

Session-2

CR Equation: The Complex-valued function $w = f(z) = u + iv$ is analytic in a domain D iff C-R equations $u_x = v_y$ & $u_y = -v_x$ are satisfied, assuming all partial derivatives $u_x, u_y, v_x, & v_y$ are continuous in D .

Cauchy-Riemann Equations in polar form

Let $z = r(\cos \theta + i \sin \theta) = re^{i\theta}$ (where $r > 0$)

& Let $w = f(z) = f(re^{i\theta}) = u + iv$

ie. $u(r, \theta) + iv(r, \theta) = f(re^{i\theta})$

$$\Rightarrow u_\theta = -rv_r \quad \text{or} \quad \boxed{v_r = -\frac{1}{r}u_\theta} \quad (7) \quad \text{Again} \quad v_\theta = ru_r$$

or $\boxed{u_r = \frac{1}{r}v_\theta}$ (8)

are called C-R equations in polar form.

Laplace's Equation

If $f(z) = u(x, y) + iv(x, y)$ is analytic in a domain D , then u & v satisfy

Laplace's equation.

i.e. $\nabla^2 u = u_{xx} + u_{yy} = 0$

& $\nabla^2 v = v_{xx} + v_{yy} = 0$

respectively in D and both have continuous 2nd order partial derivative in D .

Example: $f(z) = z + \frac{1}{z}$ is analytic ?

Ans: Let $z = r(\cos\theta + i\sin\theta) = re^{i\theta}$ and $\frac{1}{z} = \frac{1}{re^{i\theta}} = \frac{1}{r}e^{-i\theta} = \frac{1}{r}(\cos\theta - i\sin\theta)$

$$\begin{aligned}\text{Now } f(z) = z + \frac{1}{z} &= r(\cos\theta + i\sin\theta) + \frac{1}{r}(\cos\theta - i\sin\theta) \\ &= \left(r + \frac{1}{r}\right)\cos\theta + i\left(r - \frac{1}{r}\right)\sin\theta\end{aligned}$$

But we know that $f(z) = w = u + iv$

$$\text{i.e } u = \left(r + \frac{1}{r}\right)\cos\theta \text{ and } v = \left(r - \frac{1}{r}\right)\sin\theta$$

$$\frac{\partial u}{\partial r} = \left(1 - \frac{1}{r^2}\right)\cos\theta \text{ and } \frac{\partial v}{\partial r} = \left(1 + \frac{1}{r^2}\right)\sin\theta$$

$$\frac{\partial u}{\partial \theta} = -\left(r + \frac{1}{r}\right)\sin\theta \text{ and } \frac{\partial v}{\partial \theta} = \left(r - \frac{1}{r}\right)\cos\theta$$

$$\Rightarrow \frac{1}{r}\frac{\partial v}{\partial \theta} = \left(1 - \frac{1}{r^2}\right)\cos\theta = \frac{\partial u}{\partial r}$$

Again $\frac{1}{r}\frac{\partial u}{\partial \theta} = -\left(1 + \frac{1}{r^2}\right)\sin\theta = -\frac{\partial v}{\partial r}$ hence it satisfies the CR equation.

Therefore the given complex function is analytic function.

Example-2

Find the Laplacian form of the following function $u = \ln|z|$

Ans: Given that $u = \ln|z| = \ln\sqrt{x^2 + y^2}$ (where $z = x + iy$)

$$u = \frac{1}{2} \ln(x^2 + y^2)$$

$$u_x = \frac{x}{x^2 + y^2} \quad \text{and} \quad u_y = \frac{y}{x^2 + y^2}$$

$$u_{xx} = \frac{(x^2 + y^2) - x \cdot 2x}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2}$$

$$u_{yy} = \frac{(x^2 + y^2) - y \cdot 2y}{(x^2 + y^2)^2} = \frac{x^2 - y^2}{(x^2 + y^2)^2}$$

$$\text{Therefore } u_{xx} + u_{yy} = \frac{y^2 - x^2}{(x^2 + y^2)^2} + \frac{x^2 - y^2}{(x^2 + y^2)^2} = 0.$$

Hence u satisfies the Laplacian equation. Therefore u is a Harmonic Function.

Harmonic function:

The function $u(x, y)$ is said to be harmonic if it satisfies Laplace's equation i.e. $\nabla^2 u = u_{xx} + u_{yy} = 0$.

Theorem 2: If $f(z) = u(x, y) + iv(x, y)$ is an analytic function then its real and imaginary parts i.e. u & v are harmonic.

Conjugate Harmonic function: If two harmonic functions u & v satisfy Cauchy-Riemann equations in a region S , then they are the real and imaginary parts of an analytic function $f(z)$ in S , i.e. $f(z) = u + iv$ is analytic in S .

So, v is called the 'conjugate harmonic function' of u in the region S . It is true that a harmonic conjugate, when it exists, is unique except for an additive constant

Note: If u is harmonic then v is called conjugate harmonic of u and v is harmonic then u is called conjugate harmonic of v .

Example-3: Determine whether the following function is harmonic then find their conjugate harmonic function.

1. $u = \ln|z|.$

2. $u = x^2 + y^2$

3. $v = \text{Arg}(z)$

Ans: Given that $u = \ln|z| = \ln\sqrt{x^2 + y^2}$ (where $z = x + iy$)

$$u = \frac{1}{2} \ln\sqrt{x^2 + y^2}$$

$$u_x = \frac{x}{x^2 + y^2} \quad \text{and} \quad u_y = \frac{y}{x^2 + y^2}$$

$$u_{xx} = \frac{(x^2 + y^2) - x \cdot 2x}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2}$$

$$u_{yy} = \frac{(x^2 + y^2) - y \cdot 2y}{(x^2 + y^2)^2} = \frac{x^2 - y^2}{(x^2 + y^2)^2}$$

Therefore $u_{xx} + u_{yy} = \frac{y^2 - x^2}{(x^2 + y^2)^2} + \frac{x^2 - y^2}{(x^2 + y^2)^2} = 0.$

Hence u satisfy the Laplacian equation. Therefore u is Harmonic Function.

We know that $u_x = v_y$ and $u_y = -v_x$ (CR – Equation)

$$u_x = v_y = \frac{x}{x^2 + y^2}$$

$$\frac{\partial v}{\partial y} = \frac{x}{x^2 + y^2} \dots\dots\dots(1)$$

Integrating both sides of (1) partially w.r.t. y keeping x constant, we obtain

$$v = \int \frac{x}{x^2 + y^2} dy + h(x)$$

$$v = \tan^{-1} \left(\frac{y}{x} \right) + h(x) \dots \dots \dots (2)$$

Differentiating equation (2) partially w.r.t. x we get

$$\frac{\partial v}{\partial x} = \frac{1}{1 + \left(\frac{y^2}{x^2} \right)} \cdot \left(\frac{-y}{x^2} \right) + h'(x)$$

$$\Rightarrow \frac{-y}{x^2 + y^2} + h'(x) \dots \dots \dots (3)$$

Again from second CR- equation $u_y = -v_x \Rightarrow v_x = -u_y$

$$v_x = \frac{-y}{x^2 + y^2} \dots \dots \dots (4)$$

Equating (3) and (4) we get $\frac{-y}{x^2 + y^2} + h'(x) = \frac{-y}{x^2 + y^2}$

i.e $h'(x) = 0$

Again integrate both sides we get $h(x) = 0$

Put the value of $h(x)$ value in equation (2) we get the required conjugate function.

Therefore $v(x, y) = \tan^{-1} \left(\frac{y}{x} \right)$ (Ans)

Exercise -1

Are the following functions harmonic? If your answer is yes, find a corresponding analytic function

$$f(z) = u(x, y) + iv(x, y).$$

12. $u = xy$

13. $v = xy$

14. $v = -y/(x^2 + y^2)$

15. $u = \ln |z|$

16. $v = \ln |z|$

17. $u = x^3 - 3xy^2$

18. $u = 1/(x^2 + y^2)$

19. $v = (x^2 - y^2)^2$

20. $u = \cos x \cosh y$

21. $u = e^{-x} \sin 2y$