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System Modeling & Control

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Module V

Root Locus Technique



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Root Locus

Motivation

To satisfy transient performance requirements, it may be necessary to know how to choose certain controller parameters so that the resulting closed-loop poles are in the performance regions, which can be solved with Root Locus technique.

Definition

A graph displaying the roots of a polynomial equation when one of the parameters in the coefficients of the equation changes from 0 to ∞ .

Rules for Sketching Root Locus

Examples

Controller Design Using Root Locus

Letting the CL characteristic equation (CLCE) be the polynomial equation, one can use the Root Locus technique to find how a positive controller design parameter affects the resulting CL poles, from which one can choose a right value for the controller parameter.

The Root Locus Method

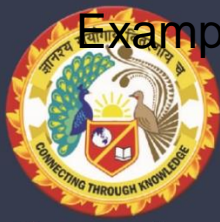
No matter what we pick K to be, the closed-loop system must always have n poles, where n is the number of poles of $G(s)$.

The root locus must have n branches, each branch starts at a pole of $G(s)$ and goes to a zero of $G(s)$.

If $G(s)$ has more poles than zeros (as is often the case), $m < n$ and we say that $G(s)$ has zeros at infinity. In this case, the limit of $G(s)$ as $s \rightarrow \infty$ is zero.

The number of zeros at infinity is $n-m$, the number of poles minus the number of zeros, and is the number of branches of the root locus that go to infinity (asymptotes).

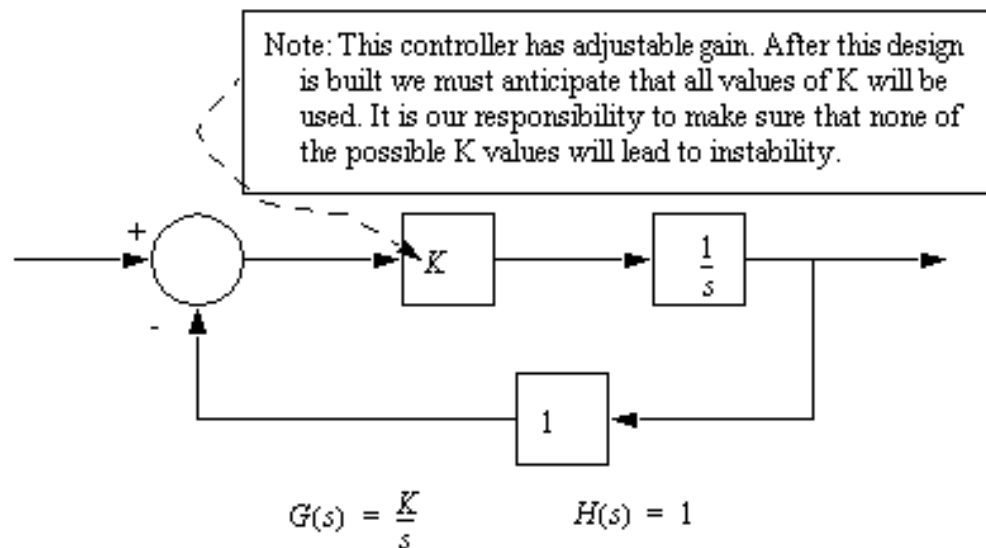
Since the root locus is actually the locations of all possible closed loop poles, from the root locus we can select a gain such that our closed-loop system will perform the way we want. If any of the selected poles are on the right half plane, the closed-loop system will be unstable. The poles that are closest to the imaginary axis have the greatest influence on the closed-loop response, so even though the system has three or four poles, it may still act like a second or even first order system depending on the location(s) of the dominant pole(s).



Example

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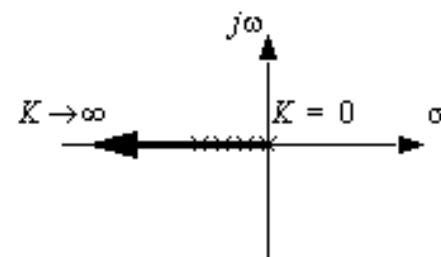


First, we must develop a transfer function for the entire control system.

$$G_g(s) = \frac{G(s)}{1 + G(s)H(s)} = \frac{\left(\frac{K}{s}\right)}{1 + \left(\frac{K}{s}\right)(1)} = \frac{K}{s + K}$$

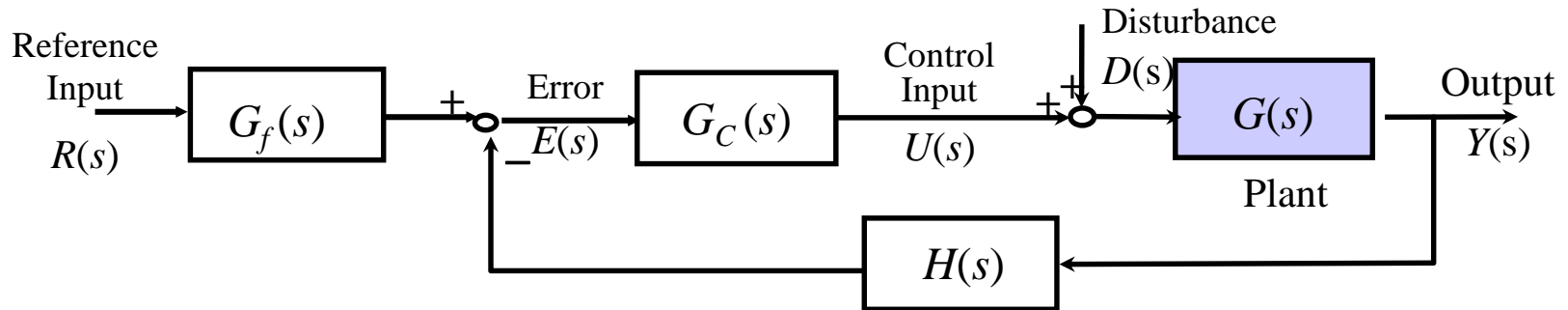
Next, we use the characteristic equation of the denominator to find the roots as the value of K varies. These can then be plotted on a complex plane. Note: the value of gain ' K ' is normally found from 0 to $+\infty$.

$s + K = 0$	K	root
	0	
	1	
	2	
	3	
	etc..	



Note: because all of the roots for all values of K are real negative this system will always be stable, and it will always tend to have a damped response. The larger the value of K , the more stable the system becomes.

Closed-Loop Characteristic Equation (CLCE)



The closed-loop transfer function $G_{YR}(s)$ is:

$$G_{YR}(s) = \frac{G(s)G_c(s)G_f(s)}{1 + G(s)G_c(s)H(s)}$$

The closed-loop characteristic equation (CLCE) is:

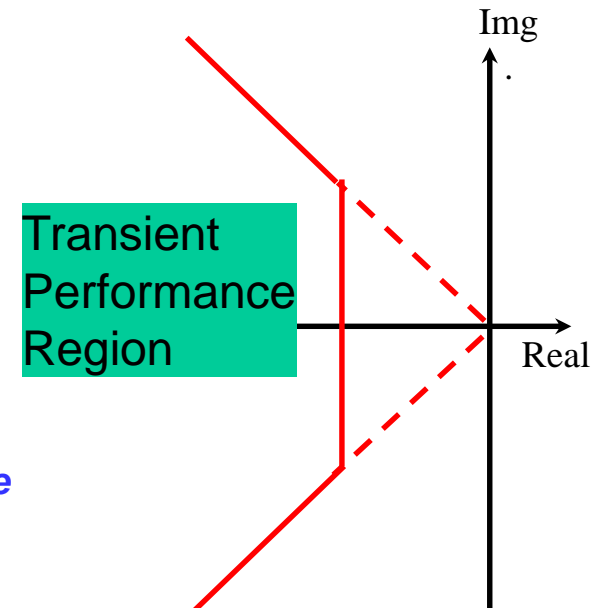
$$1 + G(s)G_c(s)H(s) = 0$$

For simplicity, assume a simple proportional feedback controller:

$$G_c(s) = K_p \quad \text{where } K_p GH = 0$$

The transient performance specifications define a region on the complex plane where the closed-loop poles should be located.

Q: How should we choose K_p such that the CL poles are within the desired performance boundary?





Motivation

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Ex: The closed-loop characteristic equation for the DC motor positioning system under proportional control is:

$$1 + K_p K_s G(s) = 0 \quad \Rightarrow \quad 1 + K_p \cdot 0.03 \cdot \frac{16}{s(0.0174s + 1)} = 0$$

Q: *How to choose K_p such that the resulting closed-loop poles are in the desired performance region?*

How do we find the roots of the equation:

as a function of the design parameter K_p ?

Graphically display the locations of the closed-loop poles for all $K_p > 0$ on the complex plane, from which we know the range of values for K_p that CL poles are in the performance region.

Root Locus – Definition

Root Locus is the method of graphically displaying the roots of a polynomial equation having the following form on the complex plane when the parameter K varies from 0 to ∞ :

$$1 + K \cdot G(s) = 0 \quad \text{or} \quad 1 + K \cdot \frac{N(s)}{D(s)} = 0$$

where $N(s)$ and $D(s)$ are known polynomials in factorized form:

$$N(s) = (s - z_1)(s - z_2) \cdots (s - z_{N_z})$$

$$D(s) = (s - p_1)(s - p_2) \cdots (s - p_{N_p})$$

Conventionally, the N_z roots of the polynomial $N(s)$, z_1, z_2, \dots, z_{N_z} , are called the finite open-loop zeros. The N_p roots of the polynomial $D(s)$, p_1, p_2, \dots, p_{N_p} , are called the finite open-loop poles.

Note: By transforming the closed-loop characteristic equation of a feedback controlled system with a single positive design parameter K into the above standard form, one can use the Root Locus technique to determine the range of K that have CL poles in the performance region.

■ Apply the following root locus sketching rules to obtain an *approximated* root locus plot.

$$1 + K \cdot \frac{N(s)}{D(s)} = 0 \quad \Rightarrow \quad 1 + K \cdot \frac{(s - z_1)(s - z_2) \cdots (s - z_{N_Z})}{(s - p_1)(s - p_2) \cdots (s - p_{N_P})} = 0$$

Rule 1: The number of branches of the root locus is equal to the number of closed-loop poles (or roots of the characteristic equation). In other words, the number of branches is equal to the number of open-loop poles or open-loop zeros, whichever is greater.

Rule 2: Root locus starts at open-loop poles (*when $K=0$*) and ends at open-loop zeros (*when $K=\infty$*). If the number of open-loop poles is greater than the number of open-loop zeros, some branches starting from finite open-loop poles will terminate at zeros at infinity (i.e., go to infinity). If the reverse is true, some branches will start at poles at infinity and terminate at the finite open-loop zeros.

Rule 3: Root locus is symmetric about the real axis, which reflects the fact that closed-loop poles appear in complex conjugate pairs.

Rule 4: Along the real axis, the root locus includes all segments that are to the left of an *odd* number of finite real open-loop poles and zeros.

Root Locus Sketching Rules

Rule 5: If number of poles N_p exceeds the number of zeros N_z , then as $K \rightarrow \infty$, $(N_p - N_z)$ branches will become asymptotic to straight lines. These straight lines intersect the real axis with angles θ_k at σ_0 .

$$\sigma_0 = \frac{\sum p_i - \sum z_i}{N_p - N_z} = \frac{\text{Sum of open-loop poles} - \text{Sum of open-loop zeros}}{\# \text{ of open-loop poles} - \# \text{ of open-loop zeros}}$$

$$\theta_k = (2k + 1) \frac{\pi}{N_p - N_z} \text{ [rad]} = (2k + 1) \frac{180^\circ}{N_p - N_z} \text{ [deg]}, \quad k = 0, 1, 2, \dots$$

If N_z exceeds N_p , then as $K \rightarrow 0$, $(N_z - N_p)$ branches behave as above.

Rule 6: Breakaway and/or break-in (arrival) points should be the solutions to the following equations:

$$\frac{d}{ds} \left(\frac{N(s)}{D(s)} \right) = 0 \quad \text{or} \quad \frac{d}{ds} \left(\frac{D(s)}{N(s)} \right) = 0$$

Root Locus Sketching Rules

Rule 7: The departure angle for a pole p_i (the arrival angle for a zero z_i) can be calculated by slightly modifying the following equation:

angle criterion →
$$\angle(s - z_1) + \angle(s - z_2) + \dots + \angle(s - z_{N_z}) - \angle(s - p_1) - \angle(s - p_2) - \dots - \angle(s - p_{N_p}) = 180^\circ$$

The departure angle q_j from the pole p_j can be calculated by replacing the term with q_j and replacing all the s 's with p_j in the other terms.

Rule 8: If the root locus passes through the imaginary axis (the stability boundary), the crossing point $j\omega$ and the corresponding gain K can be found as follows:

Replace s in the left side of the closed-loop characteristic equation with $j\omega$ to obtain the real and imaginary parts of the resulting complex number.

Set the real and imaginary parts to zero, and solve for ω and K . This will tell you at what values of K and at what points on the $j\omega$ axis the roots will cross.

Steps to Sketch Root Locus

Lesson - 25

Step 1: Transform the closed-loop characteristic equation into the standard form for sketching root locus:

$$1 + K \cdot \frac{N(s)}{D(s)} = 0 \quad \text{or} \quad 1 + K \cdot \frac{(s - z_1)(s - z_2) \cdots (s - z_{N_Z})}{(s - p_1)(s - p_2) \cdots (s - p_{N_P})} = 0$$

Step 2: Find the open-loop zeros, z_i , and the open-loop poles, p_i . Mark the open-loop poles and zeros on the complex plane. Use \bullet to represent open-loop poles and \circ to represent the open-loop zeros.

Step 3: Determine the real axis segments that are on the root locus by applying Rule 4.

Step 4: Determine the number of asymptotes and the corresponding intersection σ_0 and angles θ_k by applying Rules 2 and 5.

Step 5: (If necessary) Determine the break-away and break-in points using Rule 6.

Step 6: (If necessary) Determine the departure and arrival angles using Rule 7.

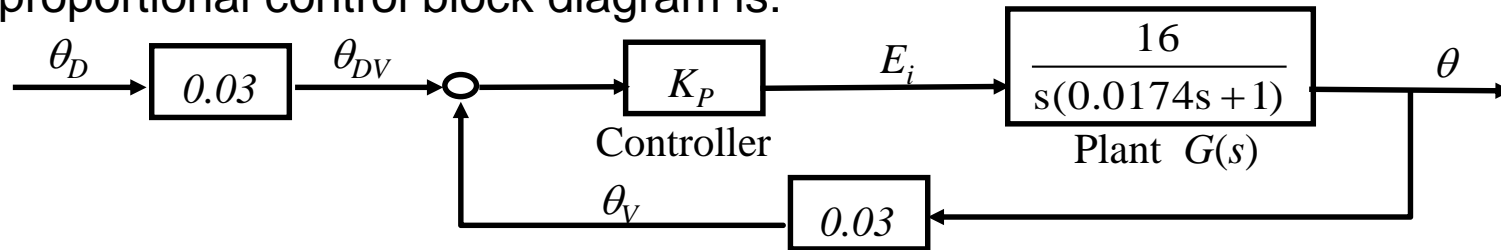
Step 7: (If necessary) Determine the imaginary axis crossings using Rule 8.

Step 8: Use the information from Steps 1-7 and Rules 1-3 to sketch the root locus.

Example 1

DC Motor Position Control

In the previous example on the printer paper advance position control, the proportional control block diagram is:



Sketch the root locus of the closed-loop poles as the proportional gain K_P varies from 0 to ∞ .

Find closed-loop characteristic equation:

$$1 + K_p G(s) H(s) = 0$$

$$1 + K_p \frac{0.48 \frac{N(s)}{s(0.0174s + 1)}}{\underbrace{D(s)}} = 0$$

Example 1

Step 1: Transform the closed-loop characteristic equation into the standard form for sketching root locus:

$$1 + \boxed{27.58K_p} \frac{1}{\underbrace{s(s + 57.47)}_{D(s)}} = 0$$

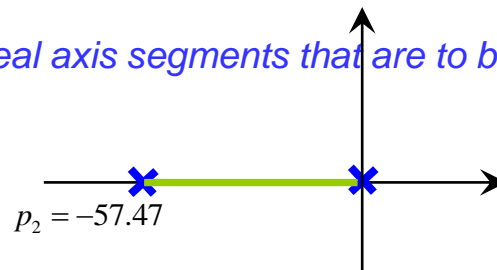
Step 2: Find the open-loop zeros, z_i , and the open-loop poles, p_i :

No open-loop zeros

open-loop poles

$$p_1 = 0, p_2 = -57.47$$

Step 3: Determine the real axis segments that are to be included in the root locus by applying Rule 4.



$$p_1 = 0$$

Example 1

Step 4: Determine the number of asymptotes and the corresponding intersection σ_0 and angles θ_k by applying Rules 2 and 5.

$$\sigma_0 = \frac{\sum p_i - \sum z_i}{N_p - N_z} = \frac{-57.47}{2} = -28.74$$

$$\theta_k = (2k + 1) \frac{\pi}{N_p - N_z} \text{ [rad]} = \begin{cases} \frac{\pi}{2} \\ \frac{3\pi}{2} \end{cases}$$

Step 5: (If necessary) Determine the break-away and break-in points using Rule 6.

$$\frac{d}{ds} \left(\frac{N(s)}{D(s)} \right) = 0 \quad \text{or} \quad \frac{d}{ds} \left(\frac{D(s)}{N(s)} \right) = 0, \quad \frac{d}{ds} \left(\frac{s(0.0174s + 1)}{0.48} \right) = 0, 0.0348s + 1 = 0, s = -28.74$$

Step 6: (If necessary) Determine the departure and arrival angles using Rule 7.

$$-\angle(p_2 - p_1) - \theta_{p_2} = 180^\circ, \theta_{p_2} = 0^\circ$$

$$-\theta_{p_1} - \angle(p_1 - p_2) = 180^\circ, \theta_{p_1} = 180^\circ$$

Step 7: (If necessary) Determine the imaginary axis crossings using Rule 8.

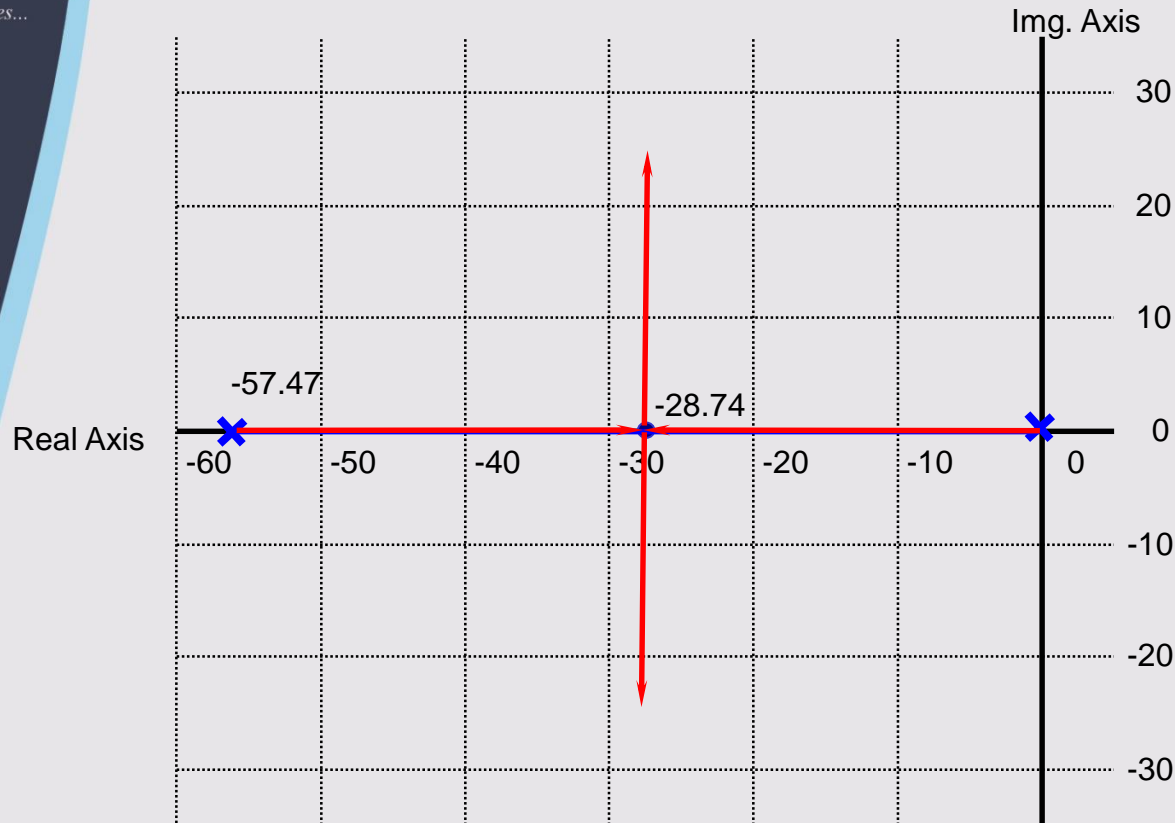
Could s be pure imaginary in this example?



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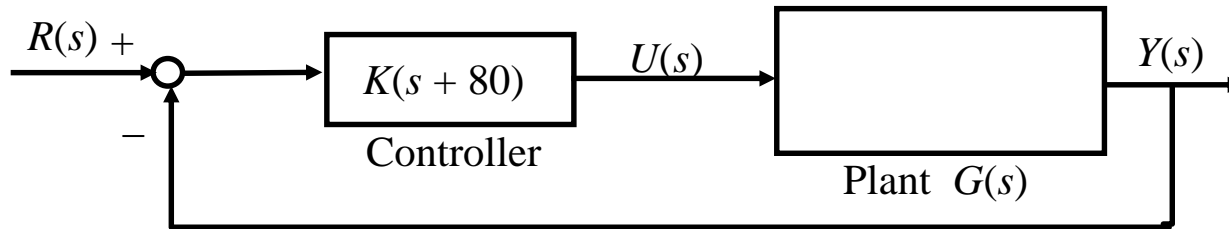
Example 1

Step 8: Use the information from Steps 1-7 and Rules 1-3 to sketch the root locus.



Example 2

A positioning feedback control system is proposed. The corresponding block diagram is:



Sketch the root locus of the closed-loop poles as the controller gain K varies from 0 to ∞ .

Find closed-loop characteristic equation:

$$1 + G_c(s)G(s)H(s) = 0$$

$$1 + K(s + 80) \frac{16}{s(0.0174s + 1)} = 0$$

Example 2

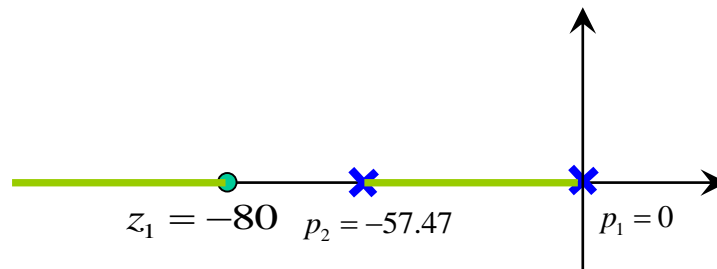
Step 1: Formulate the (closed-loop) characteristic equation into the standard form for sketching root locus:

$$1 + K \frac{\underbrace{16(s+80)}_{N(s)}}{\underbrace{s(0.0174s+1)}_{D(s)}} = 1 + 920K \frac{\underbrace{(s+80)}_{N(s)}}{\underbrace{s(s+57.47)}_{D(s)}} = 0$$

Step 2: Find the open-loop zeros, z_i , and the open-loop poles, p_i :

$$z_1 = -80 \quad \begin{array}{l} \boxed{\text{open-loop zeros}} \\ \boxed{\text{open-loop poles}} \end{array} \quad p_1 = 0, p_2 = -57.47$$

Step 3: Determine the real axis segments that are to be included in the root locus by applying Rule 4.





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Example 2

Step 4: Determine the number of asymptotes and the corresponding intersection σ_0 and angles θ_k by applying Rules 2 and 5.

$$\frac{d}{ds} \left(\frac{N(s)}{D(s)} \right) = 0 \quad \text{or} \quad \frac{d}{ds} \left(\frac{D(s)}{N(s)} \right) = 0,$$

Step 5: (If necessary) Determine the break-away and break-in points using Rule 6.

$$\frac{d}{ds} \left(\frac{(s+80)}{s(s+57.47)} \right) =$$

$$s^2 + 160s + 4600 = 0$$

$$s_1 = -122, s_2 = -37.6$$

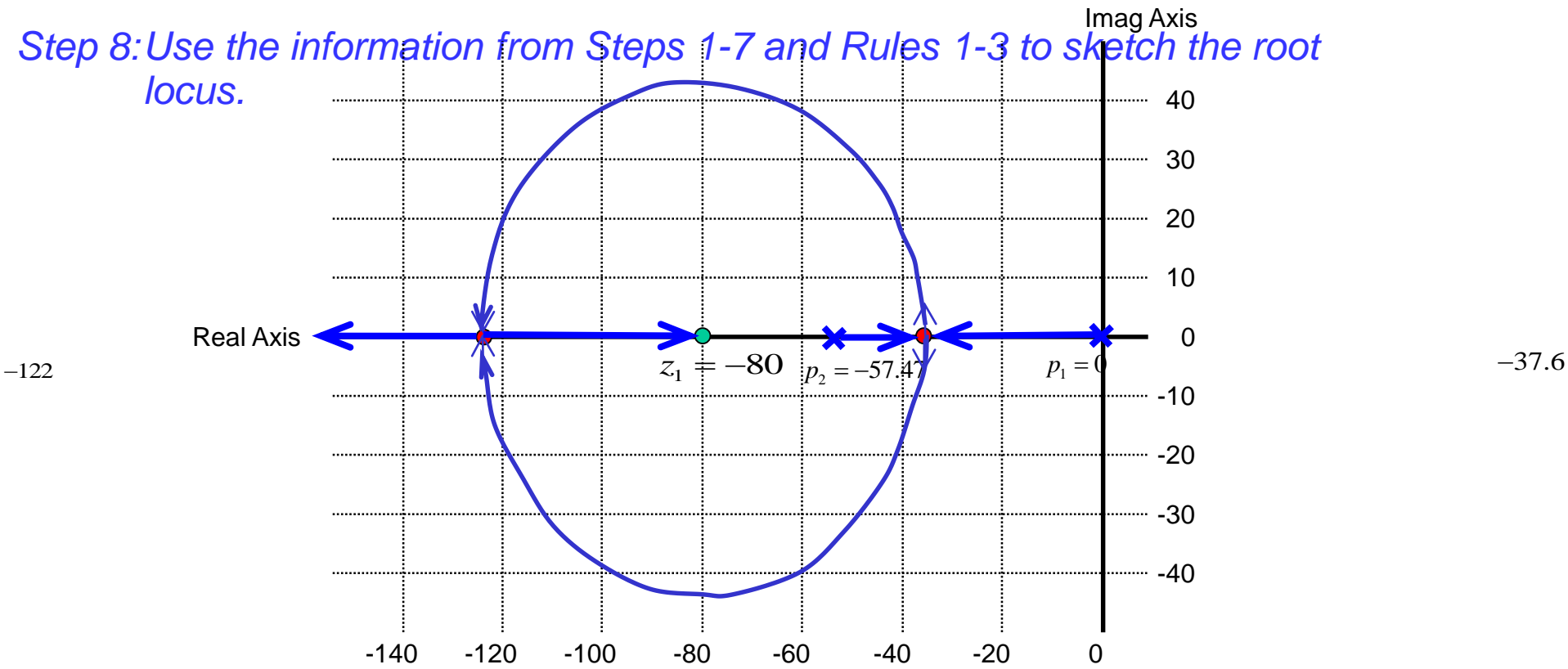
$$\frac{s(s+57.47) - (s+80)(2s+57.47)}{[s(s+57.47)]^2} = 0,$$

Example 2

Step 6:(If necessary) Determine the departure and arrival angles using Rule 7.

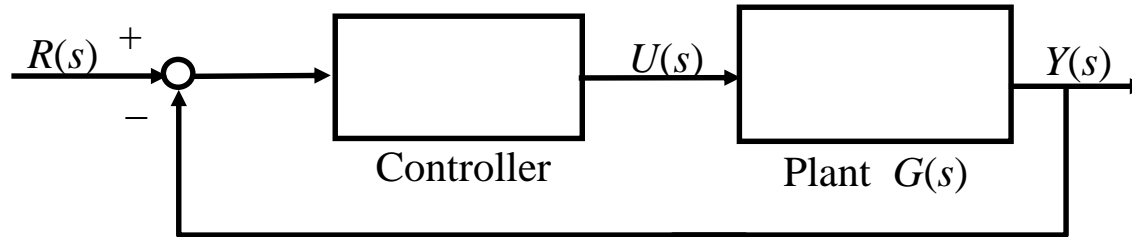
Step 7:(If necessary) Determine the imaginary axis crossings using Rule 8.

Step 8: Use the information from Steps 1-7 and Rules 1-3 to sketch the root locus.



Example 3

A feedback control system is proposed. The corresponding block diagram is:



Sketch the root locus of the closed-loop poles as the controller gain K varies from 0 to ∞ .

Find closed-loop characteristic equation:

$$1 + G_c(s)G(s)H(s) = 0$$

$$1 + \frac{K}{s+4} \frac{1}{s(s^2 + 4s + 20)} = 0$$

Example 3

Step 1: Transform the closed-loop characteristic equation into the standard form for sketching root locus:

$$1 + K \frac{1}{\underbrace{s(s^2 + 4s + 20)(s + 4)}_{D(s)}} = 0$$

Step 2: Find the open-loop zeros, z_i , and the open-loop poles, p_i :

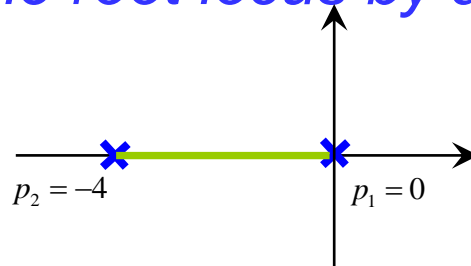
open-loop zeros

No open-loop zeros

open-loop poles

$$p_1 = 0, p_2 = -4, p_{3,4} = -2 \pm 4j$$

Step 3: Determine the real axis segments that are to be included in the root locus by applying Rule 4.



Example 3

Step 4: Determine the number of asymptotes and the corresponding intersection σ_0 and angles θ_k by applying Rules 2 and 5.

$$\sigma_0 = \frac{\sum p_i - \sum z_i}{N_P - N_Z} = \frac{0 + (-4) + (-2 + 4j) + (-2 - 4j)}{4 - 0} = -2$$

$$\theta_k = (2k + 1) \frac{\pi}{N_P - N_Z} \text{ [rad]} = \begin{cases} \frac{\pi}{4} \\ \frac{3\pi}{4} \\ \frac{5\pi}{4} \\ \frac{7\pi}{4} \end{cases}$$

Step 5: (If necessary) Determine the break-away and break-in points using Rule 6.

$$\frac{d}{ds} \left(\frac{N(s)}{D(s)} \right) = 0 \text{ or } \frac{d}{ds} \left(\frac{D(s)}{N(s)} \right) = 0,$$

$$\frac{d}{ds} \left(\frac{D(s)}{N(s)} \right) = \frac{d}{ds} \left(\frac{s(s^2 + 4s + 20)(s + 4)}{1} \right) = \frac{d}{ds} (s^4 + 8s^3 + 36s^2 + 80s)$$

$$= 4s^3 + 24s^2 + 72s + 80 = 0$$

$$s_1 = -2, s_{2,3} = -2 \pm 2.45j$$

Example 3

Step 6:(If necessary) Determine the departure and arrival angles using Rule

$$7. \sum_{i=1}^{N_z} \angle(s - z_i) - \sum_{i=1}^{N_p} \angle(s - p_i) = 180^\circ$$

$$p_1 = 0 : \quad \theta_{p_1} = 180^\circ \quad p_3 = -2 + 4j : \quad \theta_{p_3} = -90^\circ$$

$$p_2 = -4 : \quad \theta_{p_2} = 0^\circ \quad p_4 = -2 - 4j : \quad \theta_{p_4} = 90^\circ$$

Step 7:(If necessary) Determine the imaginary axis crossings using Rule 8.

$$1 + K \frac{1}{s(s^2 + 4s + 20)(s + 4)} = 0 \Rightarrow$$

$$s(s^2 + 4s + 20)(s + 4) + K = 0$$

$$\Leftrightarrow s^4 + 8s^3 + 36s^2 + 80s + K = 0$$

CLCE

$$s = j\omega$$

$$\Rightarrow (\omega^4 - 36\omega^2 + K) + (-8\omega^3 + 80\omega)j = 0$$

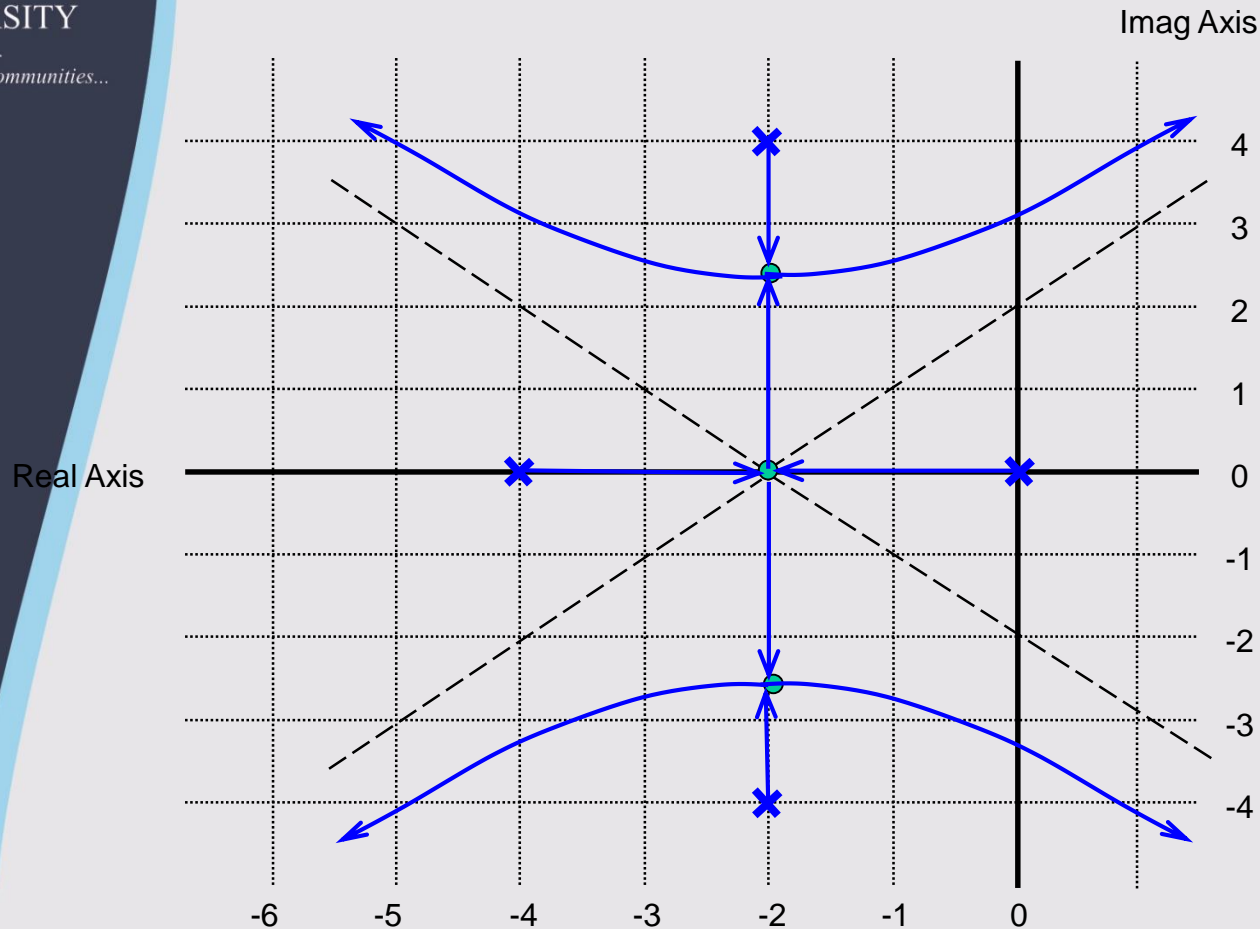
$$\Rightarrow \begin{cases} \omega^4 - 36\omega^2 + K = 0 \\ -8\omega^3 + 80\omega = 0 \end{cases} \Rightarrow \begin{cases} K_1 = 0 \\ \omega_1 = 0 \end{cases}, \begin{cases} K_2 = 260 \\ \omega_2 = \sqrt{10} = 3.16 \end{cases}$$



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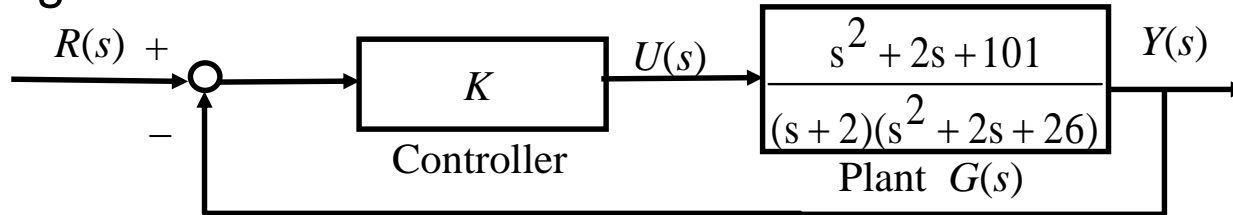
Example 3

Step 8: Use the information from Steps 1-7 and Rules 1-3 to sketch the root locus.



Example 4

A feedback control system is proposed. The corresponding block diagram is:



Sketch the root locus of the closed-loop poles as the controller gain K varies from 0 to ∞ .

Find closed-loop characteristic equation:

$$1 + K \frac{s^2 + 2s + 101}{(s + 2)(s^2 + 2s + 26)} = 0$$

Example 4

Step 1: Formulate the (closed-loop) characteristic equation into the standard form for sketching root locus:

$$1 + K \frac{\overbrace{s^2 + 2s + 101}^{N(s)}}{\underbrace{(s+2)(s^2 + 2s + 26)}_{D(s)}} = 0$$

Step 2: Find the open-loop zeros, z_i , and the open-loop poles, p_i :

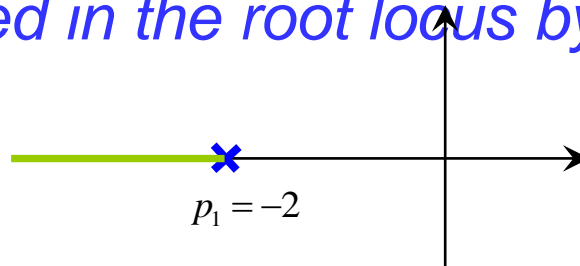
open-loop zeros

$$s^2 + 2s + 101 = (s+1)^2 + 100 = 0, z_{1,2} = -1 \pm 10j$$

open-loop poles

$$(s+2)((s+1)^2 + 25) = 0, p_1 = -2, p_{2,3} = -1 \pm 5j$$

Step 3: Determine the real axis segments that are to be included in the root locus by applying Rule 4.



Example 4

Step 4: Determine the number of asymptotes and the corresponding intersection σ_0 and angles θ_k by applying Rules 2 and 5.

One asymptote

$$N_p - N_z = 1$$

$$\theta_k = (2k + 1) \times 180^\circ = 180^\circ$$

Step 5: (If necessary) Determine the break-away and break-in points using Rule 6.

$$z_1 = -1 + 10j \quad \theta_{z_1} = 90^\circ - \tan^{-1}(10) - 90^\circ - 90^\circ = 180^\circ$$

Step 6: (If necessary) Determine the departure and arrival angles using Rule 7.

$$\theta_{z_1} = 354^\circ = -6^\circ$$

$$p_1 = -2$$

$$\theta_{p_1} = 180^\circ$$

$$p_2 = -1 + 5j$$

$$\theta_{p_2} = 11^\circ$$

$$z_2 = -1 - 10j \quad \theta_{z_2} = 6^\circ$$

$$p_3 = -1 - 5j$$

$$\theta_{p_2} = -11^\circ$$

Step 7: (If necessary) Determine the imaginary axis crossings using Rule 8.

$$(s + 2)(s^2 + 2s + 26) + K(s^2 + 2s + 101) = 0$$

$$\Leftrightarrow s^3 + (4 + K)s^2 + (30 + 2K)s + (52 + 101K) = 0$$

$$\stackrel{s=j\omega}{\Rightarrow} [(52 + 101K) - (4 + K)\omega^2] + [(30 + 2K) - \omega^2]\omega j = 0$$

$$\begin{cases} (52 + 101K) - (4 + K)\omega^2 = 0 \\ [(30 + 2K) - \omega^2]\omega = 0 \end{cases} \Rightarrow \begin{cases} \omega_1 = 0 \\ K_1 = -\frac{52}{101} \end{cases}, \begin{cases} \omega_2 = 9.5 \\ K_2 = 30.4 \end{cases}, \begin{cases} \omega_3 = 5.7 \\ K_3 = 1.1 \end{cases}$$

Example 4

Step 8: Use the information from Steps 1-7 and Rules 1-3 to sketch the root locus

Stability condition

$$0 < K < 1.1$$

or

$$K > 30.4$$

