

S-Domain Analysis

s-Domain Circuit Analysis

**Time domain
(t domain)**

↓
Linear
Circuit

↓
Differential
equation

↓
Classical
techniques

↓
Response
waveform

Laplace Transform

L

Laplace Transform

L

Inverse Transform

L^{-1}

**Complex frequency
domain (s domain)**

↓
Transformed
Circuit

↓
Algebraic
equation

↓
Algebraic
techniques

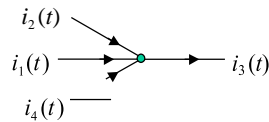
↓
Response
transform

Kirchhoff's Laws in s-Domain

t domain

s domain

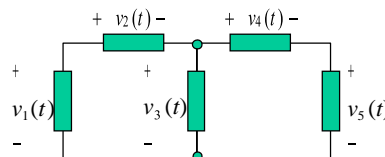
Kirchhoff's current law (KCL)



$$i_1(t) + i_2(t) - i_3(t) - i_4(t) = 0$$

$$I_1(s) + I_2(s) - I_3(s) - I_4(s) = 0$$

Kirchhoff's voltage law (KVL)



$$-v_1(t) + v_2(t) + v_3(t) = 0$$

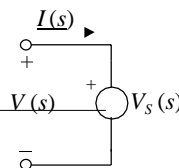
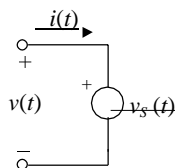
$$-V_1(s) + V_2(s) + V_3(s) = 0$$

Signal Sources in s Domain

t domain

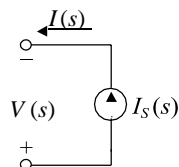
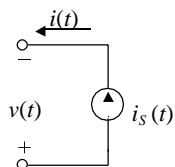
s domain

Voltage Source:
 $v(t) = v_S(t)$
 $i(t)$ = depends
 on circuit



Voltage Source:
 $V(s) = V_S(s)$
 $I(s)$ = depends
 on circuit

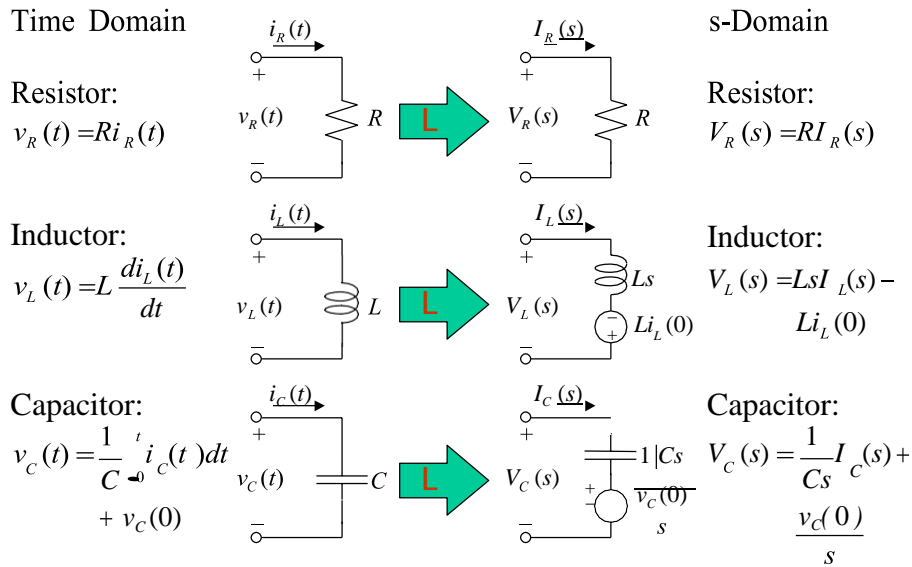
Current Source:
 $i(t) = i_S(t)$
 $v(t)$ = depends
 on circuit



Current Source:
 $I(s) = I_S(s)$
 $V(s)$ = depends
 on circuit

Time and s-Domain Element Models

Impedance and Voltage Source for Initial Conditions



Impedance and Voltage Source for Initial Conditions

- Impedance $Z(s)$

$$Z(s) = \frac{\text{voltage transform}}{\text{current transform}}$$

with all initial conditions set to zero

- Impedance of the three passive elements

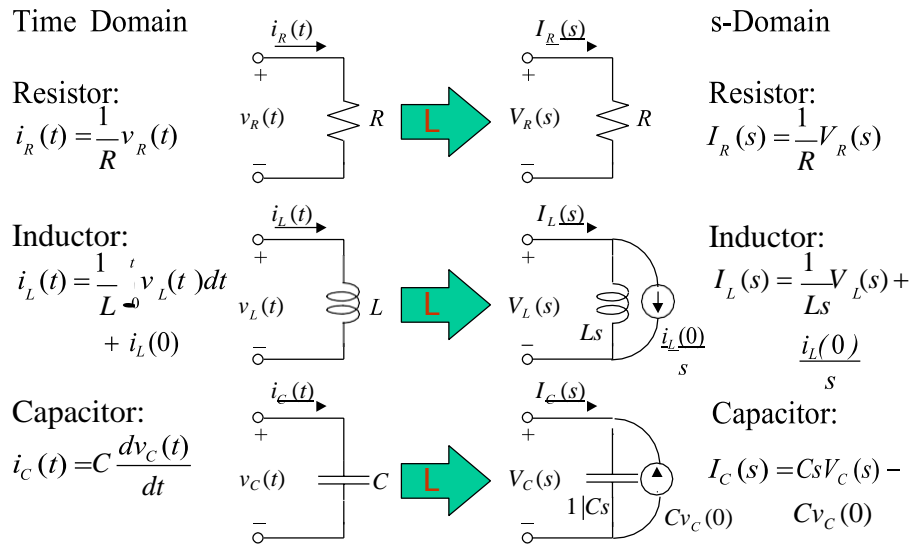
$$Z_R(s) = \frac{V_R(s)}{I_R(s)} = R$$

$$Z_L(s) = \frac{V_L(s)}{I_L(s)} = Ls \quad \text{with } i_L(0) = 0$$

$$Z_C(s) = \frac{V_C(s)}{I_C(s)} = \frac{1}{Cs} \quad \text{with } v_C(0) = 0$$

Time and s-Domain Element Models

Admittance and Current Source for Initial Conditions



Admittance and Current Source for Initial Conditions

- Admittance $Y(s)$

$$Y(s) = \frac{\text{current transform}}{\text{voltage transform}} = \frac{1}{Z(s)}$$

with all initial conditions set to zero

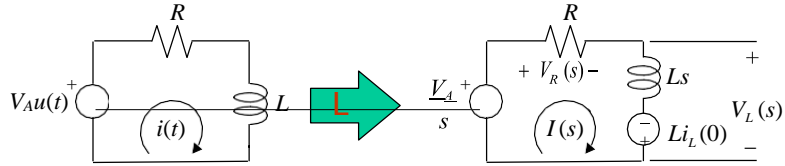
- Admittance of the three passive elements

$$Y_R(s) = \frac{I_R(s)}{V_R(s)} = \frac{1}{R}$$

$$Y_L(s) = \frac{I_L(s)}{V_L(s)} = \frac{1}{Ls} \quad \text{with } i_L(0) = 0$$

$$Y_C(s) = \frac{I_C(s)}{V_C(s)} = Cs \quad \text{with } v_C(0) = 0$$

Example: Solve for Current Waveform $i(t)$



By KVL: $-\frac{V_A}{s} + V_R(s) + V_L(s) = 0$

Resistor: $V_R(s) = RI(s)$ Inductor: $V_L(s) = LsI(s) - Li_L(0)$

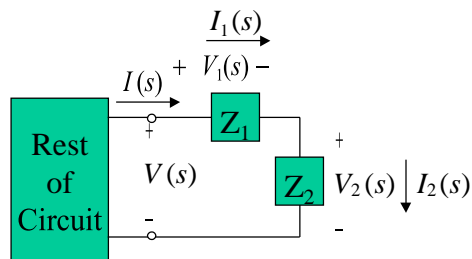
$$-\frac{V_A}{s} + RI(s) + LsI(s) - Li_L(0) = 0$$

$$I(s) = \frac{V_A |L}{s(s + R|L)} + \frac{i_L(0)}{s + R|L}$$

$$= \frac{V_A}{s} \frac{R}{R} - \frac{V_A}{s + R|L} \frac{R}{R} + \frac{i_L(0)}{s + R|L}$$

Inverse Transform: $i(t) = \left[\underbrace{\frac{V_A}{R}}_{\text{forced response}} - \underbrace{\frac{V_A}{R} e^{-\frac{R}{L}t}}_{\text{natural response}} + i_L(0)e^{-\frac{R}{L}t} \right] u(t)$

Series Equivalence and Voltage Division

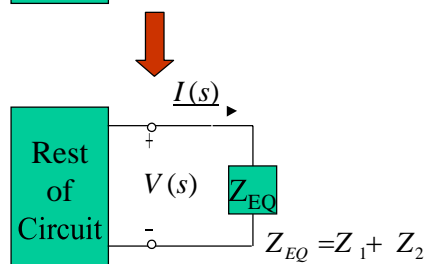


$$V_1(s) = Z_1(s)I_1(s) = Z_1(s)I(s)$$

$$V_2(s) = Z_2(s)I_2(s) = Z_2(s)I(s)$$

KVL: $V(s) = V_1(s) + V_2(s)$
 $= (Z_1(s) + Z_2(s))I(s)$

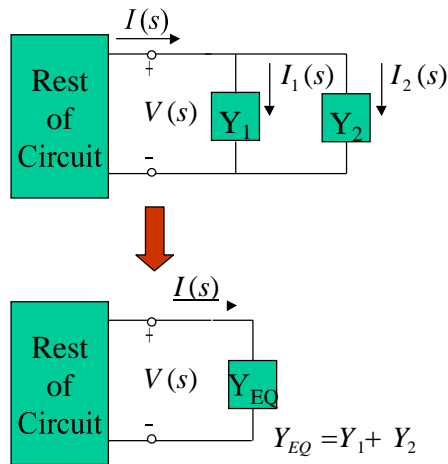
$\Rightarrow Z_{EQ}(s) = Z_1(s) + Z_2(s)$



$$V_1(s) = \frac{Z_1(s)}{Z_{EQ}(s)} V(s)$$

$$V_2(s) = \frac{Z_2(s)}{Z_{EQ}(s)} V(s)$$

Parallel Equivalence and Current Division



$$I_1(s) = Y_1(s)V(s)$$

$$I_2(s) = Y_2(s)V(s)$$

$$\text{KCL: } I(s) = I_1(s) + I_2(s) \\ = (Y_1(s) + Y_2(s))V(s)$$

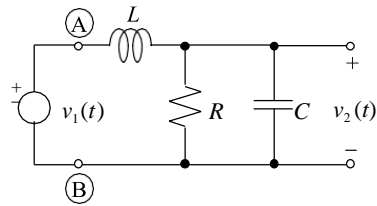
$$\rightarrow Y_{EQ}(s) = Y_1(s) + Y_2(s)$$

$$I_1(s) = \frac{Y_1(s)}{Y_{EQ}(s)} I(s)$$

$$I_2(s) = \frac{Y_2(s)}{Y_{EQ}(s)} I(s)$$

Example:

Equivalence Impedance and Admittance



Inductor current = 0 } at t = 0
capacitor voltage = 0 }

Find equivalent impedance at A and B
Solve for $v_2(t)$

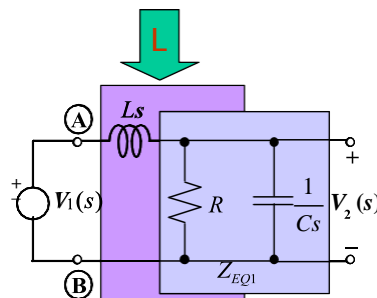
$$Y_{EQ1}(s) = \frac{1}{Z_{EQ1}(s)} = \frac{1}{R} + Cs = \frac{RCs + 1}{R}$$

$$Z_{EQ}(s) = Ls + Z_{EQ1}(s) = Ls + \frac{R}{RCs + 1}$$

$$= \frac{RLCs^2 + Ls + R}{RCs + 1}$$

$$V_2(s) = \frac{Z_{EQ1}(s)}{Z_{EQ}(s)} V_1(s)$$

$$= \frac{R}{RLCs^2 + Ls + R} V_1(s)$$



General Techniques for s-Domain Circuit Analysis

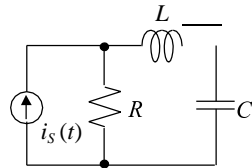
- Node Voltage Analysis (in s-domain)
 - Use Kirchhoff's Current Law (KCL)
 - Get equations of node voltages
 - Use current sources for initial conditions
 - Voltage source \longrightarrow current source
- Mesh Current Analysis (in s-domain)
 - Use Kirchhoff's Voltage Law (KVL)
 - Get equations of currents in the mesh
 - Use voltage sources for initial conditions
 - Current source \longrightarrow voltage source

(Works only for "Planar" circuits)

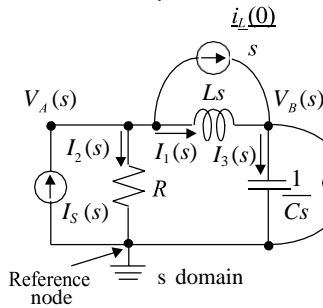
Formulating Node-Voltage Equations

- Step 0:** Transform the circuit into the s domain using current sources to represent capacitor and inductor initial conditions
- Step 1:** Select a reference node. Identify a node voltage at each of the non-reference nodes and a current with every element in the circuit
- Step 2:** Write KCL connection constraints in terms of the element currents at the non-reference nodes
- Step 3:** Use the element admittances and the fundamental property of node voltages to express the element currents in terms of the node voltages
- Step 4:** Substitute the device constraints from Step 3 into the KCL connection constraints from Step 2 and arrange the resulting equations in a standard form

Example: Formulating Node-Voltage Equations



t domain



s domain
Reference node

Step 0: Transform the circuit into the s domain using current sources to represent capacitor and inductor initial conditions

Step 1: Identify $N-1=2$ node voltages and a current with each element

Step 2: Apply KCL at nodes A and B:

$$\text{Node A: } I_S(s) - \frac{i_L(0)}{s} - I_1(s) - I_2(s) = 0$$

$$\text{Node B: } C v_C(0) + \frac{i_L(0)}{s} + I_1(s) - I_3(s) = 0$$

Step 3: Express element equations in terms of node voltages

$$I_1(s) = Y_L(s) [V_A(s) - V_B(s)] = \frac{1}{Ls} [V_A(s) - V_B(s)]$$

$$I_2(s) = Y_R(s) V_A(s) = G V_A(s) \text{ where } G = 1/R$$

$$I_3(s) = Y_C(s) V_B(s) = Cs V_B(s)$$

Formulating Node-Voltage Equations (Cont'd)

Step 2: Apply KCL at nodes A and B:

$$\text{Node A: } I_S(s) - \frac{i_L(0)}{s} - I_1(s) - I_2(s) = 0$$

$$\text{Node B: } C v_C(0) + \frac{i_L(0)}{s} + I_1(s) - I_3(s) = 0$$

Step 3: Express element equations in terms of node voltages

$$I_1(s) = Y_L(s) [V_A(s) - V_B(s)] = \frac{1}{Ls} [V_A(s) - V_B(s)]$$

$$I_2(s) = Y_R(s) V_A(s) = G V_A(s) \text{ where } G = 1/R$$

$$I_3(s) = Y_C(s) V_B(s) = Cs V_B(s)$$

Step 4: Substitute eqns. in Step 3 into eqns. in Step 2 and collect common terms to yield node-voltage eqns.

$$\text{Node A: } \left(G + \frac{1}{Ls} \right) V_A(s) - \left(\frac{1}{Ls} \right) V_B(s) = I_S(s) - \frac{i_L(0)}{s}$$

$$\text{Node B: } - \left(\frac{1}{Ls} \right) V_A(s) + \left(\frac{1}{Ls} + Cs \right) V_B(s) = C v_C(0) + \frac{i_L(0)}{s}$$

Solving s-Domain Circuit Equations

- Circuit Determinant: $\Delta(s) = \begin{vmatrix} G + 1/Ls & -1/Ls \\ -1/Ls & Cs + 1/Ls \end{vmatrix}$
 $= (G + 1/Ls)(Cs + 1/Ls) - (1/Ls)^2$
 $= \frac{GLCs^2 + Cs + G}{Ls}$

Depends on circuit element parameters: L, C, G=1/R,
not on driving force and initial conditions

- Solve for node A using Cramer's rule:

$$V_A(s) = \frac{\Delta_A(s)}{\Delta(s)} = \frac{\begin{vmatrix} I_s(s) + i_L(0) & s & -1/Ls \\ i_L(0)/s + C v_C(0) & Cs + 1/Ls & \end{vmatrix}}{\Delta(s)}$$

$$= \underbrace{\frac{(LCs^2 + 1)I_s(s)}{GLCs^2 + Cs + G}}_{\text{Zero State}} + \underbrace{\frac{-LCs i_L(0) + C v_C(0)}{GLCs^2 + Cs + G}}_{\text{Zero input}}$$

Zero State Zero input
when initial condition when input sources
sources are turned off are turned off

Solving s-Domain Circuit Eqns. (Cont'd)

- Solve for node B using Cramer's rule:

$$V_B(s) = \frac{\Delta_B(s)}{\Delta(s)} = \frac{\begin{vmatrix} G + 1/Ls & I_s(s) - i_L(0) & s \\ -1/Ls & i_L(0)/s + C v_C(0) & \end{vmatrix}}{\Delta(s)}$$

$$= \underbrace{\frac{I_s(s)}{GLCs^2 + Cs + G}}_{\text{Zero State}} + \underbrace{\frac{GL i_L(0) + (GLs + 1)C v_C(0)}{GLCs^2 + Cs + G}}_{\text{Zero input}}$$

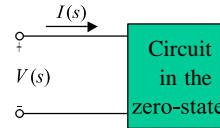
Zero State Zero input

Network Functions

Network function = $\frac{\text{Zero-state Response Transform}}{\text{Input Signal Transform}}$

- Driving-point function relates the voltage and current at a given pair of terminals called a port

$$Z(s) = \frac{V(s)}{I(s)} = \frac{1}{Y(s)}$$



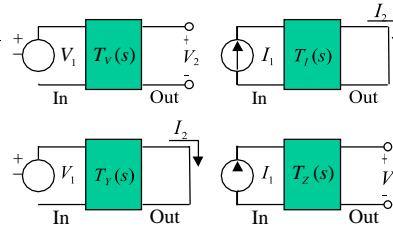
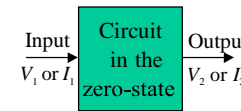
- Transfer function relates an input and response at different ports in the circuit

$$T_V(s) = \text{Voltage Transfer Function} = \frac{V_2(s)}{V_1(s)}$$

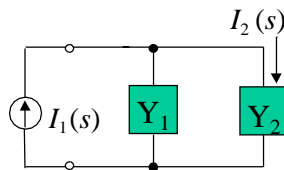
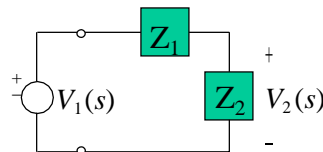
$$T_I(s) = \text{Current Transfer Function} = \frac{I_2(s)}{I_1(s)}$$

$$T_Y(s) = \text{Transfer Admittance} = \frac{I_2(s)}{V_1(s)}$$

$$T_Z(s) = \text{Transfer Impedance} = \frac{V_2(s)}{I_1(s)}$$



Calculating Network Functions



- Driving-point impedance

$$Z_{EQ}(s) = Z_1(s) + Z_2(s)$$

- Voltage transfer function:

$$V_2(s) = \left[\frac{Z_2(s)}{Z_1(s) + Z_2(s)} \right] V_1(s)$$

$$T_V(s) = \frac{V_2(s)}{V_1(s)} = \frac{Z_2(s)}{Z_1(s) + Z_2(s)}$$

- Driving-point admittance

$$Y_{EQ}(s) = Y_1(s) + Y_2(s)$$

- Voltage transfer function:

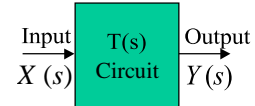
$$I_2(s) = \left[\frac{Y_2(s)}{Y_1(s) + Y_2(s)} \right] I_1(s)$$

$$T_I(s) = \frac{I_2(s)}{I_1(s)} = \frac{Y_2(s)}{Y_1(s) + Y_2(s)}$$

Impulse Response and Step Response

- Input-output relationship in s-domain

$$Y(s) = T(s)X(s)$$



- When input signal is an impulse $x(t) = \delta(t)$

$$Y(s) = T(s) \times 1 = T(s)$$

- Impulse response equals network function
- $H(s)$ = impulse response transform
- $h(t)$ = impulse response waveform

- When input signal is a step $x(t) = u(t)$

- $G(s)$ = step response transform

- $g(t)$ = step response waveform

$$G(s) = \frac{T(s)}{s} = \frac{H(s)}{s}$$

$$g(s) = \int_0^t h(t) dt, \quad h(t) (=) \frac{dg(t)}{dt}$$

(=) means equal almost everywhere, excludes those points at which $g(t)$ has a discontinuity