

Session_10

Laurent's Series

If a single valued function $f(z)$ is analytic inside and on the boundaries of a ring shaped region (annulus) S bounded by two concentric circles C_1 and C_2 of radii r_1 and r_2 ($r_1 > r_2$) respectively having centre at a , then for all z is S ,

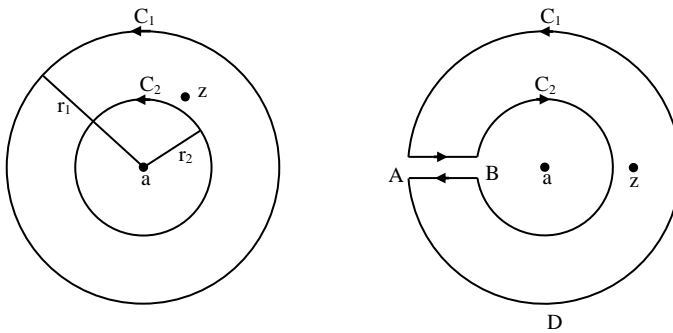
$$f(z) = A_0 + A_1(z-a) + A_2(z-a)^2 + \dots + B_1(z-a)^{-1} + B_2(z-a)^{-2} + \dots$$

$$\text{or, } f(z) = \sum_{n=0}^{\infty} A_n(z-a)^n + \sum_{n=1}^{\infty} B_n(z-a)^{-n}$$

$$\text{where, } A_n = \frac{1}{2\pi i} \oint_{C_1} \frac{f(t)}{(t-a)^{n+1}} dt ; n = 0, 1, 2, \dots$$

$$\text{and } B_n = \frac{1}{2\pi i} \oint_{C_2} \frac{f(t)}{(t-a)^{-n+1}} dt ; n = 1, 2, 3, \dots$$

Proof:



By introducing a cross AB the ring shaped region S , bounded by two circles C_1 and C_2 was converted into a simply connected region, bounded by a simple closed curve say D .

Let z be an interior point and t be a boundary point of the simple connected region.

\therefore By Cauchy's integral formula, we have

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \oint_D \frac{f(t)}{t-z} dt \\ &= \frac{1}{2\pi i} \left[\oint_{C_1} \frac{f(t)}{t-z} dt + \int_{AB} \frac{f(t)}{t-z} dt + \oint_{C_2} \frac{f(t)}{t-z} dt + \int_{BA} \frac{f(t)}{t-z} dt \right] \\ &= \frac{1}{2\pi i} \left[\oint_{C_1} \frac{f(t)}{t-z} dt + \oint_{C_2} \frac{f(t)}{t-z} dt \right] \left(\because \text{Integrals along } \right. \\ &\quad \left. AB \text{ and } BA \text{ are cancelled out} \right) \\ \Rightarrow f(z) &= \frac{1}{2\pi i} \oint_{C_1} \frac{f(t)}{t-z} dt - \frac{1}{2\pi i} \oint_{C_2} \frac{f(t)}{t-z} dt \end{aligned} \quad (5)$$

where C_1 and C_2 are traversed in anticlock-wise direction.

For the 1st integral in (5), t lies on C_1

$$\therefore |z-a| < |t-a| \quad \text{i.e., } \left| \frac{z-a}{t-a} \right| < 1$$

$$\begin{aligned} \text{Now, } \frac{1}{t-z} &= \frac{1}{(t-a)-(z-a)} = \frac{1}{t-a} \left[1 - \frac{z-a}{t-a} \right]^{-1} \\ &= \frac{1}{t-a} \left[1 + \left(\frac{z-a}{t-a} \right) + \left(\frac{z-a}{t-a} \right)^2 + \dots \right] \begin{pmatrix} \text{Expanding} \\ \text{Binomilly} \end{pmatrix} \end{aligned}$$

Multiplying both sides by $\frac{f(t)}{2\pi i}$ and integrating term by term w.r.t. t along the circle C_1 , we obtain

$$\begin{aligned} \frac{1}{2\pi i} \oint_{C_1} \frac{f(t)}{t-z} dt &= \frac{1}{2\pi i} \oint_{C_1} \frac{f(t)}{t-a} dt + \frac{z-a}{2\pi i} \oint_{C_1} \frac{f(t)}{(z-a)^2} dt \\ &\quad + \frac{(z-a)^2}{2\pi i} \oint_{C_1} \frac{f(t)}{(t-a)^3} dt + \dots \\ &= A_0 + A_1(z-a) + A_2(z-a)^2 + \dots \\ &= \sum_{n=0}^{\infty} A_n (z-a)^n \end{aligned} \quad (6)$$

$$\text{where } A_n = \frac{1}{2\pi i} \oint_{C_1} \frac{f(t)}{(t-a)^{n+1}} dt ; n=0,1,2,\dots \quad (7)$$

For the 2nd integral in (5); t lies on C_2

$$\therefore |t-a| < |z-a| \quad \text{or} \quad \left| \frac{t-a}{z-a} \right| < 1$$

$$\begin{aligned} \text{Now } \frac{1}{t-z} &= \frac{1}{(t-a)-(z-a)} = -\frac{1}{z-a} \left[1 - \frac{t-a}{z-a} \right]^{-1} \\ \Rightarrow -\frac{1}{t-z} &= \frac{1}{z-a} \left[1 + \left(\frac{t-a}{z-a} \right) + \left(\frac{t-a}{z-a} \right)^2 + \dots \right] \begin{pmatrix} \text{Expanding} \\ \text{Binomially} \end{pmatrix} \end{aligned}$$

Multiplying both sides by $\frac{f(t)}{2\pi i}$ and integrating term by term w.r.t. t along the circle C_2 , we obtain

$$\begin{aligned} -\frac{1}{2\pi i} \oint_{C_2} \frac{f(t)}{t-z} dt &= \frac{1}{2\pi i(z-a)} \oint_{C_2} f(t) dt \\ &\quad + \frac{1}{(z-a)^2} \cdot \frac{1}{2\pi i} \oint_{C_2} (t-a) f(t) dt \\ &\quad + \frac{1}{(z-a)^3} \cdot \frac{1}{2\pi i} \oint_{C_2} (t-a)^2 f(t) dt + \dots \\ &= B_1(z-a)^{-1} + B_2(z-a)^{-2} + B_3(z-a)^{-3} + \dots \\ &= \sum_{n=1}^{\infty} B_n (z-a)^{-n} \end{aligned} \quad (8)$$

$$\text{where, } B_n = \frac{1}{2\pi i} \oint_{C_2} \frac{f(t)}{(t-a)^{-n+1}} dt ; n=1,2,3,\dots \quad (9)$$

Substituting the values from (6) and (8) in (5), we obtain;

$$f(z) = \sum_{n=0}^{\infty} A_n (z-a)^n + \sum_{n=1}^{\infty} B_n (z-a)^{-n} \quad (10)$$

where A_n and B_n are given in (7) and (9).

Note:

(1) In the Laurent expansion (10) of a complex valued function $f(z)$, the first part $\sum_{n=0}^{\infty} A_n (z-a)^n$ is called the **analytic-part** of Laurent's series which is similar to the Taylor's series of the function and the remainder $\sum_{n=1}^{\infty} B_n (z-a)^{-n}$ is known as the **principal part** of the Laurent's series. The second summation is of chief interest, since it contains the singularity.

(2) In the statement of Laurent's series,

$$A_n = \frac{1}{2\pi i} \oint_{C_1} \frac{f(t)}{(t-a)^{n+1}} dt \neq \frac{f^n(a)}{n!}, \text{ because } f(z) \text{ is not given to be analytic inside } C_1$$

(3) In case $f(z)$ is analytic inside C_1 (or inside C_2), then $B_n = 0$ and

$$A_n = \frac{1}{2\pi i} \oint_{C_1} \frac{f(t)}{(t-a)^{n+1}} dt = \frac{f^n(a)}{n!} \text{ and Laurent's series reduces to Taylor's series.}$$

(4) If C is any simple closed curve lying in the ring shaped region S and enclosing the circle C_2 , then

$$\oint_{C_1} \frac{f(t)}{(t-a)^{n+1}} dt = \oint_C \frac{f(t)}{(t-a)^{n+1}} dt$$

$$\text{and } \oint_{C_2} \frac{f(t)}{(t-a)^{-n+1}} dt = \oint_C \frac{f(t)}{(t-a)^{-n+1}} dt$$

\therefore Laurent's series can be expressed as

$$f(t) = \sum_{n=-\infty}^{\infty} A_n (z-a)^n, \text{ where } A_n = \frac{1}{2\pi i} \oint_C \frac{f(t)}{(t-a)^{n+1}} dt$$

(5) The process of finding the coefficient A_n by complex integration is complicated.

In practice, we expand the function $f(z)$ by binomial theorem.