. We must always keep in mind that deriving reasonable mathematical models is the most important part of the entire analysis of control systems. Throughout this book we assume that the principle of causality applies to the systems considered. This means that the current output of the system (the output at time $t=0$) depends on the past input (the input for $t<0$) but does not depend on the future input (the input for $t>0$).



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Modeling of Mechanical System

Depending upon the motion the mechanical systems can be divided into different types.

1. Translational
2. Rotational
3. Rotational and Translational



Translational Mechanical System

Elements of Mechanical System (Translational)

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Mass:-

The ideal mass element represents a particle of mass, which is the lumped approximation of the mass of a body concentrated at the centre of mass. The mass has two terminals, one is free, attached to motional variable u and the other represents the reference.

$f(t)$ represents the applied force, $x(t)$ represents the displacement, and M represents the mass. Then, in accordance with Newton's second law

$$f(t) = Ma(t) = \frac{Mdv(t)}{dt} = \frac{Md^2x(t)}{dt^2}$$



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Damper:-

Damper is the damping elements and damping is the friction existing in physical systems whenever mechanical system moves on sliding surface. The friction encountered is of many types:

- Stiction: The force required at startup.
- Coulomb Friction: The force of sliding friction between dry surfaces. This force is substantially constant. –
- Viscous friction force: The force of friction between moving surfaces separated by viscous fluid or the force between a solid body and a fluid medium. It is linearly proportional to velocity over a certain limited velocity range.

The mathematical model of friction is given by

$$f(t) = B \left(\frac{dx_1(t)}{dt} - \frac{dx_2(t)}{dt} \right)$$



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Spring:

- The final translational mechanical element is a spring. The ideal spring gives the elastic deformation of a body. The defining equation from Hooke's law is given by

$$f(t) = K(x_1(t) - x_2(t))$$

- Note here that the force developed is directly proportional to the difference in the displacement of one end of the spring relative to the other. These equations apply for the forces and the displacements in the directions shown by the arrowheads in figure.
- If any of the directions are reversed, the sign on that term in the equations must be changed. For these mechanical elements, friction dissipates energy but cannot store it. Both mass and a spring can store energy but can not dissipate it



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Mathematical modeling of Translational Mechanical System

Element	Symbol	Force velocity	Force displacement	Force acceleration
Mass		$F = M \int \frac{dv}{dt}$	$F = M \frac{d^2x}{dt^2}$	$F = Ma$
Spring (one end fixed)		$F = k \int v dt$	$F = kx$	—
Dashpot (one end fixed)		$F = Bv$	$F = B \frac{dx}{dt}$	—
Spring (Displaystyle in both ends)		$F = k \int (v_1 - v_2) dt$	$F = k(x_1 - x_2)$	—

Mechanical Rotational System

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Mechanical rotational systems are comprised of a group of mechanical elements that only rotate. We will only consider rotations of the inertias about one axis. The choice of positive angular displacement is arbitrary; however, the equations must be consistent. Typically, $\theta=0$ is chosen when the system is in the equilibrium position (i.e., at rest).

The kinematics relationships for an inertia rotating about one axis are

$$\frac{d\theta(t)}{dt} = \dot{\theta}(t) = \omega(t) \qquad \frac{d^2\theta(t)}{dt^2} = \ddot{\theta}(t) = \dot{\omega}(t) = \alpha(t)$$

Where t = Time

α = Angular acceleration (rad/s²)

θ = Angular displacement (rad)

ω = Angular Velocity (rad/s)

Elements of Mechanical System (Rotational)

Moment of Inertia

Spring

Damper

Analogy between Mechanical and electrical quantities

It is possible to make electrical and mechanical systems using *analog*s. An analogous electrical and mechanical system will have differential equations of the same form. There are two analogs that are used to go between electrical and mechanical systems.

Electrical to Mechanical 1 (Force-Current)

Electrical to Mechanical 2 (Force-Voltage)



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The analogous quantities are given below.

Electrical Quantity	Mechanical Analog I (Force-Current)	Mechanical Analog II (Force Voltage)
Voltage, e	Velocity, v	Force, f
Current, i	Force, f	Velocity, v
Resistance, R	Lubricity, $1/B$ (Inverse friction)	Friction, B
Capacitance, C	Mass, M	Compliance, $1/K$ (Inverse spring constant)
Inductance, L	Compliance, $1/K$ (Inverse spring constant)	Mass, M
Transformer, $N_1:N_2$	Lever, $L_1:L_2$	Lever, $L_1:L_2$



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To see the analogies more clearly, examine the following table that shows the constitutive relationships for the various analogous quantities. The entries for the mechanical analogs are formed by substituting the analogous quantities into the equations for the electrical elements.

Electrical Equation	Mechanical Analog I (Force-Current)	Mechanical Analog II (Force Voltage)
$e = iR$	$v = \frac{f}{B}$	$f = vB$
$e = L \frac{di}{dt}$ $i = \frac{1}{L} \int e \cdot dt$	$v = \frac{1}{K} \frac{df}{dt}$ $f = K \int v \cdot dt = K \cdot x$	$f = M \frac{dv}{dt} = M \cdot a$ $v = \frac{1}{M} \int f \cdot dt$
$e = \frac{1}{C} \int i \cdot dt$ $i = C \frac{de}{dt}$	$v = \frac{1}{M} \int f \cdot dt$ $f = M \frac{dv}{dt} = M \cdot a$	$f = K \int v \cdot dt = K \cdot x$ $v = \frac{1}{K} \frac{df}{dt}$
power = $e \cdot i$	power = $v \cdot f$	power = $f \cdot v$
Transformer $\frac{e_1}{e_2} = \frac{N_1}{N_2} = \frac{i_2}{i_1}$	Lever $\frac{v_1}{v_2} = \frac{L_1}{L_2} = \frac{f_2}{f_1}$	Lever $\frac{f_1}{f_2} = \frac{L_2}{L_1} = \frac{v_2}{v_1}$
capacitor energy $\frac{1}{2} C \cdot e^2$	mass energy $\frac{1}{2} M \cdot v^2$	spring energy $\frac{1}{2} K \cdot x^2 = \frac{1}{2} K \cdot \left(\frac{f}{K} \right)^2 = \frac{1}{2} \frac{f^2}{K}$

Converting Between Systems

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Electrical to Mechanical 1 (Force-Current).

The important relationship when converting from a circuit to the Mechanical 1 analog is that between

Kirchoff's Current Law and D'Alemberts Law (with inertial forces included).

Electrical	Mechanical 1
$\sum_{\text{node}} \text{current} = 0$	$\sum_{\text{object}} \text{forces} = 0$

One deficiency in this analogy is that it only works easily for capacitors that are grounded. This can

Be seen by the analogies between energy in a capacitor and energy in a mass, and the analogy between

electrical ground (unchangeable voltage=0) and mechanical "ground" (immoveable position).

Electrical	Mechanical 1
capacitor energy $\frac{1}{2} C \cdot e^2$	mass energy $\frac{1}{2} M \cdot v^2$
voltage of ground=0 (you can apply any current to ground and voltage remains 0)	velocity of ground=0 (you can apply any force to ground and velocity remains 0)

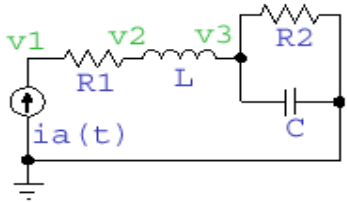
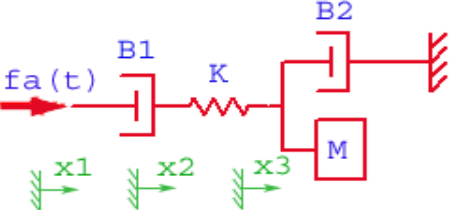


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Procedure for Conversion from Electrical to Mechanical 1

Example: Conversion from Electrical to Mechanical 1 -- Mathematical Method

<p>Start with an electrical circuit. Label all node voltages.</p>	
<p>Write a node equations for each node voltage</p>	$\sum_{\text{node } v_1} i = 0 = i_a + \frac{(e_2 - e_1)}{R_1}$ $\sum_{\text{node } v_2} i = 0 = \frac{(e_1 - e_2)}{R_1} + \frac{1}{L} \int (e_3 - e_2) dt$ $\sum_{\text{node } v_3} i = 0 = \frac{1}{L} \int (e_2 - e_3) dt - \frac{e_3}{R_2} - C \frac{de_3}{dt}$
<p>Re-write the equations using analogs (make making substitutions from the table of analogous quantities), with each electrical node being replaced by a position.</p>	$\sum_{x_1} f = 0 = f_a + (v_2 - v_1)B_1$ $\sum_{x_2} f = 0 = (v_1 - v_2)B_1 + K \int (v_3 - v_2) dt$ $= (v_1 - v_2)B_1 + K(x_3 - x_2)$ $\sum_{x_3} f = 0 = K \int (v_2 - v_3) dt - v_3 B_2 - Ma_3$ $= K(x_2 - x_3) - v_3 B_2 - Ma_3$
<p>Draw the mechanical system that corresponds with the equations.</p>	



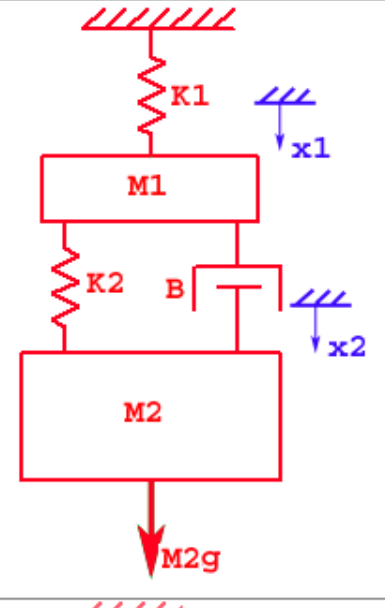
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Mechanical 1 (Force-Current) to Electrical.

The procedure to go from Mechanical 1 to Electrical is simply the reverse of Electrical to Mechanical 1. A simple visual method can be used where force generators are replaced by current sources, friction elements by resistors, springs by inductors, and masses by capacitors (which are grounded). Each position becomes a node in the circuit.

Example: Conversion from Mechanical 1 to Electrical -- Visual Method

Start with the mechanical system. Label all positions.

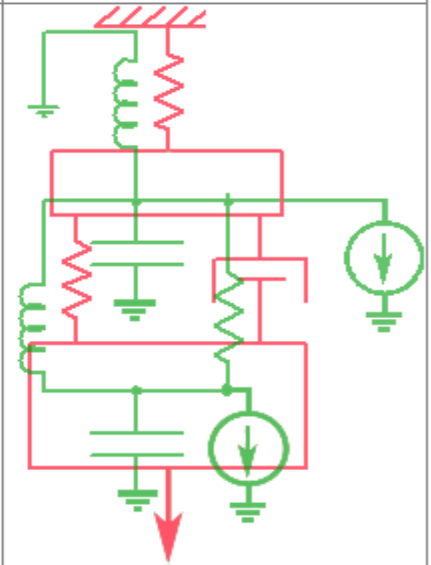




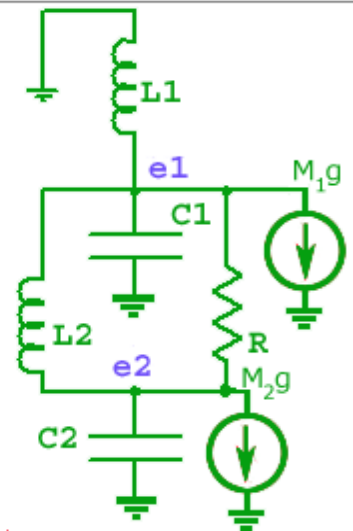
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Draw over circuit, replacing mechanical elements with their analogs; force generators by current sources, input velocities by voltage sources, friction elements by resistors, springs by inductors, and masses by capacitors (which are grounded). Each position becomes a node.



Label nodes and electrical elements as they were in the original mechanical system.





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Electrical to Mechanical 2 (Force-Voltage).

The important relationship when converting from a circuit to the Mechanical 2 analog is that between Kirchoff's Voltage Law and D'Alemberts Law (with inertial forces included)

Electrical	Mechanical 2
$\sum_{\text{loop}} \text{voltage} = 0$	$\sum_{\text{object}} \text{forces} = 0$

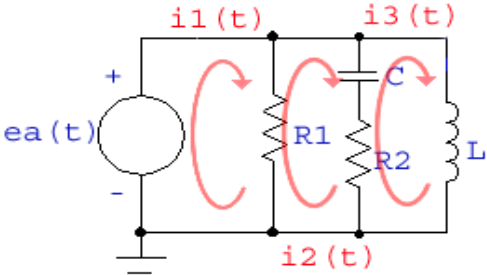
One deficiency in this analogy is that it only works easily for inductors with only one current defined through them. This can be seen by the analogies between energy in an inductor and energy in a mass

Electrical	Mechanical 2
inductor energy	mass energy
$\frac{1}{2} L \cdot i^2$	$\frac{1}{2} M \cdot v^2$

Procedure for Conversion from Electrical to Mechanical 2

Converting a circuit diagram to a mechanical 2 analog uses a similar procedure as electrical to mechanical 1 except that the voltages around a loop summed to zero (instead of the sum of currents at a node) is analogous to the sum of forces at a point being summed.

Example: Conversion from Electrical to Mechanical 2 -- Mathematical Method

<p>Start with an electrical circuit. Label all currents. Choose currents so that only one current flows through inductors.</p>	
<p>Write a loop equations for each loop.</p>	$\sum_{\text{loop } i_1} v = 0 = e_a + (i_2 - i_1)R_1$ $\sum_{\text{loop } i_2} v = 0 = (i_1 - i_2)R_1 + \frac{1}{C} \int (i_3 - i_2) dt + (i_3 - i_2)R_2$ $\sum_{\text{loop } i_3} v = 0 = \frac{1}{C} \int (i_2 - i_3) dt + (i_2 - i_3)R_2 - L \frac{di_3}{dt}$
<p>Re-write the equations using analogs (make making substitutions from the table of analogous quantities), with each electrical loop being replaced by a position.</p>	$\sum_{x_1} f = 0 = f_a + (v_2 - v_1)B_1$ $\sum_{x_2} f = 0 = (v_1 - v_2)B_1 + K \int (v_3 - v_2) dt + (v_3 - v_2)B_2$ $= (v_1 - v_2)B_1 + K(x_3 - x_2) + (v_3 - v_2)B_2$ $\sum_{x_3} f = 0 = K \int (v_2 - v_3) dt + (v_2 - v_3)B_2 - M \frac{dv_3}{dt}$